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Continuity for Multilinear Integral Operators on Some Hardy and Herz Type Spaces

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Abstract

The continuity for some multilinear operators generated by certain integral operators and Lipschitz functions on some Hardy and Herz-type spaces are obtained. The operators include Littlewood-Paley operators, Marcinkiewicz operators and Bochner-Riesz operator.

Keywords: *Multilinear operator; Lipschitz function; Hardy space; Herz space; Herz type Hardy space; Littlewood-Paley operator; Marcinkiewicz operator; Bochner-Riesz operator.*
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1 Introduction and Preliminaries

As the development of singular integral operators T, their commutators and multilinear operators have been well studied (see [1],[4-7]). From [8] and [9], we know that the commutators and multilinear operators generated by T and the BMO functions are bounded on $L^p(R^n)$ for $1 < p < \infty$. Chanillo (see [2]) proves a similar result when T is replaced by the fractional integral operator. However, it was observed that the commutators and multilinear operators are not bounded, in general, from $H^p(R^n)$ to $L^p(R^n)$ for $0 < p \leq 1$. But, the boundedness holds if the BMO functions are replaced by the the Lipschitz functions (see [3], [11], [16] and [19]). This show the difference of the BMO functions and the Lipschitz functions. The purpose of this paper is to establish the continuity properties for some multilinear operators generated by certain non-convolution type integral operators and Lipschitz functions on some Hardy and Herz-type spaces. The operators include Littlewood-Paley operators, Marcinkiewicz operators and Bochner-Riesz operator.

First, let us introduce some notations(see [10], [17-21]). Throughout this paper, Q will denote a cube of R^n with sides parallel to the axes. For a cube Q and a locally integrable function f, let $f_Q = |Q|^{-1}\int_Q f(x)dx$. Denote the Hardy spaces by $H^p(R^n)$. It is well known that $H^p(R^n)(0 < p \le 1)$ has the atomic decomposition characterization (see [20],[21]). For $\beta > 0$, the Lipschitz space $Lip_\beta(R^n)$ is the space of functions f such that (see [19])

$$||f||_{Lip_\beta} = \sup_{x,h\in R^n,\ h>0} |f(x+h) - f(x)|/|h|^\beta < \infty.$$

Definition 1.1 Let $0 < p, q < \infty$, $\alpha \in R$. For $k \in Z$, define $B_k = \{x \in R^n : |x| \le 2^k\}$ and $C_k = B_k \setminus B_{k-1}$. Denote by χ_k the characteristic function of C_k and χ_0 the characteristic function of B_0.

(1) The homogeneous Herz space is defined by

$$\dot{K}_q^{\alpha,p}(R^n) = \{f \in L^q_{loc}(R^n \setminus \{0\}) : ||f||_{\dot{K}_q^{\alpha,p}} < \infty\},$$

where

$$||f||_{\dot{K}_q^{\alpha,p}} = \left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p}||f\chi_k||^p_{L^q}\right]^{1/p};$$

(2) The nonhomogeneous Herz space is defined by

$$K_q^{\alpha,p}(R^n) = \{f \in L^q_{loc}(R^n) : ||f||_{K_q^{\alpha,p}} < \infty\},$$

where

$$||f||_{K_q^{\alpha,p}} = \left[\sum_{k=1}^{\infty} 2^{k\alpha p}||f\chi_k||^p_{L^q} + ||f\chi_0||^p_{L^q}\right]^{1/p}.$$

Definition 1.2 Let $\alpha \in R$, $0 < p, q < \infty$.

(1) The homogeneous Herz type Hardy space is defined by

$$H\dot{K}_q^{\alpha,p}(R^n) = \{f \in S'(R^n) : G(f) \in \dot{K}_q^{\alpha,p}(R^n)\},$$

and

$$||f||_{H\dot{K}_q^{\alpha,p}} = ||G(f)||_{\dot{K}_q^{\alpha,p}};$$

(2) The nonhomogeneous Herz type Hardy space is defined by

$$HK_q^{\alpha,p}(R^n) = \{f \in S'(R^n) : G(f) \in K_q^{\alpha,p}(R^n)\},$$

and

$$||f||_{HK_q^{\alpha,p}} = ||G(f)||_{K_q^{\alpha,p}};$$

where $G(f)$ is the grand maximal function of f.

The Herz type Hardy spaces have the atomic decomposition characterization.

Definition 1.3 Let $\alpha \in R$, $1 < q < \infty$. A function $a(x)$ on R^n is called a central (α, q)-atom (or a central (a, q)-atom of restrict type), if

1) Supp$a \subset B(0, r)$ for some $r > 0$ (or for some $r \geq 1$),

2) $||a||_{L^q} \leq |B(0, r)|^{-\alpha/n}$,

3) $\int a(x)x^\gamma dx = 0$ for $|\gamma| \leq [\alpha - n(1 - 1/q)]$.

Lemma 1.1(see[10],[18]]) Let $0 < p < \infty$, $1 < q < \infty$ and $\alpha \geq n(1 - 1/q)$. A temperate distribution f belongs to $H\dot{K}_q^{\alpha,p}(R^n)$(or $HK_q^{\alpha,p}(R^n)$) if and only if there exist central (α, q)-atoms(or central (α, q) -atoms of restrict type)a_j supported on $B_j = B(0, 2^j)$ and constants λ_j, $\sum_j |\lambda_j|^p < \infty$ such that $f = \sum_{j=-\infty}^{\infty} \lambda_j a_j$(or $f = \sum_{j=0}^{\infty} \lambda_j a_j$)in the $S'(R^n)$ sense, and

$$||f||_{H\dot{K}_q^{\alpha,p}}(\text{ or } ||f||_{HK_q^{\alpha,p}}) \approx \left(\sum_j |\lambda_j|^p\right)^{1/p}.$$

2 Theorems

In this paper, we will study a class of multilinear operators related to some integral operators, whose definitions are follows.

Fixed $0 \leq \delta < n$ and $\varepsilon > 0$. Let m_i be the positive integers($i = 1, \cdots, l$), $m_1 + \cdots + m_l = m$ and A_i be the functions on R^n $(i = 1, \cdots, l)$. Set

$$R_{m_i+1}(A_i; x, y) = A_i(x) - \sum_{|\gamma| \leq m_i} \frac{1}{\gamma!} D^\gamma A_i(y)(x - y)^\gamma$$

and

$$Q_{m_i+1}(A_i; x, y) = R_{m_i}(A_i; x, y) - \sum_{|\gamma| = m_i} \frac{1}{\gamma!} D^\gamma A_i(x)(x - y)^\gamma.$$

Let $F_t(x, y)$ define on $R^n \times R^n \times [0, +\infty)$. Set

$$F_t(f)(x) = \int_{R^n} F_t(x, y) f(y) dy$$

and

$$F_t^A(f)(x) = \int_{R^n} \frac{\prod_{j=1}^l R_{m_j+1}(A_j; x, y)}{|x - y|^m} F_t(x, y) f(y) dy$$

for every bounded and compactly supported function f. Let H be the Banach space $H = \{h : ||h|| < \infty\}$ such that, for each fixed $x \in R^n$, $F_t(f)(x)$ and $F_t^A(f)(x)$ may be viewed as a mapping from $[0, +\infty)$ to H. Then, the multilinear operator related to F_t is defined by

$$T^A(f)(x) = ||F_t^A(f)(x)||,$$

where F_t satisfies:
$$||F_t(x,y)|| \le C|x-y|^{-n+\delta}$$
and
$$||F_t(y,x)-F_t(z,x)|| \le C|y-z|^{\varepsilon}|x-z|^{-n-\varepsilon+\delta}$$
if $2|y-z| \le |x-z|$. Let $T(f)(x) = ||F_t(f)(x)||$. We also consider the variant of T^A, which is defined by

$$\tilde{T}^A(f)(x) = ||\tilde{F}_t^A(f)(x)||,$$

where
$$\tilde{F}_t^A(f)(x) = \int_{R^n} \frac{\prod_{i=1}^l Q_{m_i+1}(A_i;x,y)}{|x-y|^m} F_t(x,y)f(y)dy.$$

Note that when $m=0$, T^A is just higher order commutator of the operators T and A(see [1],[12-14],[19]), while when $m>0$, it is non-trivial generalizations of the commutator. It is well known that multilinear operators are of great interest in harmonic analysis and have been widely studied by many authors when A has derivatives of order m in $BMO(R^n)$(see [4-6],[9]). The purpose of this paper is to prove the continuity properties of the multilinear operators T^A and $\tilde{T}^A$ on Hardy and Herz-type spaces. In Section 4, some examples of Theorems in this paper are given.

We shall prove the following theorems in Section 3.

Theorem 2.1 Let $0<\beta\le 1$, $0\le\delta<n-l\beta$ and $D^\gamma A_i \in Lip_\beta(R^n)$ for all γ with $|\gamma| = m_i$ and $i=1,\cdots,l$.

(a) Suppose that T^A maps $L^s(R^n)$ continuously into $L^r(R^n)$ for any $1<r<n/\mu$ and $1/s = 1/r-\mu/n$. If $\max(n/(n+\beta), n/(n+\varepsilon)) < p \le 1$, $1/p-1/q = (\delta+l\beta)/n$, then T^A maps $H^p(R^n)$ continuously into $L^q(R^n)$.

(b) Suppose that $\tilde{T}^A$ maps $L^s(R^n)$ continuously into $L^r(R^n)$ for any $1<r<n/\mu$ and $1/s = 1/r-\mu/n$. If $0<\beta<\min(1/l, \varepsilon/l)$, then $\tilde{T}^A$ maps $H^{n/(n+l\beta)}(R^n)$ continuously into $L^{n/(n-\delta)}(R^n)$.

Theorem 2.2 Let $0<\beta\le 1$, $0\le\delta<n-l\beta$, $0<p<\infty$, $1<q_1,q_2<\infty$, $1/q_1-1/q_2 = (\delta+l\beta)/n$ and $D^\gamma A_i \in Lip_\beta(R^n)$ for all γ with $|\gamma| = m_i$ and $i=1,\cdots,l$.

(i) Suppose that T^A maps $L^s(R^n)$ continuously into $L^r(R^n)$ for any $1<r<n/\mu$ and $1/s=1/r-\mu/n$. If $n(1-1/q_1)\le\alpha<\min(n(1-1/q_1)+l\beta, n(1-1/q_1)+\varepsilon)$, then T^A maps $H\dot{K}_{q_1}^{\alpha,p}(R^n)$ continuously into $\dot{K}_{q_2}^{\alpha,p}(R^n)$.

(ii) Suppose that $\tilde{T}^A$ maps $L^s(R^n)$ continuously into $L^r(R^n)$ for any $1<r<n/\mu$ and $1/s=1/r-\mu/n$. If $0<p\le 1$ and $0<\beta<\min(1/l,\varepsilon/l)$, then $\tilde{T}^A$ maps $H\dot{K}_{q_1}^{n(1-1/q_1)+l\beta,p}(R^n)$ continuously into $\dot{K}_{q_2}^{n(1-1/q_1)+l\beta,p}(R^n)$.

Remark. Theorem 2 also hold for the nonhomogeneous Herz and Herz type Hardy space.

3 Proofs of Theorems

We begin with a preliminary lemma.

Lemma 3.1(see [6]) Let A be a function on R^n such that $D^\gamma A \in L^q_{loc}(R^n)$ for $|\gamma| = m$ and some $q > n$. Then

$$|R_m(A;x,y)| \le C|x-y|^m \sum_{|\gamma|=m} \left(\frac{1}{|\tilde{Q}(x,y)|} \int_{\tilde{Q}(x,y)} |D^\gamma A(z)|^q dz \right)^{1/q},$$

where $\tilde{Q}(x,y)$ is the cube centered at x and having side length $5\sqrt{n}|x-y|$.

Proof of Theorem 2.1(a). It suffices to show that there exists a constant $C > 0$ such that for every H^p-atom a, there is

$$||T^A(a)||_{L^q} \le C.$$

Without loss of generality, we may assume $l = 2$. Let a be a H^p-atom, that is that a supported on a cube $Q = Q(x_0, d)$, $||a||_{L^\infty} \le |Q|^{-1/p}$ and $\int_{R^n} a(x)x^\eta dx = 0$ for $|\eta| \le [n(1/p-1)]$. We write

$$\int_{R^n} |T^A(a)(x)|^q dx = \left(\int_{|x-x_0|\le 2d} + \int_{|x-x_0|>2d} \right) |T^A(a)(x)|^q dx = I_1 + I_2.$$

For I_1, taking $q_1 > q$ and $1 < p_1 < n/(\delta + 2\beta)$ such that $1/p_1 - 1/q_1 = (\delta + 2\beta)/n$, by Hölder's inequality and the (L^{p_1}, L^{q_1})-boundedness of T^A, we get

$$I_1 \le C||T^A(a)||^q_{L^{q_1}} |2Q|^{1-q/q_1} \le C||a||^q_{L^{p_1}} |Q|^{1-q/q_1} \le C.$$

To estimate I_2, we need to estimate $T^A(a)(x)$ for $x \in (2Q)^c$. Let $\tilde{A}_i(x) = A_i(x) - \sum_{|\gamma|=m_i} \frac{1}{\gamma!}(D^\gamma A_i)_Q x^\gamma$. Then $R_{m_i}(A_i;x,y) = R_{m_i}(\tilde{A}_i;x,y)$ and $D^\gamma \tilde{A}_i(y) = D^\gamma A_i(y) - (D^\gamma A_i)_Q$. We write, by the vanishing moment of a,

$$\begin{aligned}
& F_t^A(a)(x) \\
= & \int_{R^n} \left[\frac{F_t(x,y)}{|x-y|^m} - \frac{F_t(x,x_0)}{|x-x_0|^m} \right] R_{m_1}(\tilde{A}_1;x,y) R_{m_2}(\tilde{A}_2;x,y) a(y) dy \\
& + \int_{R^n} \frac{F_t(x,x_0)}{|x-x_0|^m} [R_{m_1}(\tilde{A}_1;x,y) - R_{m_1}(\tilde{A}_1;x,x_0)] R_{m_2}(\tilde{A}_2;x,y) a(y) dy \\
& + \int_{R^n} \frac{F_t(x,x_0)}{|x-x_0|^m} [R_{m_2}(\tilde{A}_2;x,y) - R_{m_2}(\tilde{A}_2;x,x_0)] R_{m_1}(\tilde{A}_1;x,x_0) a(y) dy \\
& - \sum_{|\gamma_2|=m_2} \frac{1}{\gamma_2!} \int_{R^n} \frac{R_{m_1}(\tilde{A}_1;x,y) D^{\gamma_2} \tilde{A}_2(y)(x-y)^{\gamma_2}}{|x-y|^m} F_t(x,y) a(y) dy \\
& - \sum_{|\gamma_1|=m_1} \frac{1}{\gamma_1!} \int_{R^n} \frac{R_{m_2}(\tilde{A}_2;x,y) D^{\gamma_1} \tilde{A}_1(y)(x-y)^{\gamma_1}}{|x-y|^m} F_t(x,y) a(y) dy \\
& + \sum_{|\gamma_1|=m_1, |\gamma_2|=m_2} \frac{1}{\gamma_1!} \frac{1}{\gamma_2!} \int_{R^n} \frac{D^{\gamma_1} \tilde{A}_1(y) D^{\gamma_2} \tilde{A}_2(y)(x-y)^{\gamma_1+\gamma_2}}{|x-y|^m} F_t(x,y) a(y) dy;
\end{aligned}$$

By Lemma 3.1 and the following inequality

$$|b(x)-b_Q| \le \frac{1}{|Q|}\int_Q ||b||_{Lip_\beta}|x-y|^\beta dy \le ||b||_{Lip_\beta}(|x-x_0|+d)^\beta,$$

we get

$$|R_{m_i}(\tilde{A}_i;x,y)| \le \sum_{|\gamma|=m_i} ||D^\gamma A_i||_{Lip_\beta}(|x-y|+d)^{m_i+\beta};$$

By the formula (see [6]):

$$R_{m_i}(\tilde{A}_i;x,y)-R_{m_i}(\tilde{A}_i;x,x_0) = \sum_{|\eta|<m}\frac{1}{\eta!}R_{m_i-|\eta|}(D^\eta \tilde{A}_i;x_0,y)(x-x_0)^\eta,$$

and note that $|x-y|\sim|x-x_0|$ for $y\in Q$ and $x\in R^n\setminus 2Q$, we obtain

$$\begin{aligned}
&|T^A(a)(x)| = ||F_t^A(a)(x)|| \le C\prod_{i=1}^{2}\left(\sum_{|\gamma_i|=m_i}||D^{\gamma_i}A_i||_{Lip_\beta}\right)\int_Q\left[\frac{|y-x_0|}{|x-x_0|^{n+1-\delta-2\beta}}\right.\\
&\left.+\frac{|y-x_0|^\varepsilon}{|x-x_0|^{n+\varepsilon-\delta-2\beta}}+\frac{|y-x_0|^\beta}{|x-x_0|^{n-\delta-\beta}}+\frac{|y-x_0|^{2\beta}}{|x-x_0|^{n-\delta}}\right]|a(y)|dy\\
\le\ & C\prod_{i=1}^{2}\left(\sum_{|\gamma_i|=m_i}||D^{\gamma_i}A_i||_{Lip_\beta}\right)\left[\frac{|Q|^{1/n+1-1/p}}{|x-x_0|^{n+1-\delta-2\beta}}+\frac{|Q|^{\varepsilon/n+1-1/p}}{|x-x_0|^{n+\varepsilon-\delta-2\beta}}\right.\\
&\left.+\frac{|Q|^{\beta/n+1-1/p}}{|x-x_0|^{n-\delta-\beta}}+\frac{|Q|^{2\beta/n+1-1/p}}{|x-x_0|^{n-\delta}}\right];
\end{aligned}$$

Thus

$$\begin{aligned}
I_2 &\le \sum_{k=1}^{\infty}\int_{2^{k+1}Q\setminus 2^kQ}|T^A(a)(x)|^q dx\\
&\le C\left[\prod_{i=1}^{2}\left(\sum_{|\gamma_i|=m_i}||D^{\gamma_i}A_i||_{Lip_\beta}\right)\right]^q\\
&\quad\times\sum_{k=1}^{\infty}\left[2^{kqn(1/p-(n+1)/n)}+2^{kqn(1/p-(n+\varepsilon)/n)}+2^{kqn(1/p-(n+\beta)/n)}\right]\\
&\le C\left[\prod_{i=1}^{2}\left(\sum_{|\gamma_i|=m_i}||D^{\gamma_i}A_i||_{Lip_\beta}\right)\right]^q \le C,
\end{aligned}$$

which together with the estimate for I_1 yields the desired result.

(b). Without loss of generality, we may assume $l = 2$. It is only to prove that there exists a constant $C > 0$ such that for every $H^{n/(n+2\beta)}$-atom a supported on $Q = Q(x_0, d)$, there is

$$||\tilde{T}^A(a)||_{L^{n/(n-\delta)}} \le C.$$

We write

$$\int_{R^n} |\tilde{T}^A(a)(x)|^{n/(n-\delta)}dx = \left[\int_{|x-x_0|\le 2d} + \int_{|x-x_0|>2d}\right] |\tilde{T}^A(a)(x)|^{n/(n-\delta)}dx := J_1 + J_2.$$

For J_1, by the (L^p, L^q)-boundedness of $\tilde{T}^A$ for $1 < p < n/(\delta + 2\beta)$, $q > n/(n-\delta)$and $1/q = 1/p - (\delta + 2\beta)/n$, we get

$$J_1 \le C||\tilde{T}^A(a)||_{L^q}^{n/(n-\delta)}|2Q|^{1-n/((n-\delta)q)} \le C||a||_{L^p}^{n/(n-\delta)}|Q|^{1-n/((n-\delta)q)} \le C.$$

To obtain the estimate of J_2, we denote $\tilde{A}_i(x) = A_i(x) - \sum_{|\gamma|=m_i} \frac{1}{\gamma!}(D^\gamma A_i)_{2Q}x^\gamma$. Then $Q_{m_i}(A_i; x, y) = Q_{m_i}(\tilde{A}_i; x, y)$. We write, by the vanishing moment of a and $Q_{m_i+1}(A_i; x, y) = R_{m_i}(A_i; x, y) - \sum_{|\gamma|=m_i} \frac{1}{\gamma!}D^\gamma A_i(x)(x-y)^\gamma$, for $x \in (2Q)^c$,

$$\begin{aligned}
&\tilde{F}_t^A(a)(x)\\
= &\int_{R^n} \left[\frac{F_t(x,y)}{|x-y|^m} - \frac{F_t(x,x_0)}{|x-x_0|^m}\right] R_{m_1}(\tilde{A}_1; x, y)R_{m_2}(\tilde{A}_2; x, y)a(y)dy\\
&+ \int_{R^n} \frac{F_t(x,x_0)}{|x-x_0|^m}[R_{m_1}(\tilde{A}_1; x, y) - R_{m_1}(\tilde{A}_1; x, x_0)]R_{m_2}(\tilde{A}_2; x, y)a(y)dy\\
&+ \int_{R^n} \frac{F_t(x,x_0)}{|x-x_0|^m}[R_{m_2}(\tilde{A}_2; x, y) - R_{m_2}(\tilde{A}_2; x, x_0)]R_{m_1}(\tilde{A}_1; x, x_0)a(y)dy\\
&- \sum_{|\gamma_2|=m_2} \int_{R^n} \left[\frac{F_t(x,y)(x-y)^{\gamma_2}}{|x-y|^m} - \frac{F_t(x,x_0)(x-x_0)^{\gamma_2}}{|x-x_0|^m}\right]\\
&\times R_{m_1}(\tilde{A}_1; x, y)D^{\gamma_2}\tilde{A}_2(x)a(y)dy\\
&- \sum_{|\gamma_2|=m_2} \int_{R^n} \frac{F_t(x,x_0)(x-x_0)^{\gamma_2}}{|x-x_0|^m}[R_{m_1}(\tilde{A}_1; x, y) - R_{m_1}(\tilde{A}_1; x, x_0)]\\
&\times D^{\gamma_2}\tilde{A}_2(x)a(y)dy\\
&- \sum_{|\gamma_1|=m_1} \int_{R^n} \left[\frac{F_t(x,y)(x-y)^{\gamma_1}}{|x-y|^m} - \frac{F_t(x,x_0)(x-x_0)^{\gamma_1}}{|x-x_0|^m}\right]\\
&\times R_{m_2}(\tilde{A}_2; x, y)D^{\gamma_1}\tilde{A}_1(x)a(y)dy
\end{aligned}$$

$$
\begin{aligned}
&-\sum_{|\gamma_1|=m_1}\int_{R^n}\frac{F_t(x,x_0)(x-x_0)^{\gamma_1}}{|x-x_0|^m}[R_{m_2}(\tilde{A}_2;x,y)-R_{m_2}(\tilde{A}_2;x,x_0)]\\
&\times D^{\gamma_1}\tilde{A}_1(x)a(y)dy\\
&+\sum_{|\gamma_1|=m_1,|\gamma_2|=m_2}\int_{R^n}\left[\frac{F_t(x,y)(x-y)^{\gamma_1+\gamma_2}}{|x-y|^m}-\frac{F_t(x,x_0)(x-x_0)^{\gamma_1+\gamma_2}}{|x-x_0|^m}\right]\\
&\times D^{\gamma_1}\tilde{A}_1(x)D^{\gamma_2}\tilde{A}_2(x)a(y)dy,
\end{aligned}
$$

then, similar to the proof of **(a)**, we obtain

$$
\begin{aligned}
&|\tilde{T}^A(a)(x)|\\
\le\;& C\prod_{i=1}^2\left(\sum_{|\gamma_i|=m_i}||D^{\gamma_i}A_i||_{Lip_\beta}\right)\int_Q\left[\frac{|y-x_0|}{|x-x_0|^{n+1-\delta-2\beta}}+\frac{|y-x_0|^\varepsilon}{|x-x_0|^{n+\varepsilon-\delta-2\beta}}\right]|a(y)|dy\\
\le\;& C\prod_{i=1}^2\left(\sum_{|\gamma_i|=m_i}||D^{\gamma_i}A_i||_{Lip_\beta}\right)\left[\frac{|Q|^{(1-2\beta)/n}}{|x-x_0|^{n+1-\delta-2\beta}}+\frac{|Q|^{(\varepsilon-2\beta)/n}}{|x-x_0|^{n+\varepsilon-\delta-2\beta}}\right],
\end{aligned}
$$

thus

$$
J_2\le C\left[\prod_{i=1}^2\left(\sum_{|\gamma_i|=m_i}||D^{\gamma_i}A_i||_{Lip_\beta}\right)\right]^{n/(n-\delta)}\sum_{k=1}^\infty[2^{kn(2\beta-1)/(n-\delta)}+2^{kn(2\beta-\varepsilon)/(n-\delta)}]\le C,
$$

which together with the estimate for J_1 yields the desired result. This completes the proof of Theorem 2.1.

Proof of Theorem 2.2(i). Without loss of generality, we may assume $l=2$. Let $f\in H\dot{K}_{q_1}^{\alpha,p}(R^n)$ and $f(x)=\sum_{j=-\infty}^\infty\lambda_j a_j(x)$ be the atomic decomposition for f as in Lemma 1.1. We write

$$
\begin{aligned}
||T^A(f)||^p_{\dot{K}_{q_2}^{\alpha,p}}\;\le\;&\sum_{k=-\infty}^\infty 2^{k\alpha p}\left(\sum_{j=-\infty}^{k-3}|\lambda_j|||T^A(a_j)\chi_k||_{L^{q_2}}\right)^p\\
&+\sum_{k=-\infty}^\infty 2^{k\alpha p}\left(\sum_{j=k-2}^{\infty}|\lambda_j|||T^A(a_j)\chi_k||_{L^{q_2}}\right)^p=K_1+K_2.
\end{aligned}
$$

For K_2, by the (L^{q_1}, L^{q_2}) boundedness of T^A, we have

$$\begin{aligned}
K_2 &\leq C \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left(\sum_{j=k-2}^{\infty} |\lambda_j| ||a_j||_{L^{q_1}} \right)^p \\
&\leq \begin{cases} C \sum_{j=-\infty}^{\infty} |\lambda_j|^p \left(\sum_{k=-\infty}^{j+2} 2^{(k-j)\alpha p} \right), & 0 < p \leq 1 \\ C \sum_{j=-\infty}^{\infty} |\lambda_j|^p \left(\sum_{k=-\infty}^{j+2} 2^{(k-j)\alpha p/2} \right) \left(\sum_{k=-\infty}^{j+2} 2^{(k-j)\alpha p'/2} \right)^{p/p'}, & p > 1 \end{cases} \\
&\leq C \sum_{j=-\infty}^{\infty} |\lambda_j|^p \leq C ||f||^p_{H\dot{K}^{\alpha,p}_{q_1}}.
\end{aligned}$$

For K_1, similar to the proof of Theorem 2.1 **(a)**, we get, for $x \in C_k$, $j \leq k-3$,

$$\begin{aligned}
&|T^A(a_j)(x)| \\
\leq\ & C \left(\frac{|B_j|^{1/n}}{|x|^{n+1-\delta-2\beta}} + \frac{|B_j|^{\varepsilon/n}}{|x|^{n+\varepsilon-\delta-2\beta}} + \frac{|B_j|^{\beta/n}}{|x|^{n-\delta-\beta}} + \frac{|B_j|^{2\beta/n}}{|x|^{n-\delta}} \right) \int_{R^n} |a_j(y)| dy \\
\leq\ & C \left(\frac{2^{j(1+n(1-1/q_1)-\alpha)}}{|x|^{n+1-\delta-2\beta}} + \frac{2^{j(\varepsilon+n(1-1/q_1)-\alpha)}}{|x|^{n+\varepsilon-\delta-2\beta}} + \frac{2^{j(\beta+n(1-1/q_1)-\alpha)}}{|x|^{n-\delta-\beta-n}} \right),
\end{aligned}$$

thus

$$\begin{aligned}
&||T^A(a_j)\chi_k||_{L^{q_2}} \\
\leq\ & C 2^{-k\alpha} \left(2^{(j-k)(1+n(1-1/q_1)-\alpha)} + 2^{(j-k)(\varepsilon+n(1-1/q_1)-\alpha)} + 2^{(j-k)(\beta+n(1-1/q_1)-\alpha)} \right);
\end{aligned}$$

To be simply, denote $W(j,k) = 2^{(j-k)(1+n(1-1/q_1)-\alpha)} + 2^{(j-k)(\varepsilon+n(1-1/q_1)-\alpha)} + 2^{(j-k)(\beta+n(1-1/q_1)-\alpha)}$ and recall that $\alpha < \min(n(1-1/q_1)+\beta, n(1-1/q_1)+\varepsilon)$, then

$$\begin{aligned}
K_1 &\leq C \sum_{k=-\infty}^{\infty} \left(\sum_{j=-\infty}^{k-3} |\lambda_j| W(j,k) \right)^p \\
&\leq \begin{cases} C \sum_{j=-\infty}^{\infty} |\lambda_j|^p \sum_{k=j+3}^{\infty} W(j,k)^p, & 0 < p \leq 1 \\ C \sum_{j=-\infty}^{\infty} |\lambda_j|^p \left[\sum_{k=j+3}^{\infty} W(j,k)^{p/2} \right] \left[\sum_{k=j+3}^{\infty} W(j,k)^{p'/2} \right]^{p/p'}, & p > 1 \end{cases} \\
&\leq C \sum_{j=-\infty}^{\infty} |\lambda_j|^p \leq C ||f||^p_{H\dot{K}^{\alpha,p}_{q_1}}.
\end{aligned}$$

These yield the desired result.

(ii). Without loss of generality, assume $l = 2$. Let $f \in H\dot{K}^{n(1-1/q_1)+2\beta,p}_{q_1}(R^n)$ and $f(x) = \sum_{j=-\infty}^{\infty} \lambda_j a_j(x)$ be the atomic decomposition for f as in Lemma

1.1. Write

$$\begin{aligned} \|\tilde{T}^A(f)\|^p_{\dot{K}_{q_2}^{n(1-1/q_1)+2\beta,p}} &\leq \sum_{k=-\infty}^{\infty} 2^{kp(n(1-1/q_1)+2\beta)} \left(\sum_{j=-\infty}^{k-3} |\lambda_j| \|\tilde{T}^A(a_j)\chi_k\|_{L^{q_2}} \right)^p \\ &\quad + \sum_{k=-\infty}^{\infty} 2^{kp(n(1-1/q_1)+2\beta)} \left(\sum_{j=k-2}^{\infty} |\lambda_j| \|\tilde{T}^A(a_j)\chi_k\|_{L^{q_2}} \right)^p \\ &= L_1 + L_2. \end{aligned}$$

For L_2, by the (L^{q_1}, L^{q_2}) boundedness of $\tilde{T}^A$, we get

$$\begin{aligned} L_2 &\leq C \sum_{k=-\infty}^{\infty} 2^{kp(n(1-1/q_1)+2\beta)} \left(\sum_{j=k-2}^{\infty} |\lambda_j| \|a_j\|_{L^{q_1}} \right)^p \\ &\leq C \sum_{j=-\infty}^{\infty} |\lambda_j|^p \left(\sum_{k=-\infty}^{j+2} 2^{(k-j)p(n(1-1/q_1)+2\beta)} \right) \\ &\leq C \sum_{j=-\infty}^{\infty} |\lambda_j|^p \leq C\|f\|^p_{H\dot{K}_{q_1}^{n(1-1/q_1)+2\beta,p}}. \end{aligned}$$

For L_1, similar to the proof of Theorem 2.1 **(b)**, we get, for $x \in C_k$, $j \leq k-3$,

$$\begin{aligned} |\tilde{T}^A(a)(x)| &\leq C \left(\frac{|B_j|^{1/n}}{|x|^{n+1-\delta-2\beta}} + \frac{|B_j|^{\varepsilon/n}}{|x|^{n+\varepsilon-\delta-2\beta}} \right) \int_{R^n} |a_j(y)| dy \\ &\leq C \left(\frac{2^{j(1-2\beta)}}{|x|^{n+1-\delta-2\beta}} + \frac{2^{j(\varepsilon-2\beta)}}{|x|^{n+\varepsilon-\delta-2\beta}} \right), \end{aligned}$$

thus

$$\begin{aligned} L_1 &\leq C \sum_{k=-\infty}^{\infty} 2^{kp(n(1-1/q_1)+2\beta)} \left(\sum_{j=-\infty}^{k-3} |\lambda_j|^p \frac{2^{j(1-2\beta)}}{2^{k(n+1-\delta-2\beta)}} + \frac{2^{j(\varepsilon-2\beta)}}{2^{k(n+\varepsilon-\delta-2\beta)}} \right)^p 2^{knp/q_2} \\ &\leq C \sum_{j=-\infty}^{\infty} |\lambda_j|^p \sum_{k=j+3}^{\infty} \left(2^{p(1-2\beta)(j-k)} + 2^{p(\varepsilon-2\beta)(j-k)} \right) \\ &\leq C \sum_{j=-\infty}^{\infty} |\lambda_j|^p \leq C\|f\|^p_{H\dot{K}_{q_1}^{n(1-1/q_1)+2\beta,p}}. \end{aligned}$$

These yield the desired result and finish the proof of Theorem 2.2.

4 Examples

Now we give some examples including Littlewood-Paley operators, Marcinkiewicz operators and Bochner-Riesz operator.

Example 1 Littlewood-Paley operator.

Fixed $\varepsilon > 0$ and $\mu > (3n+2)/n$. Let ψ be a fixed function which satisfies:

(1) $\int_{R^n} \psi(x)dx = 0$,

(2) $|\psi(x)| \le C(1+|x|)^{-(n+1)}$,

(3) $|\psi(x+y)-\psi(x)| \le C|y|^\varepsilon(1+|x|)^{-(n+1+\varepsilon)}$ when $2|y| < |x|$;

We denote that $\Gamma(x) = \{(y,t) \in R_+^{n+1} : |x-y| < t\}$ and the characteristic function of $\Gamma(x)$ by $\chi_{\Gamma(x)}$. The Littlewood-Paley multilinear operators are defined by

$$g_\psi^A(f)(x) = \left(\int_0^\infty |F_t^A(f)(x)|^2 \frac{dt}{t}\right)^{1/2},$$

$$S_\psi^A(f)(x) = \left[\int\int_{\Gamma(x)} |F_t^A(f)(x,y)|^2 \frac{dydt}{t^{n+1}}\right]^{1/2}$$

and

$$g_\mu^A(f)(x) = \left[\int\int_{R_+^{n+1}} \left(\frac{t}{t+|x-y|}\right)^{n\mu} |F_t^A(f)(x,y)|^2 \frac{dydt}{t^{n+1}}\right]^{1/2},$$

where

$$F_t^A(f)(x) = \int_{R^n} \frac{\prod_{j=1}^l R_{m_j+1}(A_j;x,y)}{|x-y|^m} \psi_t(x-y)f(y)dy,$$

$$F_t^A(f)(x,y) = \int_{R^n} \frac{\prod_{j=1}^l R_{m_j+1}(A_j;x,z)}{|x-z|^m} f(z)\psi_t(y-z)dz$$

and $\psi_t(x) = t^{-n}\psi(x/t)$ for $t > 0$. The variants of g_ψ^A, S_ψ^A and g_μ^A are defined by

$$\tilde{g}_\psi^A(f)(x) = \left(\int_0^\infty |\tilde{F}_t^A(f)(x)|^2 \frac{dt}{t}\right)^{1/2},$$

$$\tilde{S}_\psi^A(f)(x) = \left[\int\int_{\Gamma(x)} |\tilde{F}_t^A(f)(x,y)|^2 \frac{dydt}{t^{n+1}}\right]^{1/2}$$

and

$$\tilde{g}_\mu^A(f)(x) = \left[\int\int_{R_+^{n+1}} \left(\frac{t}{t+|x-y|}\right)^{n\mu} |\tilde{F}_t^A(f)(x,y)|^2 \frac{dydt}{t^{n+1}}\right]^{1/2},$$

where

$$\tilde{F}_t^A(f)(x) = \int_{R^n} \frac{\prod_{j=1}^l Q_{m_j+1}(A_j;x,y)}{|x-y|^m} \psi_t(x-y)f(y)dy$$

and

$$\tilde{F}_t^A(f)(x,y) = \int_{R^n} \frac{\prod_{j=1}^l Q_{m_j+1}(A_j;x,z)}{|x-z|^m} \psi_t(y-z)f(z)dz.$$

Set $F_t(f)(y) = f * \psi_t(y)$. We also define that

$$g_\psi(f)(x) = \left(\int_0^\infty |F_t(f)(x)|^2 \frac{dt}{t}\right)^{1/2},$$

$$S_\psi(f)(x) = \left(\int\int_{\Gamma(x)} |F_t(f)(y)|^2 \frac{dydt}{t^{n+1}}\right)^{1/2}$$

and

$$g_\mu(f)(x) = \left(\int\int_{R_+^{n+1}} \left(\frac{t}{t+|x-y|}\right)^{n\mu} |F_t(f)(y)|^2 \frac{dydt}{t^{n+1}}\right)^{1/2},$$

which are the Littlewood-Paley operators (see [21]). Let H be the space

$$H = \left\{h : ||h|| = \left(\int_0^\infty |h(t)|^2 dt/t\right)^{1/2} < \infty\right\}$$

or

$$H = \left\{h : ||h|| = \left(\int\int_{R_+^{n+1}} |h(y,t)|^2 dydt/t^{n+1}\right)^{1/2} < \infty\right\},$$

then, for each fixed $x \in R^n$, $F_t^A(f)(x)$ and $F_t^A(f)(x,y)$ may be viewed as the mapping from $[0,+\infty)$ to H, and it is clear that

$$g_\psi^A(f)(x) = ||F_t^A(f)(x)||, \quad g_\psi(f)(x) = ||F_t(f)(x)||,$$

$$S_\psi^A(f)(x) = \left|\left|\chi_{\Gamma(x)} F_t^A(f)(x,y)\right|\right|, \quad S_\psi(f)(x) = \left|\left|\chi_{\Gamma(x)} F_t(f)(y)\right|\right|$$

and

$$g_\mu^A(f)(x) = \left|\left|\left(\frac{t}{t+|x-y|}\right)^{n\mu/2} F_t^A(f)(x,y)\right|\right|,$$

$$g_\mu(f)(x) = \left|\left|\left(\frac{t}{t+|x-y|}\right)^{n\mu/2} F_t(f)(y)\right|\right|.$$

It is easily to see that g_ψ, S_ψ and g_μ satisfy the conditions of Theorem 2.1 and 2.2, thus Theorem 2.1 and 2.2 hold for g_ψ^A and $\tilde{g}_\psi^A$, S_ψ^A and $\tilde{S}_\psi^A$, g_μ^A and $\tilde{g}_\mu^A$.

Example 2 Marcinkiewicz operator.

Fixed Fix $\lambda > \max(1, 2n/(n+2))$ and $0 < \gamma \le 1$. Let Ω be homogeneous of degree zero on R^n with $\int_{S^{n-1}} \Omega(x')d\sigma(x') = 0$. Assume that $\Omega \in Lip_\gamma(S^{n-1})$. The Marcinkiewicz multilinear operators are defined by

$$\mu_\Omega^A(f)(x) = \left(\int_0^\infty |F_t^A(f)(x)|^2 \frac{dt}{t^3}\right)^{1/2},$$

$$\mu_S^A(f)(x) = \left[\int\int_{\Gamma(x)} |F_t^A(f)(x,y)|^2 \frac{dydt}{t^{n+3}}\right]^{1/2}$$

and

$$\mu_\lambda^A(f)(x) = \left[\int\int_{R_+^{n+1}} \left(\frac{t}{t+|x-y|}\right)^{n\lambda} |F_t^A(f)(x,y)|^2 \frac{dydt}{t^{n+3}}\right]^{1/2},$$

where

$$F_t^A(f)(x) = \int_{|x-y|\le t} \frac{\prod_{j=1}^l R_{m_j+1}(A_j;x,y)}{|x-y|^m} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y)dy$$

and

$$F_t^A(f)(x,y) = \int_{|y-z|\le t} \frac{\prod_{j=1}^l R_{m_j+1}(A_j;y,z)}{|y-z|^m} \frac{\Omega(y-z)}{|y-z|^{n-1}} f(z)dz;$$

The variants of μ_Ω^A, μ_S^A and μ_λ^A are defined by

$$\tilde{\mu}_\Omega^A(f)(x) = \left(\int_0^\infty |\tilde{F}_t^A(f)(x)|^2 \frac{dt}{t^3}\right)^{1/2},$$

$$\tilde{\mu}_S^A(f)(x) = \left[\int\int_{\Gamma(x)} |\tilde{F}_t^A(f)(x,y)|^2 \frac{dydt}{t^{n+3}}\right]^{1/2}$$

and

$$\tilde{\mu}_\lambda^A(f)(x) = \left[\int\int_{R_+^{n+1}} \left(\frac{t}{t+|x-y|}\right)^{n\lambda} |\tilde{F}_t^A(f)(x,y)|^2 \frac{dydt}{t^{n+3}}\right]^{1/2},$$

where

$$\tilde{F}_t^A(f)(x) = \int_{|x-y|\le t} \frac{\prod_{j=1}^l Q_{m_j+1}(A_j;x,y)}{|x-y|^m} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y)dy$$

and

$$\tilde{F}_t^A(f)(x,y) = \int_{|y-z|\le t} \frac{\prod_{j=1}^l Q_{m_j+1}(A_j;y,z)}{|y-z|^m} \frac{\Omega(y-z)}{|y-z|^{n-1}} f(z)dz.$$

Set

$$F_t(f)(x) = \int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y)dy;$$

We also define that

$$\mu_\Omega(f)(x) = \left(\int_0^\infty |F_t(f)(x)|^2 \frac{dt}{t^3}\right)^{1/2},$$

$$\mu_S(f)(x) = \left(\int \int_{\Gamma(x)} |F_t(f)(y)|^2 \frac{dydt}{t^{n+3}} \right)^{1/2}$$

and

$$\mu_\lambda(f)(x) = \left(\int \int_{R_+^{n+1}} \left(\frac{t}{t+|x-y|} \right)^{n\lambda} |F_t(f)(y)|^2 \frac{dydt}{t^{n+3}} \right)^{1/2},$$

which are the Marcinkiewicz operators(see [22]). Let H be the space

$$H = \left\{ h : ||h|| = \left(\int_0^\infty |h(t)|^2 dt/t^3 \right)^{1/2} < \infty \right\}$$

or

$$H = \left\{ h : ||h|| = \left(\int \int_{R_+^{n+1}} |h(y,t)|^2 dydt/t^{n+3} \right)^{1/2} < \infty \right\}.$$

Then, it is clear that

$$\mu_\Omega^A(f)(x) = ||F_t^A(f)(x)||, \quad \mu_\Omega(f)(x) = ||F_t(f)(x)||,$$

$$\mu_S^A(f)(x) = \left|\left|\chi_{\Gamma(x)} F_t^A(f)(x,y)\right|\right|, \quad \mu_S(f)(x) = \left|\left|\chi_{\Gamma(x)} F_t(f)(y)\right|\right|$$

and

$$\mu_\lambda^A(f)(x) = \left|\left| \left(\frac{t}{t+|x-y|} \right)^{n\lambda/2} F_t^A(f)(x,y) \right|\right|,$$

$$\mu_\lambda(f)(x) = \left|\left| \left(\frac{t}{t+|x-y|} \right)^{n\lambda/2} F_t(f)(y) \right|\right|.$$

It is easily to see that μ_Ω, μ_S and μ_λ satisfy the conditions of Theorem 2.1 and 2.2, thus Theorem 2.1 and 2.2 hold for μ_Ω^A and $\tilde{\mu}_\Omega^A$, μ_S^A and $\tilde{\mu}_S^A$, μ_λ^A and $\tilde{\mu}_\lambda^A$.

Example 3 Bochner-Riesz operator .

Let $\delta > (n-1)/2$, $B_t^\delta(f)\hat{}(\xi) = (1-t^2|\xi|^2)_+^\delta \hat{f}(\xi)$ and $B_t^\delta(z) = t^{-n}B^\delta(z/t)$ for $t > 0$. Set

$$F_{\delta,t}^A(f)(x) = \int_{R^n} \frac{\prod_{j=1}^l R_{m_j+1}(A_j;x,y)}{|x-y|^m} B_t^\delta(x-y)f(y)dy$$

and

$$\tilde{F}_{\delta,t}^A(f)(x) = \int_{R^n} \frac{\prod_{j=1}^l Q_{m_j+1}(A_j;x,y)}{|x-y|^m} B_t^\delta(x-y)f(y)dy.$$

The maximal Bochner-Riesz multilinear operator and its the variants are defined by

$$B_{\delta,*}^A(f)(x) = \sup_{t>0} |B_{\delta,t}^A(f)(x)| \ \text{ and } \ \tilde{B}_{\delta,*}^A(f)(x) = \sup_{t>0} |\tilde{B}_{\delta,t}^A(f)(x)|.$$

We also define that

$$B_{\delta,*}(f)(x) = \sup_{t>0} |B_t^\delta(f)(x)|,$$

which is the maximal Bochner-Riesz operator(see [15]). Let H be the space $H = \{h : ||h|| = \sup_{t>0} |h(t)| < \infty\}$, then

$$B_{\delta,*}^A(f)(x) = ||B_{\delta,t}^A(f)(x)||, \quad B_*^\delta(f)(x) = ||B_t^\delta(f)(x)||.$$

It is easily to see that $B_{\delta,*}$ satisfies the conditions of Theorem 2.1 and 2.2, thus Theorem 2.1 and 2.2 hold for $B_{\delta,*}^A$ and $\tilde{B}_{\delta,*}^A$.

4 Open problem

In this paper, the boundedness properties of the multilinear operators generated by certain non-convolution type integral operators and Lipschitz functions on some Hardy and Herz-type spaces are obtained. The operators include Littlewood-Paley operators, Marcinkiewicz operators and Bochner-Riesz operator.

The open problem is to study the boundedness of the multilinear operators generated by the non-convolution type integral operators and others locally integrable functions on others spaces.

References

[1] J.Alvarez, R.J.Babgy, D.S.Kurtz and C.Perez, Weighted estimates for commutators of linear operators, Studia Math., 104(1993), 195-209.

[2] S.Chanillo, A note on commutators, Indiana Univ. Math. J. 31(1982), 7-16.

[3] W.G.Chen, Besov estimates for a class of multilinear singular integrals, Acta Math. Sinica, 16(2000), 613-626.

[4] J.Cohen, A sharp estimate for a multilinear singular integral on R^n, Indiana Univ. Math. J., 30(1981), 693-702.

[5] J.Cohen and J.Gosselin, On multilinear singular integral operators on R^n, Studia Math., 72(1982), 199-223.

[6] J.Cohen and J.Gosselin, A BMO estimate for multilinear singular integral operators, Illinois J. Math., 30(1986), 445-465.

[7] R.Coifman and Y.Meyer, Wavelets, Calderón-Zygmund and multilinear operators, Cambridge Studies in Advanced Math.48, Cambridge University Press, Cambridge , 1997.

[8] R.Coifman, R.Rochberg and G.Weiss, Factorization theorems for Hardy spaces in several variables, Ann. of Math., 103(1976), 611-635.

[9] Y.Ding and S.Z.Lu, Weighted boundedness for a class rough multilinear operators, Acta Math. Sinica, 17(2001), 517-526.

[10] J.Garcia-Cuerva and M.L. Herrero, A theory of Hardy spaces associated to the Herz spaces, Proc.London Math. Soc., 69(1994), 605-628.

[11] S.Janson, Mean oscillation and commutators of singular integral operators, Ark. Math., 16(1978), 263-270.

[12] L.Z.Liu, Triebel-Lizorkin space estimates for multilinear operators of sublinear operators, Proc. Indian Acad. Sci. (Math. Sci), 113(2003), 379-393.

[13] L.Z.Liu, The continuity of commutators on Triebel-Lizorkin spaces, Integral Equations and Operator Theory, 49(2004), 65-76.

[14] L.Z.Liu, Boundedness of multilinear operator on Triebel-Lizorkin spaces, Inter J. of Math. And Math. Sci., 259-272, 5(2004).

[15] S.Z.Lu, Four lectures on real H^p spaces, World Scientific, River Edge, NI, 1995.

[16] S.Z.Lu, Q.Wu and D.C.Yang, Boundedness of commutators on Hardy type spaces, Sci.in China(ser.A), 45(2002), 984-997.

[17] S.Z.Lu and D.C.Yang, The decomposition of the weighted Herz spaces and its applications, Sci. in China (ser.A), 38(1995), 147-158.

[18] S.Z.Lu and D.C.Yang, The weighted Herz type Hardy spaces and its applications, Sci. in China(ser.A), 38(1995), 662-673.

[19] M.Paluszynski, Characterization of the Besov spaces via the commutator operator of Coifman, Rochberg and Weiss, Indiana Univ. Math. J., 44(1995), 1-17.

[20] E.M.Stein, Harmonic Analysis: real variable methods, orthogonality and oscillatory integrals, Princeton Univ. Press, Princeton NJ, 1993.

[21] A.Torchinsky, Real variable methods in harmonic analysis, Pure and Applied Math., 123, Academic Press, New York, 1986.

[22] A.Torchinsky and S.Wang, A note on the Marcinkiewicz integral, Colloq. Math., 60/61(1990), 235-243.