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A Prey-Predator Model with a Cover Linearly Varying with the Prey Population and an Alternative Food for the Predator

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Abstract

The present paper is devoted to an analytical investigation of a prey-predator model with a cover linearly varying with the size of prey is provided to protect it from the predator and the predator is provided with an alternative food in addition to the prey. The model is characterized by a couple of first order non-linear ordinary differential equations. All the five equilibrium points of the model are identified and stability criteria are discussed. Further exact solutions of perturbed equations have been derived.

Keyswords: Equilibrium points, Linear cover, Normal Study State, Prey, Predator, Stability, Trajectories.

1 Introduction

In the classical Lotka - Volterra Prey - Predator model, there is no protection for Prey from the Predator and Predator sustains on the Prey alone. When the Prey population falls below a certain level, the predator would migrate to another region in search of food and return only when the Prey-Population rises to the required level. Some of the prey-predator models were discussed by Kapur [1], Michale Olinnck [2], May [3], Varma [4] Colinvaux [5], Freedman [6], Narayan

[7]. Inspired by that, we discussed a more general model by taking an alternative food for the predator, and a cover linearly varying with the population of the prey. The model is characterized by a couple of first order ordinary differential equations. All the five equilibrium points of the model are identified and stability criteria are discussed..

2 Basic Equations

The model equations for a two species Prey - Predator system is given by the following system of first order ordinary differential equations employing the following notation:

N_1 and N_2 are the populations of the prey and predator with the natural growth rates a_1 and a_2 respectively,

α_{11} is rate of decrease of the prey due to insufficient food,

α_{12} is rate of decrease of the prey due to inhibition by the predator,

α_{21} is rate of increase of the predator due to successful attacks on the prey,

α_{22} is rate of decrease of the predator due to insufficient food other than the prey,

In addition to them

$a+bN_1$ is the prey population provided with a cover of safety with constant values for *a* and *b*, from the attacks of the predator,

$$\frac{dN_1}{dt} = a_1N_1 - \alpha_{11}N_1{}^2 - \alpha_{12}[N_1 - (a+bN_1)]N_2 \tag{2.1}$$

$$\frac{dN_2}{dt} = a_2N_2 - \alpha_{22}N_2{}^2 + \alpha_{21}[N_1 - (a+bN_1)]N_2 \tag{2.2}$$

3 Equilibrium states

The system under investigation has five equilibrium states. They are

I. The fully washed out state $\overline{N}_1 = 0; \overline{N}_2 = 0$ (3.1)

II. The state in which only the prey survives and the predators are washed out

$$\overline{N}_1 = \frac{a_1}{\alpha_{11}};\ \overline{N}_2 = 0 \tag{3.2}$$

III. The state in witch both the prey and the predators coexist

$$\overline{N}_1 = \frac{p}{2q}; \overline{N}_2 = \frac{2qr + \alpha_{21}(1-b)p}{2q\alpha_{22}} \tag{3.3}$$

and this can exist only when $p^2 = 4qa\alpha_{12}(a\alpha_{21} - a_2)$ (3.4)

IV. The state in witch both the prey and the predators co-exists

$$\overline{N}_1 = \frac{p^2 + a\alpha_{12}\, q r}{p q} ; \overline{N}_2 = \frac{pqr + \alpha_{21}(1-b)[p^2 + a\alpha_{12} qr]}{\alpha_{22}\, pq} \tag{3.5}$$

V. The state in witch both the prey and the predators co-exists

$$\overline{N}_1 = \frac{a\alpha_{12}(a\alpha_{21} - a_2)}{p} \; ; \; \overline{N}_2 = \frac{p r - a r \alpha_{12}\alpha_{21}(1-b)}{\alpha_{22}\, p} \tag{3.6}$$

The equilibrium points IV & V i.e. the equations (3.5) & (3.6) exists only when

$$p^2 > 4qa\alpha_{12}(a\alpha_{21} - a_2) \tag{3.7}$$

where

$$p = a_1\alpha_{22} - \alpha_{12}(1-b)(a_2 - 2a\alpha_{21});\; q = \alpha_{11}\alpha_{22} + \alpha_{12}\alpha_{21}(1-b)^2;\; r = a_2 - a\alpha_{21} \tag{3.8}$$

The states III, IV and V are called the "**the normal study states".**

4 The stability of the equilibrium states

Let $N = (N_1, N_2)^T = \overline{N} + U$ (4.1)

where U = $(u_1, u_2)^T$ is the perturbation over the equilibrium state $\overline{N} = (\overline{N}_1, \overline{N}_2)^T$ .The basic equations (2.1) and (2.2) are quasi-linearized to obtain the equations for the perturbed state.

$$\frac{dU}{dt} = AU \tag{4.2}$$

where

$$A = \begin{bmatrix} a_1 - 2\alpha_{11}\overline{N}_1 - \alpha_{12}(1-b)\overline{N}_2 & a\alpha_{12} - \alpha_{12}(1-b)\overline{N}_1 \\ \alpha_{21}(1-b)\overline{N}_2 & -\alpha_{22}\overline{N}_2 \end{bmatrix} \tag{4.3}$$

The characteristic equation for the system is $det\left[A - \lambda I\right] = 0$ (4.4)

The equilibrium state is stable, if both the roots of the equation (4.4) are negative in case they are real or the roots have negative real parts in case they are complex.

4. 1 Stability of the equilibrium state I

The trajectories for both the washed out state are

$$u_1 = a\alpha_{12}\frac{u_{20}\, e^{(a_2 - a\alpha_{21})t}}{a_2 - a\alpha_{21} - a_1} + \{u_{10} - \frac{a\alpha_{12}u_{20}}{a_2 - a\alpha_{21} - a_1}\}e^{a_1 t} \tag{4.5}$$

$$u_2 = u_{20}\, e^{(a_2 - a\alpha_{21})t} \tag{4.6}$$

where u_{10}, u_{20} are the initial values of u_1, u_2.

The solution curves are illustrated in figures 1 to 4

Case 1: In this case the predator dominates the prey in natural growth as well as in its initial population strength. i.e. $a_2 > a\alpha_{21}$ and $u_{10} < u_{20}$ as shown in Fig.1

Case 2: In this case the prey dominates the predator in natural growth as well as in its initial population strength. i.e. $a_2 < a\alpha_{21}$ and $u_{10} > u_{20}$ as shown in Fig. 2

Case 3: The prey dominates the predator in natural growth rate but its initial strength is less than that of the predator. i.e. $u_{10} < u_{20}$ and $a_2 > a\alpha_{21}; a_1 > a_2$ In this case, the predator outnumber the prey till the time-instant $t = t^*$ after that the prey outnumber the predator.

$$t = t^* = \frac{1}{(a_2 - a\alpha_{21} - a_1)} \left(\frac{u_{10}(a_2 - a\alpha_{21} - a_{11}) - a\alpha_{21}u_{20}}{u_{20}(a_2 - a\alpha_{21} - a_{11}) - a\alpha_{21}u_{20}} \right) \tag{4.7}$$

Case 4: The prey dominates the predator in natural growth rate but its initial strength is less than that of the predator. i.e. $u_{10} < u_{20}$ and $a_2 < a\alpha_{21}$ In this case, the predator out number the prey till the time-instant $t = t^*$ given by the equation (4.7) after that the prey outnumber the predator.

4. 2 Trajectories of perturbed species for equilibrium state I

The trajectories in the $u_1 - u_2$ plane are given by

$$u_1 = \frac{c u_2^{p_3} - q_3 u_2}{p_3 - 1} \tag{4.8}$$

where $p_3 = \dfrac{a_1}{a_2 - a\alpha_{21}}$; $\quad q_3 = \dfrac{a\alpha_{21}}{a_2 - a\alpha_{21}}$ (4.9)

and c is an arbitrary constant. These are illustrated in Fig.5.

4.3 Stability of the equilibrium state II

The trajectories for the predator washed out state are

Case A: When $d_2 = 0 \Rightarrow b = 1$

$$u_1 = \left[\frac{a\alpha_{11}\alpha_{12}}{\alpha_{11}} \right] \frac{u_{20}}{a_1} + \left[u_{10} - \left\{ \left(\frac{a\alpha_{11}\alpha_{12}}{\alpha_{11}} \right) \frac{u_{20}}{a_1} \right\} \right] e^{-a_1 t} \tag{4.10}$$

and

$$u_2 = u_{20} \tag{4.11}$$

The results are illustrated in figures 6 & 7

CaseA$_1$: When the initial strength of prey is more than the predator. i.e. $u_{10} > u_{20}$, the prey outnumber the predator till the time-instant $t = t^*$ after that the predator outnumber the prey. In the course of time u_1 is asymptotic to $u_1^* = \dfrac{a\alpha_{11}\alpha_{12}u_{20}}{a_1\alpha_{11}}$ which is illustrated in Fig.6

$$t = t^* = \frac{1}{a_1} ln \left\{ \frac{a_1 u_{10} - \left(\frac{a\alpha_{11}\alpha_{12}}{\alpha_{11}} \right) u_{20}}{u_{20} - \left(\frac{a\alpha_{11}\alpha_{12}}{\alpha_{11}} \right) u_{20}} \right\} \tag{4.12}$$

CaseA$_2$: The initial strength of the predator is more than the prey, *i.e.*$u_{10}<u_{20}$, the predator outnumber the prey and in the course of time u_1 is asymptotic to

$u_1^* = \frac{a\alpha_{11}\alpha_{12}u_{20}}{a_1\alpha_{11}}$ as is illustrated in Fig.7.

Case B: When $d_2 > 0 \Rightarrow b < 1$

$$u_1 = \left[\frac{a\alpha_{11}\alpha_{12} - a_1\alpha_{12}(1-b)}{\alpha_{11}} \right] \frac{u_{20} e^{d_2 t}}{(d_2 + a_1)} + \left[u_{10} - \left\{ \left(\frac{a\alpha_{11}\alpha_{12} - a_1\alpha_{12}(1-b)}{\alpha_{11}} \right) \frac{u_{20}}{(d_2 + a_1)} \right\} \right] e^{-a_1 t}$$

(4.13)

$$u_2 = u_{20}\, e^{d_2 t} \tag{4.14}$$

where $d_2 = \frac{a_1\alpha_{21}(1-b)}{\alpha_{11}}$ (4.15)

The solution curves are illustrated in the figures 8 to 11

Case B$_1$: The predator dominates the prey in natural growth rate but its initial strength is less than that of the prey. i.e. $u_{10}>u_{20}$. In this case, the prey outnumber the predator till the time-instant $t = t^*$ given by equation (4.16) after that the predator outnumber the prey as shown in Fig. 8

$$t = t^* = \frac{1}{a_1} ln \left\{ \frac{(a_1 + d_2)u_{10} - \left(\frac{a\alpha_{11}\alpha_{12} - a_1\alpha_{12}(1-b)}{\alpha_{11}} \right) u_{20}}{a_1 + d_2 - \left(\frac{a\alpha_{11}\alpha_{12} - a_1\alpha_{12}(1-b)}{\alpha_{11}} \right)} \right\} \tag{4.16}$$

Case B$_2$: The predator dominates the prey in natural growth as well as in its initial population strength. i.e. $u_{20}>u_{10}$; $d_2>a_1$ as shown in Fig. 9

Case B$_3$: The predator dominates the prey in natural growth as well as in its initial population strength and $d_2<a_1$. i.e.$u_{20}>u_{10}$ and $d_2<a_1$ which is illustrated in Fig. 10

Case B$_4$: Initially the prey dominates and $d_2<a_1$ i.e. $u_{20}<u_{10}$. In this case, the prey outnumber the predator till the time-instant t* given by the equation (4.16), after which the predator outnumber the prey and grows unboundedly while the prey asymptotically approaches to the equilibrium value $\overline{N}_1$ given in (3.2), as shown in Fig. 11.

Case C: $d_2 < 0 \Rightarrow b > 1$ the trajectories are as same in Case B, but the state is stable

The solution curves are illustrated in the figures 12 & 13.

Case C_1 The predator dominates the prey in natural growth as well as in its initial population strength. i.e. $u_{10}<u_{20}$ However both converge asymptotically to the equilibrium point $(\overline{N}_1, \overline{N}_2)$ given by (3.2). Hence the equilibrium point is **stable** as shown in Fig. 12

Case C_2: The prey dominates the predator in its initial strength. i.e. $u_{10}>u_{20}$. In this case $u_1(t) = u_2(t)$ is possible at time t* given by (4.16) as shown in Fig. 13. Hence the equilibrium point is **stable.**

4.4 Trajectories of perturbed species for equilibrium state II

The trajectories in the u_1-u_2 plane are given by

$$u_1 = \frac{c\,u_2^{\,p_2} - q_2 u_2}{p_2 - 1} \tag{4.17}$$

where

$$p_2 = \frac{-a_1\alpha_{11}}{a_1\alpha_{21}(1-b)}; \qquad q_2 = \frac{a\alpha_{11}\alpha_{22} - a_1\alpha_{12}(1-b)}{a_1\alpha_{21}(1-b)} \tag{4.18}$$

and $c =$ arbitrary constant. The solution curves are illustrated in Fig.14.

4.5 Stability of the equilibrium state III

The trajectories for the co-existence state are

$$u_1 = \left[\frac{\left(\lambda_1 + \alpha_{22}\overline{N}_2\right)u_{10} + \left(a\alpha_{12} - \alpha_{12}(1-b)\overline{N}_1\right)u_{20}}{\lambda_1 - \lambda_2}\right]e^{\lambda_1 t} + \left[\frac{\left(\lambda_2 + \alpha_{22}\overline{N}_2\right)u_{10} + \left(a\alpha_{12} - \alpha_{12}(1-b)\overline{N}_1\right)u_{20}}{\lambda_2 - \lambda_1}\right]e^{\lambda_2 t}$$

(4.19)

$$u_2 = \left[\frac{\left\{\lambda_1 - \left(a_1 - 2\alpha_{11}\overline{N}_1 - \alpha_{12}(1-b)\overline{N}_2\right)\right\}u_{20} + \alpha_{21}(1-b)\overline{N}_2 u_{10}}{\lambda_1 - \lambda_2}\right]e^{\lambda_1 t}$$

$$+\left[\frac{\left\{\lambda_2 - \left(a_1 - 2\alpha_{11}\overline{N}_1 - \alpha_{12}(1-b)\overline{N}_2\right)\right\}u_{20} + \alpha_{21}(1-b)\overline{N}_2 u_{10}}{\lambda_2 - \lambda_1}\right]e^{\lambda_2 t}$$

(4.20)

The solution curves are illustrated in figures 15 & 16

Case 1: The prey dominates the predator in natural growth as well as in its initial population strength. i.e. $u_{10}>u_{20}$, which is illustrated in Fig.15.

Case 2: The prey dominates the predator in natural growth rate but its initial strength is less than that of the predator. i.e. $u_{10}<u_{20}$. In this case, the predator outnumber the prey till the time-instant t*, after which the prey outnumber the predator as shown in Fig.17.

$$t = t^* = \frac{1}{\lambda_1 - \lambda_2} ln\left\{\frac{(\lambda_2 - D_1) + B_1 u_{20} - (\lambda_2 - A_1)u_{20} - C_1 u_{10}}{(\lambda_1 - D_1) + B_1 u_{20} - (\lambda_1 - A_1)u_{20} - B_1 u_{10}}\right\} \tag{4.21}$$

where

$$A_1 = a_1 - 2\alpha_{11}\overline{N}_1 - \alpha_{12}(1-b)\overline{N}_2 ; \qquad B_1 = a\,\alpha_{12} - \alpha_{12}(1-b)\overline{N}_1 ;$$
$$C_1 = \alpha_{21}(1-b)\overline{N}_2 ; \qquad D_1 = -\alpha_{22}\overline{N}_2 \tag{4.22}$$

Case3: When $(A_1 + D_1)^2 < 4(A_1 D_1 - B_1 C_1)$, (4.23)

the roots are complex with negative real part. Hence the equilibrium state is **stable.** The solution curves are illustrated in Fig.17

4.6 Trajectories of perturbed species for equilibrium state III

The trajectories in the u_1-u_2 plane are given by

$$cu_2 = \left(\frac{u_2}{u_1} - v_1\right)^{\left(\frac{\alpha_{12}(1-b)\overline{N}_2 v_1 - \alpha_{22}\overline{N}_2}{v_2 - v_1}\right)} \left(\frac{u_2}{u_1} - v_2\right)^{\left(\frac{\alpha_{12}(1-b)\overline{N}_2 v_2 - \alpha_{22}\overline{N}_2}{v_1 - v_2}\right)} \tag{4.24}$$

where v_1 & v_2 are roots of the equation

$$\alpha_{12}(1-b)\overline{N}_2 v^2 - \left(a_1 - 2\alpha_{11}\overline{N}_1 - \alpha_{12}(1-b)\overline{N}_2 + \alpha_{22}\overline{N}_2\right)v - a\alpha_{12} - \alpha_{12}(1-b)\overline{N}_1 = 0 \tag{4.25}$$

When $(A_1 + D_1)^2 < 4(A_1 D_1 - B_1 C_1)$, the roots are complex with negative real part the curve is a concentric spiral as shown in Fig 18 Hence the equilibrium state is **stable.**

One can easily find the similarities in the results for equilibrium states IV & V as observed in equilibrium state III.

5. Conclusions

On rearranging, the basic equations under investigation can be written as

$$\frac{dN_1}{dt} = N_1\left\{a_1 - \alpha_{11}N_1 - \alpha_{12}(1-b)N_2\right\} + \alpha_{12}aN_2 \tag{5.1}$$

$$\frac{dN_2}{dt} = N_2\left\{(a_2 - a) - \alpha_{22}N_2 + \alpha_{21}(1-b)N_1\right\} \tag{5.2}$$

Case I: When $a = 0$ *and* $b = 0$, the basic equations are reduced to

$$\frac{dN_1}{dt} = N_1\left\{a_1 - \alpha_{11}N_1 - \alpha_{12}N_2\right\} \tag{5.3}$$

$$\frac{dN_2}{dt} = N_2\left\{a_2 - \alpha_{22}N_2 + \alpha_{21}N_1\right\} \tag{5.4}$$

i.e. the model reduced to a Prey- Predator model with a limited alternative food for the Predator.

This system will have four equilibrium states. Out of these four, three states are unstable and the co-existence state is stable. Narayan & Ramacharyulu [8] discussed this model in detail.

Case II: When $a = 0$ *and* $b \neq 0$, the basic equations under investigation are reduced to

$$\frac{dN_1}{dt} = N_1\left\{a_1 - \alpha_{11}N_1 - \alpha_{12}(1-b)N_2\right\} \tag{5.5}$$

$$\frac{dN_2}{dt} = N_2\left\{a_2 - \alpha_{22}N_2 + \alpha_{21}(1-b)N_1\right\} \tag{5.6}$$

i.e. the model reduced to a Prey- Predator model with a limited cover proportional to the population of the prey and the predator is provided with an alternative food.

This system will also have four equilibrium states. Out of these four, three states are unstable and the normal steady state is stable. Narayan & Ramacharyulu [9] discussed this model in detail.

Case III: When $a \neq 0$ *and* $b = 0$, the model equations under consideration are reduced to

$$\frac{dN_1}{dt} = N_1\left\{a_1 - \alpha_{11}N_1 - \alpha_{12}N_2\right\} + \alpha_{12}aN_2 \tag{5.67}$$

$$\frac{dN_2}{dt} = N_2\left\{(a_2 - a) - \alpha_{22}N_2 + \alpha_{21}N_1\right\}, \tag{5.8}$$

which does not represent a Prey- Predator model.

6 Trajectories

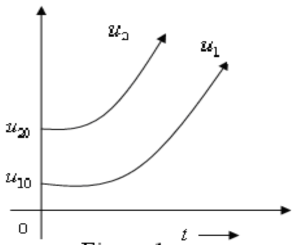
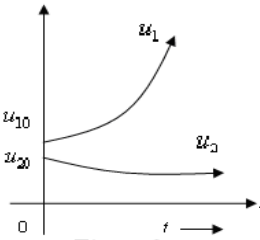


Figure.1



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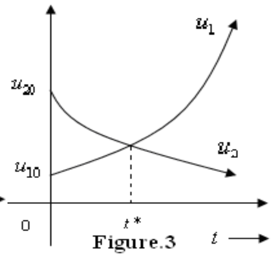

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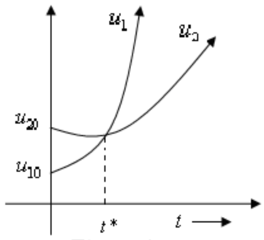

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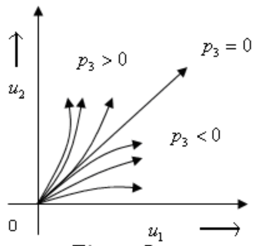

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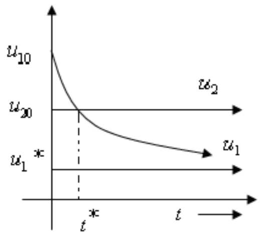

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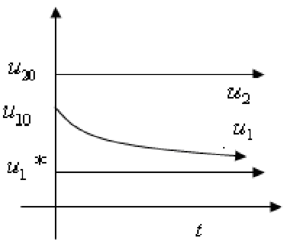

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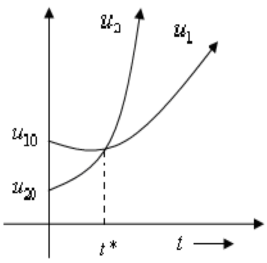

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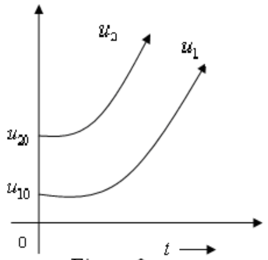

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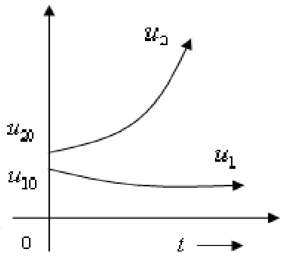

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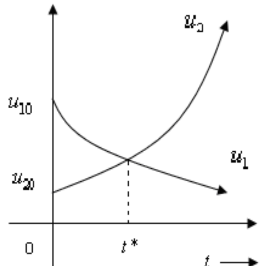

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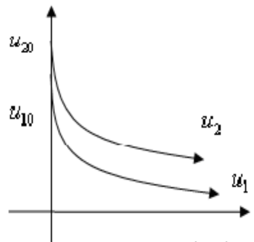

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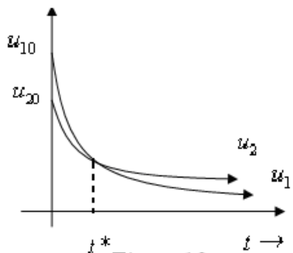

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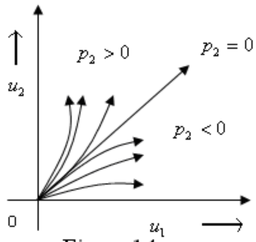

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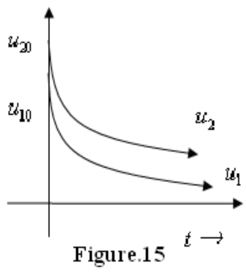

Figure.15

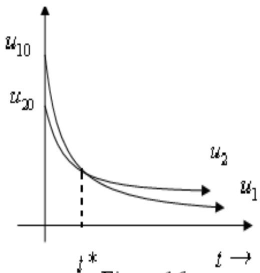

Figure.16

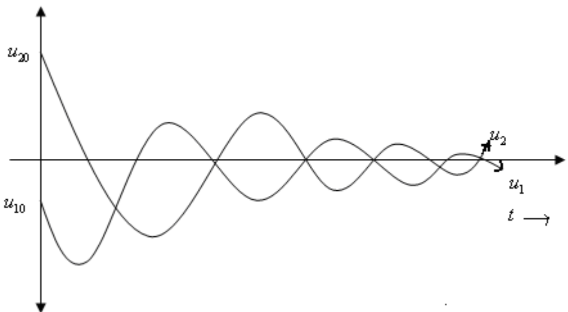

Figure.17

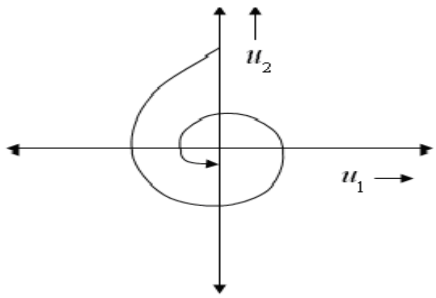

Figure.18

7 Future Works (Open Problems)

The present paper is devoted to an analytical investigation of a Prey-Predator model with a cover linearly varying with the population of prey, to protect it from the attacks of the predator and the predator is provided with an alternative food in addition to the prey.

Prey- Predator models can be studied by introducing a constant cover for the prey, harvesting of both the species, time delay for interaction etc. One can develop a Prey-Predator model by introducing age structured population.

In the present model only one normal steady state is studied and it can be extended by comparing the "three co-existence equilibrium states". By taking numerical illustration, one can examine which normal steady state is more stable than the others. Also harvesting may be introduced in this problem. Lypunov's function can be constructed to study the global stability of the model, and for each equilibrium state, we may develop a threshold theorem and the results can be analyzed. The stability may be studied by Kolmogrove's limit cycles.

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