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Lacunary Statistical Convergence on Probabilistic Normed Spaces

Mohamad Rafi Segi Rahmat

School of Applied Mathematics,
The University of Nottingham Malaysia Campus,
Jalan Broga, 43500 Semenyih,Selangor Darul Ehsan.
Mohd.Rafi@nottingham.edu.my

Abstract

In this paper, we study the concepts of lacunary statistical convergent and lacunary statistical Cauchy sequences in probabilistic normed spaces and prove some basic properties.

Keywords: *lacunary sequence; lacunary statistical convergence; probabilistic norm; probabilistic normed spaces.*

1 Introduction

Probabilistic normed (PN) spaces are real linear spaces in which the norm of each vector is an appropriate probability distribution function rather than a number. Such spaces were introduced by Šerstnev in 1963 [15]. In [1] Alsina, Schweizer and Sklar gave a new definition of PN spaces which includes Šerstnev's as a special case and leads naturally to the identification of the principle class of PN spaces, the Menger spaces. In [2], the continuity properties of probabilistic norms and the vector space operations (vector addition and scalar multiplication) are studied in details and it is shown that a PN space endowed with the strong topology turns out to be a topological vector space under certain conditions. A detailed history and the development of the subject up to 2006 can be found in [14]. Şençimen and Pehlivan [13] extended the results in paper [2] to a more general type of continuity, namely, the statistical continuity of probabilistic norms and vector space operations via the concepts of strong statistical convergence (see also [12, 8, 7]).

Since the concept of Lacunary statistical convergence is a generalization of the concept of statistical convergence (see [4, 5, 10]), it seems reasonable to think if the concept of lacunary statistical convergence and lacunary statistical

Cauchy sequences (see [3, 6, 9]) via the concepts of lacunary statistical strong convergence and lacunary statistical Cauchy can be extended to probabilistic normed spaces and in that case how the basic properties are effected. Since the study of convergence in PN spaces is fundamental to probabilistic functional analysis, we feel that the concept of lacunary statistical convergence and lacunary statistical Cauchy in a PN space would provide a more general framework for the subject.

2 Preliminaries

We recall some basic definition and results concerning PN spaces, see [1, 11]. A distance distribution function is a nondecreasing function F defined on $\mathbb{R}^+ = [0, +\infty]$, with $F(0) = 0$ and $F(+\infty) = 1$, and is left-continuous on $(0, +\infty)$. The set of all distance distribution functions will be denoted by Δ^+. The elements of Δ^+ are partially order by the usual pointwise ordering of functions and has both a maximal element ε_0 and a minimal element ε_∞: these are given, respectively, by

$$\varepsilon_0(x) = \begin{cases} 0, & x \leq 0, \\ 1, & x > 0. \end{cases} \quad \text{and} \quad \varepsilon_\infty(x) = \begin{cases} 0, & x < +\infty, \\ 1, & x = +\infty. \end{cases}$$

There is a natural topology on Δ^+ that is induced by the modified Lévy metric d_L (see, [11], Sec. 4.2). Convergence with respect to the metric d_L is equivalent to weak convergence of distribution functions, i.e., $\{F_n\}$ in Δ^+ and F in Δ^+, the sequence $\{d_L(F_n, F)\}$ converges to 0 if and only if the sequence $\{F_n(x)\}$ converges to $F(x)$ at every point of continuity of the limit function F. Moreover, the metric space (Δ^+, d_L) is compact and complete.

A *triangle function* is a binary operation τ on Δ^+ that is commutative, associative, non-decreasing in each place, and has ε_0 as an identity element. Continuity of triangle function means uniform continuity with respect to the natural product topology on $\Delta^+ \times \Delta^+$.

Definition 2.1 *A probabilistic normed space (briefly, a PN space) is a quadruple (V, η, τ, τ^*) where V is a real linear space, τ and τ^* are continuous triangle functions, and, η be is a mapping from V into the space of distribution functions Δ^+ such that - writing N_p for $\eta(p)$-for all p, q in V, the following conditions hold:*
(N1) $N_p = \varepsilon_0$ if and only if $p = \theta$, the null vector in V,
(N2) $N_{-p} = N_p$,
(N3) $N_{p+q} \geq \tau(N_p, N_q)$,
(N4) $N_p \leq \tau^*(N_{\alpha p}, N_{(1-\alpha)p})$, for all α in $[0, 1]$.

It follows from (N1), (N2), (N3) that if $\mathcal{F}\colon S\times S\to\Delta^+$ is defined via

$$\mathcal{F}(p,q)=F_{pq}=N_{p-q}, \tag{1}$$

then $(V,\mathcal{F},\tau)$ is a PM space ([11], Chap. 8). Furthermore, since τ is continuous, the system of neighborhoods $\{\mathcal{N}_p(t)\colon p\in V, t>0\}$, where

$$\mathcal{N}_p(t)=\{q\in V\colon d_L(F_{pq},\varepsilon_0)<t\}=\{q\in V\colon F_{pq}(t)>1-t\} \tag{2}$$

determine a first-countable and Hausdorff topology on V, called the *strong topology.* Thus, the strong topology can be completely specified in terms of the convergence of sequences. Throughout this paper, V denotes a PN space endowed with the strong topology, written additively, which satisfies the first axiom of countability.

A sequence $\{p_n\}$ in V *converges strongly* to a point $p\in V$, and we write $\lim_n p_n=p$, if for any $t>0$ there is a positive integer m such that $p_n\in\mathcal{N}_p(t)$ for all $n\geq m$. Similarly, a sequence $\{p_n\}$ in V is a strong Cauchy sequence if for any $t>0$ there is a positive integer i such that $(p_n,p_m)\in\mathcal{U}(t)$ for all $n,m\geq i$, where

$$\mathcal{U}(t)=\{(p,q)\in V\times V\colon d_L(F_{pq},\varepsilon_0)<t\}$$

for any $t>0$ is called the *strong vicinity*(see [11]).

Definition 2.2 *[8] A sequence $\{p_k\}$ in V is statistically strong convergent to θ the null vector in V provided that for every $t>0$*

$$\lim_{n\to\infty}\frac{1}{n}|\{k\leq n\colon d_L(N_{p_k},\varepsilon_0)\geq t\}|=0.$$

In this case we write $S-\lim_k p_k=\theta$ or $p_k\to\theta(S)$.

We shall use S to denote the set of all statistically strong convergent sequences in V. Of course, there is nothing special about θ as a limit; if one wishes to consider the convergence of the sequence $\{p_n\}$ to the vector p, then it suffices to consider the sequence $\{p_n-p\}$ and its convergence to θ.

The *statistical strong Cauchy* sequence in PN space can be defined in a similar way as

Definition 2.3 *A sequence $\{p_k\}$ in V is a statistically strong Cauchy sequence if there there exists a positive integer m such that*

$$\lim_{n\to\infty}\frac{1}{n}|\{k\leq n\colon (p_k-p_m)\notin\mathcal{U}(t)\}|=0.$$

3 Lacunary statistical convergence and some basic properties

By a lacunary sequence $\vartheta = \{k_r\}$; $r = 0, 1, 2, \cdots$, where $k_0 = 0$, we means an increasing sequence of nonnegative integers with $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. The interval determined by ϑ will be denoted by $I_r = (k_{r-1}, k_r]$ and $q_r = \frac{k_r}{k_r - 1}$. A real number sequence $\{x_k\}$ is said to be *lacunary statistically convergent* (briefly S_ϑ-convergent) to $a \in \mathbb{R}$ provided that for each $t > 0$

$$\lim_{r\to\infty} \frac{1}{h_r} |\{k \in I_r \colon |x_k - a| \geq t\}| = 0.$$

A sequence $\{x_k\}$ is a cauchy sequence if there exists a subsequence $\{x_{k'(r)}\}$ of $\{x_k\}$ such that $k'(r) \in I_r$ for each r, $\lim_{r\to\infty} x_{k'(r)} = l$ and for every $t > 0$

$$\lim_{r\to\infty} \frac{1}{h_r} |\{k \in I_r \colon |x_k - x_{k'(r)}| \geq t\}| = 0.$$

Using these concepts, we extend the lacunary statistical convergence and lacunary statistical Cauchy to the setting of sequences in a PN space endowed with the strong topology as follows.

Definition 3.1 *Let ϑ be a lacunary sequence. A sequence $\{p_k\}$ in V is said to be lacunary statistically strong convergent (briefly, S_ϑ-strong convergent) to θ in V if for each $t > 0$,*

$$\lim_{r\to\infty} \frac{1}{h_r} |\{k \in I_r \colon d_L(N_{p_k}, \varepsilon_0) \geq t\}| = 0 \tag{3}$$

or, equivalently,

$$\lim_{r\to\infty} \frac{1}{h_r} |\{k \in I_r \colon p_k \notin \mathcal{N}_\theta(t)\}| = 0, \tag{4}$$

where $\mathcal{N}_\theta(t) = \{p \in V \colon N_p(t) > 1 - t\}$ is the neighborhood of θ. In this case, we write $S_\vartheta - \lim p_k = p$ or $p_k \to \theta(S_\vartheta)$, and we will call θ, as the lacunary strong limit of the sequence $\{p_k\}$. We shall use S_ϑ to denote the set of all lacunary strong convergent sequences from V.

Of course, there is nothing special about θ as a limit; if one wishes to consider the S_ϑ-strong convergent of the sequence $\{p_n\}$ to the vector p, then it suffices to consider the sequence $\{p_n - p\}$ and its S_ϑ-strong convergent to θ.

Theorem 3.2 *Let ϑ be a lacunary sequence. If $\{p_k\}$ is a S_ϑ-strong convergent sequence in V, then its limit is unique.*

PROOF. Suppose the sequence $\{p_k\}$ is S_ϑ-strong convergent to two distinct points p and q (say). Since $p \neq q$, we have $N_{p-q} \neq \varepsilon_0$, whence $t = d_L(N_{p-q}, \varepsilon_0) > 0$. Set

$$K_1 = \{k \in I_r \colon p - p_k \in \mathcal{N}_\theta(t/2)\},$$
$$K_2 = \{k \in I_r \colon p_k - q \in \mathcal{N}_\theta(t/2)\}.$$

Then, clearly $\lim_{r\to\infty} \frac{|K_1 \cap K_2|}{h_r} = 1$, so $K_1 \cap K_2$ is a nonempty set. Let $m \in K_1 \cap K_2$, then $d_L(N_{p-p_m}, \varepsilon_0) < t/2$ and $d_L(N_{p_m - q}, \varepsilon_0) < t/2$. By uniform continuity of τ, we have

$$d_L(N_{p-q}, \varepsilon_0) \leq d_L(\tau(N_{p-p_m}, N_{p_m-q}), \varepsilon_0) < t = d_L(N_{p-q}, \varepsilon_0),$$

a contradiction to the fact that $K_1 \cap K_2$ is a nonempty set. Therefore $p = q$ and the proof is completed. □

Theorem 3.3 *For any lacunary sequence ϑ, $S_\vartheta \subseteq S$ if* $\limsup_r q_r < \infty$.

PROOF. If $\limsup_r q_r < \infty$ then there exists a $\gamma > 0$ such that $q_r < \gamma$ for all r. Let $S_\vartheta - \lim_k p_k = \theta$. We are going to prove that $S - \lim_k p_k = \theta$. Set $K_r = |\{k \in I_r \colon p_k \notin \mathcal{N}_\theta(t)\}|$. Then, by definition, for a given $t > 0$, there exists $r_0 \in \mathbb{N}$ such that

$$\frac{K_r}{h_r} < \frac{t}{2\gamma} \text{ for all } r \geq r_0.$$

Let $M = \max\{K_r \colon 1 \leq r \leq r_0\}$ and let $n \in \mathbb{N}$ such that $k_{r-1} < n \leq k_r$. Then we can write

$$\begin{aligned}
\frac{1}{n}|\{k \leqslant n \colon p_k \notin \mathcal{N}_\theta(t)\}| &\leqslant \frac{1}{k_{r-1}}|\{k \leqslant k_r \colon p_k \notin \mathcal{N}_\theta(t)\}| \\
&= \frac{1}{k_{r-1}}\{K_1 + K_2 + \cdots + K_{r_0} + \cdots + K_r\} \\
&\leqslant \frac{M}{k_{r-1}} \cdot r_0 + \frac{t}{2\gamma} \cdot q_r \\
&\leqslant \frac{M}{k_{r-1}} \cdot r_0 + \frac{t}{2}
\end{aligned}$$

and the result follows immediately. □

Theorem 3.4 *For any lacunary sequence ϑ, $S \subseteq S_\vartheta$ if* $\limsup_r q_r > 1$.

PROOF. If $\limsup_r q_r > 1$ then there exists a $\xi > 0$ and a positive integer r_0 such that $q_r \geq 1 + \xi$ for all $r \geq r_0$. Hence for $r \geq r_0$,

$$\frac{h_r}{k_r} = 1 - \frac{k_{r-1}}{k_r} = \frac{\xi}{1 + \xi}.$$

Let $S-\lim_k p_k=\theta$. Then for every $t>0$ and for every $r\geq r_0$, we have

$$\begin{aligned}\frac{1}{k_r}|\{k\leq k_r\colon p_k\notin \mathcal{N}_\theta(t)\}| &\geq \frac{1}{k_r}|\{k\in I_r\colon p_k\notin \mathcal{N}_\theta(t)\}|\\ &\geq \frac{\xi}{1+\xi}\cdot\frac{1}{h_r}|\{k\in I_r\colon p_k\notin \mathcal{N}_\theta(t)\}|.\end{aligned}$$

Therefore $S_\vartheta-\lim_k p_k=\theta$. □

Corollary 3.5 *Let ϑ be a lacunary sequence, then $S=S_\vartheta$ if*

$$1<\liminf_r q_r\leq \limsup_r <\infty.$$

Proof. By combining the Theorem 3.3 and Theorem 3.4. □

Definition 3.6 *Let ϑ be a lacunary sequence. A sequence $\{p_k\}$ in V is said to be S_ϑ-strong Cauchy sequence if there exists a subsequence $\{p_{k'(r)}\}$ of $\{p_k\}$ such that $k'(r)\in I_r$ for each r, $\lim_{r\to\infty}p_{k'(r)}=p$ and for every $t>0$*

$$\lim_{r\to\infty}\frac{1}{h_r}|\{k\in I_r\colon p_k-p_{k'(r)}\notin \mathcal{N}_\theta(t)\}|=0, \tag{5}$$

Theorem 3.7 *The sequence $\{p_k\}$ in V is S_ϑ-strong convergent if and only if it is S_ϑ-strong Cauchy sequence in V.*

Proof. Let $S_\vartheta-\lim_k p_k=\theta$ and write

$$K_n=\{k\in\mathbb{N}\colon p_k\in\mathcal{N}_\theta(1/n)\},$$

for each $n\in\mathbb{N}$. Then, obviously $K_{n+1}\subseteq K_n$ for each n and $\lim_{r\to\infty}\frac{|K_n\cap I_r|}{h_r}=1$. This implies that there exists m_1 such that $r\geq m_1$ and $\frac{|K_1\cap I_r|}{h_r}>0$, i.e., $K_1\cap I_r\neq\emptyset$. We next choose $m_2\geq m_1$ such that $r\geq m_2$ implies that $K_2\cap I_r\neq\emptyset$. Thus for each r satisfying $m_1\leq r\leq m_2$, we choose $k'(r)\in I_r$ such that $k'(r)\in I_r\cap K_1$, i.e., $p_{k'(r)}\in\mathcal{N}_\theta(1)$. In general we choose $m_{n+1}>m_n$ such that $r\geq m_{n+1}$ implies that $k'(r)\in I_r\cap K_n$, i.e., $p_{k'(r)}\in\mathcal{N}_\theta(1/n)$. Thus $k'(r)\in I_r$, for each r and $p_{k'(r)}\in\mathcal{N}_\theta(1/n)$ implies that $\lim_r p_{k'(r)}=\theta$. Furthermore, for $t>0$ and the uniform continuity of τ implies that

$$\begin{aligned}\{k\in I_r\colon d_L(N_{p_k-p_{k'(r)}},\varepsilon_0)\geq t\} &\subseteq \{k\in I_r\colon d_L(\tau(N_{p_k},N_{p_{k'(r)}}),\varepsilon_0)\geq t\}\\ &\subseteq \{k\in I_r\colon d_L(N_{p_k},\varepsilon_0)\geq t/2\}\cup\{k\in I_r\colon d_L(N_{p_{k'(r)}},\varepsilon_0)\geq t/2\}.\end{aligned}$$

The above inclusion implies

$$\begin{aligned}\frac{1}{h_r}|\{k\in I_r\colon p_k-p_{k'(r)}\notin\mathcal{N}_\theta(t)\}| &\leq \frac{1}{h_r}|\{k\in I_r\colon p_k\notin\mathcal{N}_\theta(t/2)\}|\\ &+ \frac{1}{h_r}|\{k\in I_r\colon p_{k'(r)}\notin\mathcal{N}_\theta(t/2)\}|.\end{aligned}$$

Since $S_\vartheta - \lim_k p_k = \theta$ and $\lim_r p_{k'(r)} = \theta$, it follows that $\{p_k\}$ is a S_ϑ-strong Cauchy sequence.

Conversely, suppose that $\{p_k\}$ is a S_ϑ-strong Cauchy sequence. For every $t > 0$ and the uniform continuity of τ, we have

$$\begin{aligned}\{k \in I_r \colon d_L(N_{p_k}, \varepsilon_0) \geq t\} &\subseteq \{k \in I_r \colon d_L(\tau(N_{p_k - p_{k'(r)}}, N_{p_{k'(r)}}), \varepsilon_0) \geq t\} \\ &\subseteq \{k \in I_r \colon d_L(N_{p_k - p_{k'(r)}}, \varepsilon_0) \geq t/2\} \cup \{k \in I_r \colon d_L(N_{p_{k'(r)}}, \varepsilon_0) \geq t/2\}.\end{aligned}$$

The above inclusion implies

$$\begin{aligned}\frac{1}{h_r}|\{k \in I_r \colon p_k \notin \mathcal{N}_\theta(t)\}| &\leq \frac{1}{h_r}|\{k \in I_r \colon p_k - p_{k'(r)} \notin \mathcal{N}_\theta(t/2)\}| \\ &+ \frac{1}{h_r}|\{k \in I_r \colon p_{k'(r)} \notin \mathcal{N}_\theta(t/2)\}|\end{aligned}$$

for which it follows that $S_\vartheta - \lim_k p_k = \theta$. □

Corollary 3.8 *If $\{p_k\}$ in V is a S_ϑ-strong convergent sequence, then $\{p_k\}$ has a strong convergent subsequence.*

Proof. The proof is an immediate consequence of Theorem 3.4. □

4 Conclusion

We study the concepts of lacunary statistical convergent and lacunary statistical Cauchy sequences in probabilistic normed spaces and proved several important properties of sequences in probabilistic normed spaces.

5 Open Problem

It can be easily proved that if a sequence $\{p_n\}$ in V is a strong convergent sequence in V then it is a S_ϑ-strong convergent sequence in V. But the converse is not necessarily true. Find a suitable condition(s) so that the converse of the above proposition valid.

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