Int. J. Open Problems Compt. Math., Vol. 2, No. 3, September 2009
ISSN 1998-6262;
www.i-csrs.org

Spacelike Salkowski and anti-Salkowski Curves With a Spacelike Principal Normal in Minkowski 3-Space

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Abstract

A century ago, Salkowski [9] introduced a family of curves with constant curvature but non-constant torsion (Salkowski curves) and a family of curves with constant torsion but non-constant curvature (anti-Salkowski curves). In this paper, we adapt definition of such curves to spacelike curves in Minkowski 3-space. Thereafter, we introduce an explicit parametrization of a spacelike Salkowski curves with a spacelike principal normal vector and a spacelike anti-Salkowski curves with a spacelike principal normal vector in Minkowski 3-space. Moreover, we characterize them as a space curve with constant curvature or constant torsion and whose normal vector makes a constant angle with a fixed straight line.

MSC: *53C40, 53C50*
Keywords: *Salkowski curves; constant curvature and torsion; Minkowski 3-space; spacelike curves.*

1 Introduction

Salkowski (resp. anti-Salkowski) curves in Euclidean space $\mathbf{E}^3$ are generally known as family of curves with constant curvature (resp. torsion) but non-constant torsion (resp. curvature) with an explicit parametrization [7, 9]. They were defined in an earlier paper [9] and retrieved, as an example of tangentially cubic curves, in a first version of Pottmann and Hofer [8]. Recently, Monterde

[7] studied some of characterizations of these curves and he prove that the normal vector makes a constant angle with a fixed straight line.

Analogously, we define in this paper, Salkowski curves and anti-Salkowski curves in Minkowski space $\mathbf{E}_1^3$. Also, we introduce the explicit parametrization of a spacelike Salkowski curves with a spacelike principal normal vector and spacelike anti-Salkowski curves with a spacelike principal normal vector in Minkowski space $\mathbf{E}_1^3$ and we study some characterizations of these curves.

2 Preliminaries

First, we briefly present theory of the curves in Minkowski 3-space as follows:

The Minkowski three-dimensional space $\mathbf{E}_1^3$ is the real vector space $\mathbb{R}^3$ endowed with the standard flat Lorentzian metric given by:

$$\langle , \rangle = -dx_1^2 + dx_2^2 + dx_3^2,$$

where (x_1, x_2, x_3) is a rectangular coordinate system of $\mathbf{E}_1^3$. If $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ are arbitrary vectors in $\mathbf{E}_1^3$, we define the (Lorentzian) vector product of $\mathbf{u}$ and $\mathbf{v}$ as the following:

$$u \times v = -\begin{vmatrix} -i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}.$$

An arbitrary vector $\mathbf{v} \in \mathbf{E}_1^3$ is said to be a spacelike if $\langle \mathbf{v}, \mathbf{v} \rangle > 0$ or $\mathbf{v} = 0$, timelike if $\langle \mathbf{v}, \mathbf{v} \rangle < 0$, and lightlike (or null) if $\langle \mathbf{v}, \mathbf{v} \rangle = 0$ and $\mathbf{v} \neq 0$. The norm (length) of a vector $\mathbf{v}$ is given by $\parallel \mathbf{v} \parallel = \sqrt{|\langle \mathbf{v}, \mathbf{v} \rangle|}$. An arbitrary regular (smooth) curve $\alpha : I \subset \mathbb{R} \to \mathbf{E}_1^3$ is locally spacelike if all of its velocity vectors $\alpha'(t)$ are spacelike for each $t \in I \subset \mathbb{R}$. If α is spacelike, there exists a change of the parameter t, namely, $s = s(t)$, such that $\parallel \alpha'(s) \parallel = 1$. We say then that α is a unit speed curve [1, 2, 6].

Given a unit speed curve α in Minkowski space $\mathbf{E}_1^3$ it is possible to define a Frenet frame $\{\mathbf{T}(s), \mathbf{N}(s), \mathbf{B}(s)\}$ associated for each point s [5, 10]. Here $\mathbf{T}$, $\mathbf{N}$ and $\mathbf{B}$ are the tangent, principal normal and binormal vector field, respectively. We suppose that α is a spacelike curve with a spacelike principal normal vector $\mathbf{N}$. Then $\mathbf{T}'(s) \neq 0$ is a spacelike vector independent with $\mathbf{T}(s)$. We define the curvature of α at s as $\kappa(s) = |\mathbf{T}'(s)|$. The principal normal vector $\mathbf{N}(s)$ and the binormal vector $\mathbf{B}(s)$ are defined as

$$\mathbf{N}(s) = \frac{\mathbf{T}'(s)}{\kappa(s)} = \frac{\alpha''}{|\alpha''|}, \ \ \mathbf{B}(s) = \mathbf{T}(s) \times \mathbf{N}(s),$$

where the vector $\mathbf{B}(s)$ is unitary and timelike. For each s, $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ is an orthonormal base of $\mathbf{E}_1^3$ which is called the Frenet trihedron of α. We define the torsion of α at s as:

$$\tau(s) = -\langle \mathbf{N}'(s), \mathbf{B}(s) \rangle.$$

Then the Frenet formula is

$$\begin{bmatrix} \mathbf{T}' \\ \mathbf{N}' \\ \mathbf{B}' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & \tau & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{bmatrix}, \tag{1}$$

where

$$\langle \mathbf{T}, \mathbf{T} \rangle = \langle \mathbf{N}, \mathbf{N} \rangle = 1, \langle \mathbf{B}, \mathbf{B} \rangle = -1, \langle \mathbf{T}, \mathbf{N} \rangle = \langle \mathbf{N}, \mathbf{B} \rangle = \langle \mathbf{B}, \mathbf{T} \rangle = 0.$$

3 Spacelike Salkowski curves with a spacelike principal normal and some characterizations

In this section, we introduce the explicit parametrization of a spacelike Salkowski curves with a spacelike principal normal vector in Minkowski space $\mathbf{E}_1^3$ as the following:

Definition 3.1 *For any $m \in R$ with $m > 1$, let us define the space curve*

$$\gamma_m(t) = \frac{n}{4m}\Bigg(\begin{array}{l} 2\cosh[t] - \frac{1+n}{1-2n}\cosh[(1-2n)t] - \frac{1-n}{1+2n}\cosh[(1+2n)t], \\ 2\cosh[t] - \frac{1+n}{1-2n}\sinh[(1-2n)t] - \frac{1-n}{1+2n}\sinh[(1+2n)t], \\ \frac{1}{m}\cosh[2nt]\Bigg), \end{array} \tag{2}$$

with $n = \frac{m}{\sqrt{m^2-1}}$.

We will call a spacelike Salkowski curve with a spacelike principal normal vector in Minkowski space $\mathbf{E}_1^3$. One can see a special example ($m = 15$ and $t \in [-5, 5]$) of such curves in the right figure 1.

The geometric elements of this curve γ_m are the following:

(1): $\langle \gamma_m', \gamma_m' \rangle = \frac{\cosh^2[nt]}{m^2-1}$, so $\|\gamma_m'\| = \frac{\cosh[nt]}{\sqrt{m^2-1}}$

(2): The arc-length parameter is $s = \frac{\sinh[nt]}{m}$.

(3): The curvature $\kappa(t) = 1$ and the torsion $\tau(t) = \tanh[nt]$.

(4): The Frenet frame is

$$\begin{aligned} \mathbf{T}(t) = & \Big(n\cosh[t]\sinh[nt] - \sinh[t]\cosh[nt], \\ & n\sinh[t]\sinh[nt] - \cosh[t]\cosh[nt], \tfrac{n}{m}\sinh[nt]\Big), \\ \mathbf{N}(t) = & \tfrac{n}{m}\Big(\cosh[t], \sinh[t], m\Big), \\ \mathbf{B}(t) = & \Big(\sinh[t]\sinh[nt] - n\cosh[t]\cosh[nt], \\ & \cosh[t]\sinh[nt] - n\sinh[t]\cosh[nt], -\tfrac{n}{m}\cosh[nt]\Big). \end{aligned} \tag{3}$$

From the expression of the normal vector, see Equation (3), we can see that the normal indicatrix, or nortrix, of a Salkowski curve (2) in Minkowski space $\mathbf{E}_1^3$ describes a parallel of the unit sphere. The hyperbolic angle between the normal vector and the vector $(0, 0, 1)$ is constant and equal to $\phi = \pm\text{arccosh}[n]$. This fact is reminiscent of what happens with another important class of curves, the general helices in Minkowski space $\mathbf{E}_1^3$. Such a condition implies that the tangent indicatrix, or tantrix, describes a parallel in the unit sphere.

Lemma 3.2 *Let $\alpha : I \to \mathbf{E}_1^3$ be a spacelike curve with a spacelike principal normal vector parameterized by arc-length with $\kappa = 1$. The normal vector make a constant hyperbolic angle, ϕ, with a fixed straight line in space if and only if $\tau(s) = \pm\frac{s}{\sqrt{s^2+\tanh^2[\phi]}}$.*

proof: ($\Rightarrow$) Let $\mathbf{d}$ be the unitary spacelike fixed vector makes a constant hyperbolic angle ϕ with the normal vector $\mathbf{N}$. Therefore

$$\langle \mathbf{N}, \mathbf{d} \rangle = \cosh[\phi]. \tag{4}$$

Differentiating Equation (4) and using Frenet's equations, we get

$$\langle -\mathbf{T} + \tau\mathbf{B}, \mathbf{d} \rangle = 0. \tag{5}$$

Therefore,

$$\langle \mathbf{T}, \mathbf{d} \rangle = \tau\langle \mathbf{B}, \mathbf{d} \rangle.$$

If we put $\langle \mathbf{B}, \mathbf{d} \rangle = b$, we can write

$$\mathbf{d} = \tau\, b\, \mathbf{T} + \cosh[\phi]\mathbf{N} + b\,\mathbf{B}.$$

From the unitary of the vector $\mathbf{d}$ we get $b = \pm\frac{\sinh[\phi]}{\sqrt{1-\tau^2}}$. Therefore, the vector $\mathbf{d}$ can be written as

$$\mathbf{d} = \pm\frac{\tau\,\sinh[\phi]}{\sqrt{1-\tau^2}}\mathbf{T} + \cosh[\phi]\mathbf{N} \pm \frac{\sinh[\phi]}{\sqrt{1-\tau^2}}\mathbf{B}. \tag{6}$$

If we differentiate Equation (5) again, we obtain

$$\langle \dot{\tau}\mathbf{B} + (\tau^2 - 1)\mathbf{N}, \mathbf{d}\rangle = 0. \tag{7}$$

Equations (6) and (7) lead to the differential equation

$$\pm \tanh[\phi]\frac{\dot{\tau}}{(1-\tau^2)^{3/2}} + 1 = 0.$$

Integration the above equation, we get

$$\pm \tanh[\phi]\frac{\tau}{\sqrt{1-\tau^2}} + s + c = 0. \tag{8}$$

where c is an integration constant. The integration constant can disappear with a parameter change $s \longrightarrow s - c$. Finally, to solve (8) with τ as unknown we express the desired result.

($\Leftarrow$) Suppose that $\tau = \pm\frac{s}{\sqrt{s^2+\tanh^2[\phi]}}$ and let us consider the spacelike vector

$$\mathbf{d} = \cosh[\phi]\Big(s\,\mathbf{T} + \mathbf{N} \mp \sqrt{s^2 + \tanh^2[\phi]}\,\mathbf{B}\Big).$$

We will prove that the vector $\mathbf{d}$ is a constant vector. Indeed, applying Frenet formula

$$\dot{\mathbf{d}} = \cosh[\phi]\Big(\mathbf{T} + s\kappa\mathbf{N} - \mathbf{T} + \tau\mathbf{B} \mp \frac{s}{\sqrt{s^2+\tanh^2[\phi]}}\mathbf{B} \mp \tau\sqrt{s^2+\tanh^2[\phi]}\mathbf{N}\Big) = 0$$

Therefore, $\mathbf{d}$ is constant and $\langle \mathbf{N}, \mathbf{d}\rangle = \cosh[\phi]$. This concludes the proof of Lemma (3.2).

Once the intrinsic or natural equations of a curve have been determined, the next step is to integrate Frenet formula with $\kappa = 1$ and

$$\tau = \pm\frac{s}{\sqrt{s^2+\tanh^2[\phi]}} = \pm\frac{\frac{s}{\tanh[\phi]}}{\sqrt{\left(\frac{s}{\tanh[\phi]}\right)^2 + 1}} = \pm\tanh\Big[\operatorname{arcsinh}[\frac{s}{\tanh[\phi]}]\Big]. \tag{9}$$

Theorem 3.3 *A spacelike curve has a spacelike principal normal vector in Minkowski space* $\mathbf{E}_1^3$ *with* $\kappa = 1$ *and such that their normal vector make a constant angle with a fixed straight line is, up a rigid motion of the space or up to the antipodal map,* $p \longrightarrow -p$*, spacelike Salkowski curve with a spacelike principal normal vector.*

Proof: We know from Definition 3.1 that the arc-length parameter of a Salkowski curve (2) is $s = \int_0^t \|\gamma_m'(u)\|du = \frac{1}{m}\sinh[nt]$. Therefore, $t = \frac{1}{n}\operatorname{arcsinh}[ms]$. In terms of the arc-length curvature and torsion are then

$$\kappa(s) = 1, \quad \tau(s) = \tanh[\operatorname{arcsinh}[ms]],$$

the same intrinsic equations, with $m = \coth[\phi]$ and $n = \frac{m}{\sqrt{m^2-1}} = \cosh[\phi]$ (compare with the positive case in Equation (9)), as the ones shown in Lemma 3.2.

For the negative case in Equation (9), let us recall that if a curve α has torsion τ_α, then the curve $\beta(t) = -\alpha(t)$ has as torsion $\tau_\beta(t) = -\tau_\alpha(t)$, whereas curvature is preserved.

Therefore, the fundamental theorem of curves in Minkowski space states in our situation that, up a rigid motion or up to the antipodal map, the curves we are looking for are spacelike Minkowski curves with a spacelike principal normal vector.

4 Spacelike anti-Salkowski curves with a spacelike principal normal

As an additional material we will show in this section how to build, from a curve in Minkowski space $\mathbf{E}_1^3$ of constant curvature, another curve of constant torsion.

Let us recall that a curve $\alpha : I \to \mathbf{E}_1^3$, is 2-regular at a point t_0 if $\alpha'(t_0) \neq 0$ and if $\kappa_\alpha(t_0) \neq 0$.

Lemma 4.1 *Let $\alpha : I \to \mathbf{E}_1^3$ be a regular spacelike curve with a spacelike principal normal vector parameterized by arc-length with curvature κ_α, torsion τ_α and Frenet frame $\{\mathbf{T}_\alpha, \mathbf{N}_\alpha, \mathbf{B}_\alpha\}$. Let us $\beta(t) = \int_0^t \mathbf{T}_\alpha(u)\|\mathbf{B}_\alpha^{'}(u)\|\,du$. If $s_\alpha \in I$ satisfies $\tau_\alpha(s_\alpha) \neq 0$, the curve β is 2-regular at s_β and*

$$\kappa_\beta = \frac{\kappa_\alpha}{\tau_\alpha},\ \ \tau_\beta = 1,\ \ \mathbf{T}_\beta = \mathbf{T}_\alpha,\ \ \mathbf{N}_\beta = \mathbf{N}_\alpha,\ \ \mathbf{B}_\beta = \mathbf{B}_\alpha.$$

Proof: In order to obtain the tangent vector of β let us compute

$$\mathbf{T}_\beta(s_\beta) = \dot{\beta}(s_\beta) = \frac{d\beta}{dt}\frac{dt}{ds_\beta} = \mathbf{T}_\alpha\|\mathbf{B}_\alpha'(t)\|\frac{dt}{ds_\beta}.$$

From the above equation, we get

$$\frac{ds_\beta}{dt} = \|\mathbf{B}_\alpha'(t)\| = \Big\|\frac{\mathbf{B}_\alpha}{ds_\alpha}\frac{ds_\alpha}{dt}\Big\| = \tau_\alpha\frac{ds_\alpha}{dt}, \tag{10}$$

and

$$\mathbf{T}_\beta(s_\beta) = \mathbf{T}_\alpha(s_\alpha).$$

Differentiation the above equation using Frenet's Equations (1) we obtain

$$\dot{\mathbf{T}}_\beta(s_\beta) = \frac{d\mathbf{T}_\alpha}{ds_\alpha}\frac{ds_\alpha}{dt}\frac{dt}{ds_\beta}.$$

Using Frenet's Equations (1) and Equation (10), the above equation writes

$$\kappa_\beta \, \mathbf{N}_\beta(s_\beta) = \frac{\kappa_\alpha}{\tau_\alpha} \mathbf{N}_\alpha(s_\alpha)$$

From the above equation, we get

$$\kappa_\beta = \frac{\kappa_\alpha}{\tau_\alpha},$$

and

$$\mathbf{N}_\beta(s_\beta) = \mathbf{N}_\alpha(s_\alpha).$$

So we have

$$\mathbf{B}_\beta(s_\beta) = \mathbf{T}_\beta(s_\beta) \times \mathbf{N}_\beta(s_\beta) = \mathbf{T}_\alpha(s_\alpha) \times \mathbf{N}_\alpha(s_\alpha) = \mathbf{B}_\alpha(s_\alpha).$$

Differentiating the above equation with respect to s_β we get $\tau_\beta = 1$.

Let us apply the previous result to the curve γ_m defined in Equation (2) we have the explicit parametrization of a spacelike anti-Salkowski curve as follows:

$$\beta_m(t) = \tfrac{n}{4m}\Bigg(\begin{array}{l} \tfrac{1-n}{1+2n}\sinh[(1+2n)t] + \tfrac{1+n}{1-2n}\sinh[(1-2n)t] + 2n\sinh[t], \\ \tfrac{1-n}{1+2n}\cosh[(1+2n)t] - \tfrac{1+n}{1-2n}\cosh[(1-2n)t] + 2n\cosh[t], \\ \tfrac{1}{m}(2nt - \sinh[2nt])\Bigg), \end{array} \tag{11}$$

where $n = \frac{m}{\sqrt{m^2-1}}$. Let us call these curves by the name spacelike anti-Salkowski curves with a spacelike principal normal vector. The presence of the non-trigonometric term $2nt$ in the third component of β_m makes that the change of variable studied in Section 2 for Salkowski curves does not work for anti-Salkowski. Moreover, an example ($m = 3.5$ and $t \in [-3, 3]$) of such curves can be seen in the left figure 1.

Applying Lemma 4.1 we get the following

Proposition 4.2 *The curves β_m in Equation (11) are curves of constant torsion equal to 1 and non-constant curvature equal to* $\coth[nt]$.

Finally, we state here the following

Lemma 4.3 *Let $\alpha : I \to \mathbf{E}_1^3$ be a regular spacelike curve with a spacelike principal normal vector parameterized by arc-length with curvature κ_α, torsion τ_α and Frenet frame $\{\mathbf{T}_\alpha, \mathbf{N}_\alpha, \mathbf{B}_\alpha\}$. Let us consider the curve $\beta(t) = \int_0^t \mathbf{T}_\alpha(u)\|\mathbf{T}_\alpha^{'}(u)\|\, du$. Then at a parameter $s_\alpha \in I$ such that $\kappa_\alpha(s_\alpha) \neq 0$, the curve β is 2-regular at s_β and*

$$\kappa_\beta = 1, \;\; \tau_\beta = \frac{\tau_\alpha}{\kappa_\alpha}, \;\; \mathbf{T}_\beta = \mathbf{T}_\alpha, \;\; \mathbf{N}_\beta = \mathbf{N}_\alpha, \;\; \mathbf{B}_\beta = \mathbf{B}_\alpha.$$

Proof: The proof of this Lemma is similar as the proof of Lemma 4.1.

Theorem 4.4 *The spacelike curve with $\tau = 1$ and such that their principal normal vectors make a constant hyperbolic angle with a fixed straight line are the spacelike anti-Salkowski curves defined in Equation (11).*

Proof: Let α be a spacelike curve with $\tau = 1$ and let $\beta(t) = \int_0^t \mathbf{T}_\alpha(u)\|\mathbf{T}'_\alpha(u)\|\, du$. By Lemma 4.3, β is a curve with constant curvature $\kappa = 1$, non-constant torsion $\tau = \frac{1}{\kappa_\alpha}$ and with the same principal normal vector. Therefore, β is a Salkowski curve and α is an anti-Salkowski curve in Minkowski 3-space.

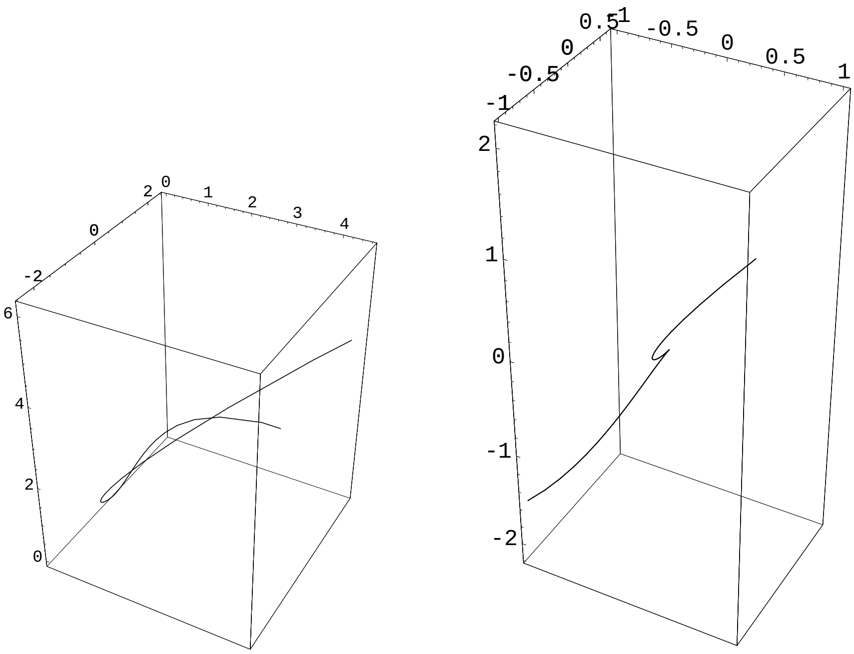

Figure 1: Spacelike Salkowski (right) and anti-Salkowski (left) curves.

5 Conclusion

In this paper, we defined the Salkowski and anti-Salkowski curves in Minkowski space $\mathbf{E}_1^3$. Next, we introduced an explicit parametrization of a spacelike Salkowski curves with a spacelike principal normal vector and a spacelike anti-Salkowski curves with a spacelike principal normal vector in Minkowski 3-space. Moreover, we characterized them as a space curve with constant curvature or constant torsion and whose principal normal vector makes a constant angle with a fixed straight line.

6 Open Problems

In recent years, by the coming of the theory of relativity, researchers treated some of classical differential geometry topics to extend analogous problems to Lorentzian manifolds. In a similar way, we study a classical topic in Minkowski space. In this work, we have investigated spacelike Salkowski curves in Minkowski space. We have given some explicit characterizations of these curves in terms of Frenet's equations. Additionally, problems such as; investigation of timelike Salkowski curves or extending such kind curves to Minkowski space-time or higher dimensional Euclidean spaces can be presented as further researches.

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