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Finite Dimensional Intuitionistic Fuzzy Normed Linear Space

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Abstract

Following the definition of intuitionistic fuzzy n-norm [3], we have introduced the definition of intuitionistic fuzzy norm (in short IFN) over a linear space and there after a few results on intuitionistic fuzzy normed linear space and finite dimensional intuitionistic fuzzy normed linear space. Lastly, we have introduced the definitions of intuitionistic fuzzy continuity and sequentially intuitionistic fuzzy continuity and proved that they are equivalent.

Keywords: *Fuzzy set, Membership function, Non - membership function, Intuitionistic fuzzy set, Fuzzy Norm, Intuitionistic fuzzy norm.*

1 Introduction

The authors T. Bag and S. K. Samanta [5] introduced the definition of fuzzy norm over a linear space following the definition S. C. Cheng and J. N. Moordeson [4] and they have studied finite dimensional fuzzy normed linear spaces. Also the definition of intuitionistic fuzzy n-normed linear space was introduced in the paper [3] and established a sufficient condition for an intuitionistic fuzzy n-normed linear space to be complete. In this paper, following the definition of intuitionistic fuzzy n-norm [3], the definition of intuitionistic fuzzy norm (in short IFN) is defined over a linear space. There after a sufficient condition is given for an intuitionistic fuzzy normed linear space to be complete and also it is proved that a finite dimensional intuitionistic fuzzy norm linear space is complete. In such spaces, it is established that a necessary and sufficient condition for a subset to be compact. Thereafter following the definition of fuzzy continuous mapping [6], the definition of intuitionistic fuzzy continuity, strongly intuitionistic fuzzy continuity and sequentially intuitionistic fuzzy continuity are defined and proved that the concept of intuitionistic fuzzy continuity and sequentially intuitionistic fuzzy continuity are equivalent. There after it is shown that intuitionistic fuzzy continuous image of a compact set is again a compact set.

Definition 1 *[3] A binary operation* $* : [0,1] \times [0,1] \longrightarrow [0,1]$ *is continuous* t *- norm if* $*$ *satisfies the following conditions* :
(i) $*$ *is commutative and associative ,*
(ii) $*$ *is continuous ,*
(iii) $a * 1 = a \quad \forall\, a \,\varepsilon\, [0,1]$,
(iv) $a * b \leq c * d$ *whenever* $a \leq c$, $b \leq d$ *and* $a, b, c, d \,\varepsilon\, [0,1]$.

Definition 2 *[3] A binary operation* $\diamond : [0,1] \times [0,1] \longrightarrow [0,1]$ *is continuous* t *- co - norm if* $\diamond$ *satisfies the following conditions* :
(i) $\diamond$ *is commutative and associative ,*
(ii) $\diamond$ *is continuous ,*
(iii) $a \diamond 0 = a \quad \forall\, a \,\varepsilon\, [0,1]$,
(iv) $a \diamond b \leq c \diamond d$ *whenever* $a \leq c$, $b \leq d$ *and* $a, b, c, d \,\varepsilon\, [0,1]$.

Remark 1 *[3]* (a) *For any* $r_1, r_2 \,\varepsilon\, (0,1)$ *with* $r_1 > r_2$ *, there exist* $r_3, r_4 \,\varepsilon\, (0,1)$ *such that* $r_1 * r_3 > r_2$ *and* $r_1 > r_4 \diamond r_2$.

(b) *For any* $r_5 \,\varepsilon\, (0,1)$ *, there exist* $r_6, r_7 \,\varepsilon\, (0,1)$ *such that* $r_6 * r_6 \geq r_5$ *and* $r_7 \diamond r_7 \leq r_5$.

Definition 3 *[3] Let* E *be any set. An* ***intuitionistic fuzzy set*** A *of* E *is an object of the form* $A = \{ (x, \mu_A(x), \nu_A(x)) : x \,\varepsilon\, E \}$ *, where the functions* $\mu_A : E \longrightarrow [0,1]$ *and* $\nu_A : E \longrightarrow [0,1]$ *denotes the degree of membership and the non - membership of the element* $x \,\varepsilon\, E$ *respectively and for every* $x \,\varepsilon\, E$, $0 \leq \mu_A(x) + \nu_A(x) \leq 1$.

Definition 4 *[3] If* A *and* B *are two intuitionistic fuzzy sets of a non - empty set* E *, then* $A \subseteq B$ *if and only if for all* $x \,\varepsilon\, E$,

$$\mu_A(x) \leq \mu_B(x) \text{ and } \nu_A(x) \geq \nu_B(x);$$

$A = B$ *if and only if for all* $x \varepsilon E$,

$$\mu_A(x) = \mu_B(x) \text{ and } \nu_A(x) = \nu_B(x);$$

$$\overline{A} = \{ (x, \nu_A(x), \mu_A(x)) : x \varepsilon E \};$$

$$A \cap B = \{ (x, \min(\mu_A(x), \mu_B(x)), \max(\nu_A(x), \nu_B(x))) : x \varepsilon E \};$$

$$A \cup B = \{ (x, \max(\mu_A(x), \mu_B(x)), \min(\nu_A(x), \nu_B(x))) : x \varepsilon E \}.$$

Definition 5 *Let* $*$ *be a continuous* t *- norm,* $\diamond$ *be a continuous* t *- co - norm and* V *be a linear space over the field* F $(= \mathbb{R}$ *or* $\mathbb{C})$. *An* ***intuitionistic fuzzy norm*** *or in short* ***IFN*** *on* V *is an object of the form* $A = \{ ((x, t), N(x, t), M(x, t)) : (x, t) \varepsilon V \times \mathbb{R}^+ \}$, *where* N, M *are fuzzy sets on* $V \times \mathbb{R}^+$, N *denotes the degree of membership and* M *denotes the degree of non - membership* $(x, t) \varepsilon V \times \mathbb{R}^+$ *satisfying the following conditions :*

(i) $N(x, t) + M(x, t) \leq 1 \quad \forall (x, t) \varepsilon V \times \mathbb{R}^+;$

(ii) $N(x, t) > 0;$

(iii) $N(x, t) = 1$ *if and only if* $x = \underline{0};$

(iv) $N(cx, t) = N(x, \frac{t}{|c|}) \quad c \neq 0, c \varepsilon F;$

(v) $N(x, s) * N(y, t) \leq N(x + y, s + t);$

(vi) $N(x, \cdot)$ *is non - decreasing function of* $\mathbb{R}^+$ *and* $\lim_{t \to \infty} N(x, t) = 1;$

(vii) $M(x, t) > 0;$

$(viii)$ $M(x, t) = 0$ *if and only if* $x = \underline{0};$

(ix) $M(cx, t) = M(x, \frac{t}{|c|}) \quad c \neq 0, c \varepsilon F;$

(x) $M(x, s) \diamond M(y, t) \geq M(x + y, s + t);$

(xi) $M(x, \cdot)$ *is non - increasing function of* $\mathbb{R}^+$ *and* $\lim_{t \to \infty} M(x, t) = 0.$

Example 1 *Let* $(V = \mathbb{R}, \|\cdot\|)$ *be a normed linear space where* $\|x\| = |x| \;\; \forall\; x \,\varepsilon\, \mathbb{R}$. *Define* $a * b = \min\{a, b\}$ *and* $a \diamond b = \max\{a, b\}$ *for all* $a, b \,\varepsilon\, [0, 1]$. *Also define* $N(x, t) = \frac{t}{t + k|x|}$ *and* $M(x, t) = \frac{k|x|}{t + k|x|}$ *where* $k > 0$. *We now consider* $A = \{((x, t), N(x, t), M(x, t)) : (x, t) \,\varepsilon\, V \times \mathbb{R}^+\}$. *Here* A *is an IFN on* V.

Proof: Obviously follows from the calculation of the example 3.2 [3].

Definition 6 *If* A *is an IFN on* V *(a linear space over the field* F $(= \mathbb{R}$ *or* $\mathbb{C})$ *) then* (V, A) *is called an intuitionistic fuzzy normed linear space or in short IFNLS.*

Definition 7 *[3] A sequence* $\{x_n\}_n$ *in an IFNLS* (V, A) *is said to converge to* $x \,\varepsilon\, V$ *if given* $r > 0$, $t > 0$, $0 < r < 1$ *there exists an integer* $n_0 \,\varepsilon\, \mathbb{N}$ *such that* $N(x_n - x, t) > 1 - r$ *and* $M(x_n - x, t) < r$ *for all* $n \geq n_0$.

Theorem 1 *In an IFNLS* (V, A), *a sequence* $\{x_n\}_n$ *converges to* $x \,\varepsilon\, V$ *if and only if* $\lim_{n \to \infty} N(x_n - x, t) = 1$ *and* $\lim_{n \to \infty} M(x_n - x, t) = 0$.

Proof: The proof directly follows from the proof of the theorem 3.4 [3] .

Theorem 2 *If a sequence* $\{x_n\}_n$ *in an IFNLS* (V, A) *is convergent , it's limit is unique .*

Proof: Let $\lim_{n \to \infty} x_n = x$ and $\lim_{n \to \infty} x_n = y$. Also let $s, t \,\varepsilon\, \mathbb{R}^+$. Now ,

$$\lim_{n \to \infty} x_n = x \;\Rightarrow\; \begin{cases} \lim_{n \to \infty} N(x_n - x, t) = 1 \\ \lim_{n \to \infty} M(x_n - x, t) = 0 \end{cases}$$

$$\lim_{n \to \infty} x_n = y \;\Rightarrow\; \begin{cases} \lim_{n \to \infty} N(x_n - y, t) = 1 \\ \lim_{n \to \infty} M(x_n - y, t) = 0 \end{cases}$$

$$\begin{aligned} N(x - y, s + t) &= N(x - x_n + x_n - y, s + t) \\ &\geq N(x - x_n, s) * N(x_n - y, t) \\ &= N(x_n - x, s) * N(x_n - y, t) \end{aligned}$$

Taking limit , we have

$$N(x - y, s + t) \geq \lim_{n \to \infty} N(x_n - x, s) * \lim_{n \to \infty} N(x_n - y, t) = 1$$

$$\Longrightarrow N(x-y, s+t) = 1 \Longrightarrow x-y = \underline{0} \Longrightarrow x = y$$

This completes the proof .

Theorem 3 *If* $\lim_{n \to \infty} x_n = x$ *and* $\lim_{n \to \infty} y_n = y$ *then* $\lim_{n \to \infty} x_n + y_n = x + y$ *in an IFNLS* (V, A) .

Proof: Let $s, t \,\varepsilon\, \mathbb{R}^+$. Now ,

$$\lim_{n \to \infty} x_n = x \Rightarrow \begin{cases} \lim_{n \to \infty} N(x_n - x, t) = 1 \\ \lim_{n \to \infty} M(x_n - x, t) = 0 \end{cases}$$

$$\lim_{n \to \infty} y_n = y \Rightarrow \begin{cases} \lim_{n \to \infty} N(y_n - y, t) = 1 \\ \lim_{n \to \infty} M(y_n - y, t) = 0 \end{cases}$$

Now,

$$\begin{aligned} N((x_n + y_n) - (x+y), s+t) &= N((x_n - x) + (y_n - y), s+t) \\ &\geq N(x_n - x, s) * N(y_n - y, t) \end{aligned}$$

Taking limit, we have

$$\begin{aligned} &\lim_{n\to\infty} N((x_n + y_n) - (x+y), s+t) \\ &\qquad \geq \lim_{n\to\infty} N(x_n - x, s) * \lim_{n\to\infty} N(y_n - y, t) \\ &\qquad = 1 * 1 = 1 \end{aligned}$$

$$\Rightarrow \lim_{n\to\infty} N((x_n + y_n) - (x+y), s+t) = 1$$

Again,

$$\begin{aligned} M((x_n + y_n) - (x+y), s+t) &= M((x_n - x) + (y_n - y), s+t) \\ &\leq M(x_n - x, s) \diamond M(y_n - y, t) \end{aligned}$$

Taking limit, we have

$$\begin{aligned} &\lim_{n\to\infty} M((x_n + y_n) - (x+y), s+t) \\ &\qquad \leq \lim_{n\to\infty} M(x_n - x, s) \diamond \lim_{n\to\infty} M(y_n - y, t) \\ &\qquad = 0 \diamond 0 = 0 \end{aligned}$$

$$\Rightarrow \lim_{n\to\infty} M((x_n + y_n) - (x+y), s+t) = 0$$

Thus, we see that $\lim_{n \to \infty} x_n + y_n = x + y$.

Theorem 4 *If* $\lim_{n \to \infty} x_n = x$ *and* $c\,(\neq 0)\,\varepsilon\, F$ *then* $\lim_{n \to \infty} c\,x_n = c\,x$ *in an IFNLS* (V, A) .

Proof: Obvious.

Theorem 5 *In an IFNLS (V, A), every subsequence of a convergent sequence converges to the limit of the sequence .*

Proof: Obvious.

Definition 8 *A sequence $\{x_n\}_n$ in an IFNLS (V, A) is said to be a* ***Cauchy sequence*** *if $\lim_{n \to \infty} N(x_{n+p} - x_n, t) = 1$ and $\lim_{n \to \infty} M(x_{n+p} - x_n, t) = 0$, $p = 1, 2, 3, \cdots, t > 0$.*

Theorem 6 *In an IFNLS (V, A), every convergent sequence is a Cauchy sequence.*

Proof: Let $\{x_n\}_n$ be a convergent sequence in the IFNLS (V, A) with $\lim_{n \to \infty} x_n = x$. Let $s, t \,\varepsilon\, \mathbb{R}^+$ and $p = 1, 2, 3, \cdots$, we have

$$\begin{aligned} N(x_{n+p} - x_n, s + t) &= N(x_{n+p} - x + x - x_n, s + t) \\ &\geq N(x_{n+p} - x, s) * N(x - x_n, t) \\ &= N(x_{n+p} - x, s) * N(x_n - x, t) \end{aligned}$$

Taking limit , we have

$$\begin{aligned} &\lim_{n \to \infty} N(x_{n+p} - x_n, s + t) \\ &\qquad \geq \lim_{n \to \infty} N(x_{n+p} - x, s) * \lim_{n \to \infty} N(x_n - x, t) \\ &\qquad = 1 * 1 = 1 \end{aligned}$$

$\Rightarrow \lim_{n \to \infty} N(x_{n+p} - x_n, s + t) = 1 \quad \forall\, s, t \,\varepsilon\, \mathbb{R}^+$ and $p = 1, 2, 3, \cdots$

Again ,

$$\begin{aligned} M(x_{n+p} - x_n, s + t) &= M(x_{n+p} - x + x - x_n, s + t) \\ &\leq M(x_{n+p} - x, s) \diamond M(x - x_n, t) \\ &= M(x_{n+p} - x, s) \diamond M(x_n - x, t) \end{aligned}$$

Taking limit , we have

$$\begin{aligned} &\lim_{n \to \infty} M(x_{n+p} - x_n, s + t) \\ &\qquad \leq \lim_{n \to \infty} M(x_{n+p} - x, s) \diamond \lim_{n \to \infty} M(x_n - x, t) \\ &\qquad = 0 \diamond 0 = 0 \end{aligned}$$

$\Rightarrow \lim_{n \to \infty} M(x_{n+p} - x_n, s + t) = 0 \quad \forall\, s, t \,\varepsilon\, \mathbb{R}^+$ and $p = 1, 2, 3, \cdots$

Thus, $\{x_n\}_n$ is a Cauchy sequence in the IFNLS (V, A).

Note 1 *The converse of the above theorem is not necessarily true . It is verified by the following example .*

Example 2 *Let $(V, \|\cdot\|)$ be a normed linear space and define $a * b = \min\{a, b\}$ and $a \diamond b = \max\{a, b\}$ for all $a, b \,\varepsilon\, [0, 1]$. For all*

$t > 0$, define $N(x, t) = \frac{t}{t + k\,\|x\|}$ and $M(x, t) = \frac{k\,\|x\|}{t + k\,\|x\|}$ where $k > 0$. It is easy to see that $A = \{((x, t), N(x, t), M(x, t)) : (x, t)\ \varepsilon\ V \times \mathbb{R}^+\}$ is an IFN on V. We now show that
(a) $\{x_n\}_n$ is a Cauchy sequence in $(V, \|\cdot\|)$ if and only if $\{x_n\}_n$ is a Cauchy sequence in the IFNLS (V, A).
(b) $\{x_n\}_n$ is a convergent sequence in $(V, \|\cdot\|)$ if and only if $\{x_n\}_n$ is a convergent sequence in the IFNLS (V, A).

Proof: (a) Let $\{x_n\}_n$ be a Cauchy sequence in $(V, \|\cdot\|)$ and $t > 0$.

$$\Longleftrightarrow \lim_{n \to \infty} \|x_{n+p} - x_n\| = 0 \ for\ p = 1, 2, \cdots$$

$$\Longleftrightarrow \lim_{n \to \infty} \frac{t}{t + k\,\|x_{n+p} - x_n\|} = 1 \ and \ \lim_{n \to \infty} \frac{k\,\|x_{n+p} - x_n\|}{t + k\,\|x_{n+p} - x_n\|} = 0$$

$$\Longleftrightarrow \lim_{n \to \infty} N(x_{n+p} - x_n, t) = 1 \ and \ \lim_{n \to \infty} M(x_{n+p} - x_n, t) = 0$$

$\Longleftrightarrow$ $\{x_n\}_n$ is a Cauchy sequence in (V, A)

(b) Let $\{x_n\}_n$ be a convergent sequence in $(V, \|\cdot\|)$ and $t > 0$.

$$\Longleftrightarrow \lim_{n \to \infty} \|x_n - x\| = 0$$

$$\Longleftrightarrow \lim_{n \to \infty} \frac{t}{t + k\,\|x_n - x\|} = 1 \ and \ \lim_{n \to \infty} \frac{k\,\|x_n - x\|}{t + k\,\|x_n - x\|} = 0$$

$$\Longleftrightarrow \lim_{n \to \infty} N(x_n - x, t) = 1 \ and \ \lim_{n \to \infty} M(x_n - x, t) = 0$$

$\Longleftrightarrow$ $\{x_n\}_n$ is a convegent sequence in (V, A).

Theorem 7 *Let (V, A) be an IFNLS, such that every Cauchy sequence in (V, A) has a convergent subsequence. Then (V, A) is complete.*

Proof: Let $\{x_n\}_n$ be a Cauchy sequence in (V, A) and $\{x_{n_k}\}_k$ be a subsequence of $\{x_n\}_n$ that converges to $x\ \varepsilon\ V$ and $t > 0$. Since $\{x_n\}_n$ is a Cauchy sequence in (V, A) , we have

$$\lim_{n, k \to \infty} N\left(x_n - x_k, \frac{t}{2}\right) = 1 \ and \ \lim_{n, k \to \infty} M\left(x_n - x_k, \frac{t}{2}\right) = 0$$

Again since $\{x_{n_k}\}_k$ converges to x , we have

$$\lim_{k \to \infty} N\left(x_{n_k} - x, \frac{t}{2}\right) = 1 \ and \ \lim_{k \to \infty} M\left(x_{n_k} - x, \frac{t}{2}\right) = 0$$

Now,

$$\begin{aligned} N(x_n - x, t) &= N(x_n - x_{n_k} + x_{n_k} - x, t) \\ &\geq N\left(x_n - x_{n_k}, \tfrac{t}{2}\right) * N\left(x_{n_k} - x, \tfrac{t}{2}\right) \end{aligned}$$

$\Longrightarrow \lim_{n \to \infty} N(x_n - x, t) = 1$

Again, we see that

$$\begin{aligned} M(x_n - x, t) &= M(x_n - x_{n_k} + x_{n_k} - x, t) \\ &\leq M\left(x_n - x_{n_k}, \tfrac{t}{2}\right) \diamond M\left(x_{n_k} - x, \tfrac{t}{2}\right) \end{aligned}$$

$\Longrightarrow \lim_{n \to \infty} M(x_n - x, t) = 0$

Thus, $\{x_n\}_n$ converges to x in (V, A) and hence (V, A) is complete.

Theorem 8 *Let (V, A) be an IFNLS , we further assume that ,*

(xii) $\left.\begin{array}{l} a \diamond a = a \\ a * a = a \end{array}\right\} \ \forall\ a\ \varepsilon\ [0, 1]$

$(xiii)$ $N(x, t) > 0\ \forall\ t > 0 \Longrightarrow x = \underline{0}$

(xiv) $M(x, t) > 0\ \forall\ t > 0 \Longrightarrow x = \underline{0}$

Define $\|x\|_\alpha^1 = \wedge\{t : N(x, t) \geq \alpha\}$ and $\|x\|_\alpha^2 = \vee\{t : M(x, t) \leq \alpha\}$, $\alpha\ \varepsilon\ (0, 1)$. Then both $\{\|x\|_\alpha^1 : \alpha\ \varepsilon\ (0, 1)\}$ and $\{\|x\|_\alpha^2 : \alpha\ \varepsilon\ (0, 1)\}$ are ascending family of norms on V . We call these norms as α - norm on V corresponding to the IFN A on V .

Proof: Let $\alpha\ \varepsilon\ (0, 1)$. To prove $\|x\|_\alpha^1$ is a norm on V , we will prove the followings :

(1) $\|x\|_\alpha^1 \geq 0 \quad \forall\ x\ \varepsilon\ V$;

(2) $\|x\|_\alpha^1 = 0 \Longleftrightarrow x = \underline{0}$;

(3) $\|c\,x\|_\alpha^1 = |c|\ \|x\|_\alpha^1$;

(4) $\|x + y\|_\alpha^1 \leq \|x\|_\alpha^1 + \|y\|_\alpha^1$.

The proof of (1) , (2) and (3) directly follows from the proof of the theorem 2.1 [5] . So, we now prove (4) .
$\|x\|_\alpha^1 + \|y\|_\alpha^1 = \wedge\{s : N(x, s) \geq \alpha\} + \wedge\{t : N(y, t) \geq \alpha\} = \wedge\{s + t : N(x, s) \geq \alpha, N(y, t) \geq \alpha\} = \wedge\{s + t : N(x, s) * N(y, t) \geq \alpha * \alpha\} \geq \wedge\{s + t : N(x + y, s + t) \geq \alpha\} = \|x + y\|_\alpha^1$, which proves (4) . Let $0 < \alpha_1 < \alpha_2 < 1$. $\|x\|_{\alpha_1}^1 = \wedge\{t : N(x, t) \geq \alpha_1\}$ and $\|x\|_{\alpha_2}^1 = \wedge\{t : N(x, t) \geq \alpha_2\}$. Since $\alpha_1 < \alpha_2$, $\{t : N(x, t) \geq \alpha_2\} \subset \{t : N(x, t) \geq \alpha_1\} \Longrightarrow \wedge\{t : N(x, t) \geq \alpha_2\} \geq \wedge\{t : N(x, t) \geq \alpha_1\} \Longrightarrow \|x\|_{\alpha_2}^1 \geq \|x\|_{\alpha_1}^1$. Thus, we see that $\{\|x\|_\alpha^1 : \alpha\ \varepsilon\ (0, 1)\}$ is an ascending

family of norms on V .
Now we shall prove that $\{ \, \| x \|_{\alpha}^{2} \, : \, \alpha \, \varepsilon \, (0 \, , \, 1) \, \}$ is also an ascending family of norms on V. Let $\alpha \, \varepsilon \, (0 \, , \, 1)$ and $x \, , \, y \, \varepsilon \, V$. It is obvious that $\| x \|_{\alpha}^{2} \geq 0$. Let $\| x \|_{\alpha}^{2} = 0$. Now, $\| x \|_{\alpha}^{2} = 0 \implies \vee \{ t \, : \, M(x \, , \, t) \leq \alpha \} = 0 \implies M(x \, , \, t) > \alpha > 0 \;\; \forall \; t > 0 \implies x = \underline{0}$. Conversely, we assume that $x = \underline{0} \implies M(x \, , \, t) = 0 \;\; \forall \; t > 0 \implies \vee \{ t \, : \, M(x \, , \, t) \leq \alpha \} = 0 \implies \| x \|_{\alpha}^{2} = 0$.
It is easy to see that $\| c \, x \|_{\alpha}^{2} = |c| \, \| x \|_{\alpha}^{2} \;\; \forall \; c \, \varepsilon \, F$.
$\| x \|_{\alpha}^{2} + \| y \|_{\alpha}^{2} = \vee \{ s \, : \, M(x \, , \, s) \leq \alpha \} + \vee \{ t \, : \, M(y \, , \, t) \leq \alpha \} = \vee \{ s + t \, : \, M(x \, , \, s) \leq \alpha \, , \, M(y \, , \, t) \leq \alpha \} = \vee \{ s + t \, : \, M(x \, , \, s) \diamond M(y \, , \, t) \leq \alpha \diamond \alpha \} \geq \vee \{ s + t \, : \, M(x + y \, , \, s + t) \leq \alpha \} = \| x + y \|_{\alpha}^{2}$, that is $\| x + y \|_{\alpha}^{2} \leq \| x \|_{\alpha}^{2} + \| y \|_{\alpha}^{2} \;\; \forall \; x \, , \, y \, \varepsilon \, V$.
Let $0 < \alpha_1 < \alpha_2 < 1$. Therefore, $\| x \|_{\alpha_1}^{2} = \vee \{ t \, : \, M(x \, , \, t) \leq \alpha_1 \}$ and $\| x \|_{\alpha_2}^{2} = \vee \{ t \, : \, M(x \, , \, t) \leq \alpha_2 \}$. Since $\alpha_1 < \alpha_2$, we have
$\{ t \, : \, M(x \, , \, t) \leq \alpha_1 \} \subset \{ t \, : \, M(x \, , \, t) \leq \alpha_2 \}$
$\implies \vee \{ t \, : \, M(x \, , \, t) \leq \alpha_1 \} \leq \vee \{ t \, : \, M(x \, , \, t) \leq \alpha_2 \}$
$\implies \| x \|_{\alpha_1}^{2} \leq \| x \|_{\alpha_2}^{2}$. Thus we see that $\{ \, \| x \|_{\alpha}^{2} \, : \, \alpha \, \varepsilon \, (0 \, , \, 1) \, \}$ is an ascending family of norms on V .

Lemma 1 *[5] Let $(V \, , \, A)$ be an IFNLS satisfying the condition $(Xiii)$ and $\{ x_1 \, , \, x_2 \, , \, \cdots \, , \, x_n \}$ be a finite set of linearly independent vectors of V . Then for each $\alpha \, \varepsilon \, (0 \, , \, 1)$ there exists a constant $C_\alpha > 0$ such that for any scalars $\alpha_1 \, , \, \alpha_2 \, , \, \cdots \, , \, \alpha_n$,*

$$\| \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n \|_{\alpha}^{1} \geq C_\alpha \sum_{i=1}^{n} | \alpha_i |$$

where $\| \cdot \|_{\alpha}^{1}$ is defined in the previous theorem.

Theorem 9 *Every finite dimensional IFNLS satisfying the conditions (Xii) and $(Xiii)$ is complete .*

Proof: Let $(V \, , \, A)$ be a finite dimensional IFNLS satisfying the conditions (Xii) and $(Xiii)$. Also, let $\dim V = k$ and $\{ e_1 \, , \, e_2 \, , \, \cdots \, , \, e_k \}$ be a basis of V. Consider $\{ x_n \}_n$ as an arbitrary Cauchy sequence in $(V \, , \, A)$. Let $x_n = \beta_1^{(n)} \, e_1 + \beta_2^{(n)} \, e_2 + \cdots + \beta_k^{(n)} \, e_k$ where $\beta_1^{(n)} \, , \, \beta_2^{(n)} \, , \, \cdots \, , \, \beta_k^{(n)}$ are suitable scalars. Then by the same calculation of the theorem 2.4 [5], there exist $\beta_1 \, , \, \beta_2 \, , \, \cdots \, , \, \beta_k \, \varepsilon \, F$ such that the sequence $\{ \beta_i^{(n)} \}_n$ converges to β_i for $i = 1 \, , \, 2 \, , \, \cdots \, , \, k$. Clearly $x = \sum_{i=1}^{k} \beta_i \, e_i \, \varepsilon \, V$. Now, for all $t > 0$,

$$N(x_n - x, t) = N(\sum_{i=1}^{k} \beta_i^{(n)} e_i - \sum_{i=1}^{k} \beta_i e_i, t)$$
$$= N(\sum_{i=1}^{k} (\beta_i^{(n)} - \beta_i) e_i, t)$$
$$\geq N((\beta_1^{(n)} - \beta_1) e_1, \tfrac{t}{k}) * \cdots * N((\beta_k^{(n)} - \beta_k) e_k, \tfrac{t}{k})$$
$$= N(e_1, \tfrac{t}{k\,|\beta_1^{(n)} - \beta_1|}) * \cdots * N(e_k, \tfrac{t}{k\,|\beta_k^{(n)} - \beta_k|})$$

Since $\lim_{n \to \infty} \frac{t}{k\left|\beta_i^{(n)} - \beta_i\right|} = \infty$, we see that $\lim_{n \to \infty} N(e_i, \frac{t}{k\,|\beta_i^{(n)} - \beta_i|}) = 1$

$\Longrightarrow \lim_{n \to \infty} N(x_n - x, t) \geq 1 * \cdots * 1 = 1 \quad \forall\, t > 0$

$\Longrightarrow \lim_{n \to \infty} N(x_n - x, t) = 1 \quad \forall\, t > 0$

Again, for all $t > 0$,

$$M(x_n - x, t) = M(\sum_{i=1}^{k} \beta_i^{(n)} e_i - \sum_{i=1}^{k} \beta_i e_i, t)$$
$$= M(\sum_{i=1}^{k} (\beta_i^{(n)} - \beta_i) e_i, t)$$
$$\leq M((\beta_1^{(n)} - \beta_1) e_1, \tfrac{t}{k}) \diamond \cdots \diamond M((\beta_k^{(n)} - \beta_k) e_k, \tfrac{t}{k})$$
$$= M(e_1, \tfrac{t}{k\,|\beta_1^{(n)} - \beta_1|}) \diamond \cdots \diamond M(e_k, \tfrac{t}{k\,|\beta_k^{(n)} - \beta_k|})$$

Since $\lim_{n \to \infty} \frac{t}{k\left|\beta_i^{(n)} - \beta_i\right|} = \infty$, we see that $\lim_{n \to \infty} M(e_i, \frac{t}{k\,|\beta_i^{(n)} - \beta_i|}) = 0$

$\Longrightarrow \lim_{n \to \infty} M(x_n - x, t) \leq 1 \diamond \cdots \diamond 1 = 1 \quad \forall\, t > 0$

$\Longrightarrow \lim_{n \to \infty} M(x_n - x, t) = 0 \quad \forall\, t > 0$.

Thus, we see that $\{x_n\}_n$ is an arbitrary Cauchy sequence that converges to $x \,\varepsilon\, V$, hence the IFNLS (V, A) is complete .

Definition 9 *Let (V, A) be an IFNLS. A subset P of V is said to be **closed** if for any sequence $\{x_n\}_n$ in P converges to $x \,\varepsilon\, P$, that is,*

$$\lim_{n \to \infty} N(x_n - x, t) = 1, \text{and} \lim_{n \to \infty} M(x_n - x, t) = 0 \Longrightarrow x \,\varepsilon\, P.$$

Definition 10 *Let (V, A) be an IFNLS. A subset Q of V is said to be the **closure** of $P\ (\subset V)$ if for any $x \,\varepsilon\, Q$, there exists a sequence $\{x_n\}_n$ in P such that*

$$\lim_{n \to \infty} N(x_n - x, t) = 1, \text{and} \lim_{n \to \infty} M(x_n - x, t) = 0 \quad \forall\, t \,\varepsilon\, \mathbb{R}^+.$$

We denote the set Q by $\overline{P}$.

Definition 11 *A subset* P *of an IFNLS is said to be* ***bounded*** *if and only if there exist* $t > 0$ *and* $0 < r < 1$ *such that*

$$N(x, t) > 1 - r \text{ and } M(x, t) < r \quad \forall\, x \,\varepsilon\, P.$$

Definition 12 *Let* (V, A) *be an IFNLS. A subset* P *of of* V *is said to be* ***compact*** *if any sequence* $\{x_n\}_n$ *in* P *has a subsequence converging to an element of* P *.*

Theorem 10 *Let* (V, A) *be an IFNLS satisfying the condition* (Xii) *. Every Cauchy sequence in* (V, A) *is bounded .*

Proof: Let $\{x_n\}_n$ be a Cauchy sequence in the IFNLS (V, A) . Then we have

$$\left.\begin{array}{l}\lim_{n \to \infty} N(x_{n+p} - x_n, t) = 1 \\ \lim_{n \to \infty} M(x_{n+p} - x_n, t) = 0\end{array}\right\} \forall\, t > 0,\ p = 1, 2, \cdots.$$

Choose a fixed r_0 with $0 < r_0 < 1$. Now we see that

$$\lim_{n \to \infty} N(x_n - x_{n+p}, t) = 1 > r_0 \;\forall\, t > 0,\ p = 1, 2, \cdots$$

$\Longrightarrow$ *For* $t' > 0 \;\exists\; n_0 = n_0(t')$ such that $N(x_n - x_{n+p}, t') > r_0 \;\forall\, n \geq n_0, p = 1, 2, \cdots$
Since, $\lim_{t \to \infty} N(x, t) = 1$, we have for each x_i , $\exists\, t_i > 0$ such that

$$N(x_i, t) > r_0 \quad \forall\, t > t_i, i = 1, 2, \cdots$$

Let $t_0 = t' + \max\{t_1, t_2, \cdots, t_{n_0}\}$. Then ,

$$\begin{aligned} N(x_n, t_0) &\geq N(x_n, t' + t_{n_0}) \\ &= N(x_n - x_{n_0} + x_{n_0}, t' + t_{n_0}) \\ &\geq N(x_n - x_{n_0}, t') * N(x_{n_0}, t_{n_0}) \\ &> r_0 * r_0 = r_0 \quad \forall\, n > n_0 \end{aligned}$$

Thus , we have

$$N(x_n, t_0) > r_0 \quad \forall\, n > n_0$$

Also , $N(x_n, t_0) \geq N(x_n, t_n) > r_0$ *forall* $n = 1, 2, \cdots, n_0$
So, we have

$$N(x_n, t_0) > r_0 \quad \forall\, n = 1, 2, \cdots \quad \cdots \quad (1)$$

Now, $\lim_{n \to \infty} M(x_n - x_{n+p}, t) = 0 < (1 - r_0) \;\forall\, t > 0, p = 1, 2, \cdots$
$\Longrightarrow$ *For* $t' > 0 \;\exists\; n'_0 = n'_0(t')$ such that $M(x_n - x_{n+p}, t') < (1 - r_0) \;\forall\, n \geq n'_0, p = 1, 2, \cdots$

Since, $\lim\limits_{t \to \infty} M(x, t) = 0$, we have for each x_i, $\exists\, t'_i > 0$ such that

$$M(x_i, t) < (1 - r_0) \quad \forall\ t > t'_i,\ i = 1, 2, \cdots$$

Let $t'_0 = t' + \max\{t'_1, t'_2, \cdots, t'_{n_0}\}$. Then,

$$\begin{aligned} M(x_n, t'_0) &\leq M(x_n, t' + t'_{n_0}) \\ &= M(x_n - x_{n'_0} + x_{n'_0}, t' + t'_{n_0}) \\ &\leq M(x_n - x_{n'_0}, t') \diamond M(x_{n'_0}, t'_{n_0}) \\ &< (1 - r_0) \diamond (1 - r_0) = (1 - r_0) \quad \forall\ n > n'_0 \end{aligned}$$

Thus, we have

$$M(x_n, t'_0) < (1 - r_0) \quad \forall\ n > n'_0$$

Also, $M(x_n, t'_0) \leq M(x_n, t'_n) < (1 - r_0)$ *forall* $n = 1, 2, \cdots, n'_0$

So, we have

$$M(x_n, t'_0) < (1 - r_0) \quad \forall\ n = 1, 2, \cdots \qquad \cdots \quad (2)$$

Let $t''_0 = \max\{t_0, t'_0\}$. Hence from (1) and (2) we see that

$$\left.\begin{array}{l} N\left(x_n, t''_0\right) > r_0 \\ M\left(x_n, t''_0\right) < (1 - r_0) \end{array}\right\} \ \forall\ n = 1, 2, \cdots$$

This implies that $\{x_n\}_n$ is bounded in (V, A).

Theorem 11 *In a finite dimensional IFNLS* (V, A) *satisfying the conditions* (Xii), $(Xiii)$ *and* (Xiv), *a subset* P *of* V *is compact if and only if* P *is closed and bounded in* (V, A).

Proof: $\Longrightarrow$ *part* : Proof of this part directly follows from the proof of the theorem 2.5 [5].

$\Longleftarrow$ *part* : In this part, we suppose that P is closed and bounded in the finite dimensional IFNLS (V, A). To show P is compact, consider $\{x_n\}_n$, an arbitrary sequence in P. Since V is finite dimensional, let $\dim V = n$ and $\{e_1, e_2, \cdots, e_n\}$ be a basis of V. So, for each x_k, $\exists\ \beta_1^k, \beta_2^k, \cdots, \beta_n^k\ \varepsilon\ F$ such that

$$x_k = \beta_1^k e_1 + \beta_2^k e_2 + \cdots + \beta_n^k e_n,\ k = 1, 2, \cdots$$

Since P is bounded, $\{x_k\}_k$ is also bounded. So, $\exists\ t_0 > 0$ and r_0 where $0 < r_0 < 1$ such that

$$\left.\begin{array}{l} N(x_k, t_0) > 1 - r_0 = \alpha_0 \\ M(x_k, t_0) < r_0 \end{array}\right\} \ \forall\ k \qquad \cdots \qquad (1)$$

Let $\|x\|_\alpha = \wedge\{t : N(x, t) \geq \alpha\}$, $\alpha\ \varepsilon\ (0, 1)$. So,we have

$$\|x\|_{\alpha_0} \leq t_0 \qquad \cdots \qquad (2)\ (By (1))$$

Since $\{ e_1 , e_2 , \cdots , e_n \}$ is linearly independent, by Lemma (1), $\exists$ a constant $c > 0$ such that $\forall\ k = 1, 2, \cdots$,

$$\| x_k \|_{\alpha_0} = \| \sum_{i=1}^{n} \beta_i^k e_i \|_{\alpha_0} > c \sum_{i=1}^{n} | \beta_i^k | \quad \cdots \quad (3)$$

From (2) and (3) we have

$$\sum_{i=1}^{n} | \beta_i^k | \leq \frac{t_0}{c} \qquad for \qquad k = 1, 2, \cdots$$

$\Longrightarrow$ *Foreach* i,

$$| \beta_i^k | \leq \sum_{i=1}^{n} | \beta_i^k | \leq \frac{t_0}{c} \qquad for \qquad k = 1, 2, \cdots$$

$\Longrightarrow \{ \beta_i^k \}_k$ is a bounded sequence for each $i = 1, 2, \cdots, n$
$\Longrightarrow \{ \beta_i^k \}_k$ has a convergent subsequence say $\{ \beta_i^{k_l} \}_l$.
$\Longrightarrow \{ \beta_1^{k_l} \}_l , \{ \beta_2^{k_l} \}_l , \cdots , \{ \beta_n^{k_l} \}_l$ are all convergent .
Let $x_{k_l} = \beta_1^{k_l} e_1 + \beta_2^{k_l} e_2 + \cdots + \beta_n^{k_l} e_n$ and $\beta_1 = \lim_{n \to \infty} \beta_1^{k_l}$, $\beta_2 = \lim_{n \to \infty} \beta_2^{k_l}$, $\cdots$, $\beta_n = \lim_{n \to \infty} \beta_n^{k_l}$ and $x = \beta_1 e_1 + \beta_2 e_2 + \cdots + \beta_n e_n$.
Now $\forall\ t > 0$, we have

$$\begin{aligned} N (x_{k_l} - x , t) &= N (\sum_{i=1}^{n} \beta_i^{k_l} e_i - \sum_{i=1}^{n} \beta_i e_i , t) \\ &= N (\sum_{i=1}^{n} (\beta_i^{k_l} - \beta_i) e_i , t) \\ &\geq N ((\beta_1^{k_l} - \beta_1) e_1 , \tfrac{t}{n}) * \cdots * N ((\beta_n^{k_l} - \beta_n) e_n , \tfrac{t}{n}) \\ &= N (e_1 , \tfrac{t}{n \, | \beta_1^{k_l} - \beta_1 |}) * \cdots * N (e_n , \tfrac{t}{n \, | \beta_n^{k_l} - \beta_n |}) \end{aligned}$$

Since $\lim_{l \to \infty} \frac{t}{n \left| \beta_i^{k_l} - \beta_i \right|} = \infty$, we see that $\lim_{l \to \infty} N (e_i , \frac{t}{n \, | \beta_i^{k_l} - \beta_i |}) = 1$
$\Longrightarrow \lim_{l \to \infty} N (x_{k_l} - x , t) \geq 1 * \cdots * 1 = 1 \quad \forall\ t > 0$
$\Longrightarrow \lim_{l \to \infty} N (x_{k_l} - x , t) = 1 \quad \forall\ t > 0 \quad \cdots \quad (4)$
Again, for all $t > 0$,

$$\begin{aligned} M (x_{k_l} - x , t) &= M (\sum_{i=1}^{n} \beta_i^{k_l} e_i - \sum_{i=1}^{n} \beta_i e_i , t) \\ &= M (\sum_{i=1}^{n} (\beta_i^{k_l} - \beta_i) e_i , t) \\ &\leq M ((\beta_1^{k_l} - \beta_1) e_1 , \tfrac{t}{n}) \diamond \cdots \diamond M ((\beta_n^{k_l} - \beta_n) e_n , \tfrac{t}{n}) \\ &= M (e_1 , \tfrac{t}{n \, | \beta_1^{k_l} - \beta_1 |}) \diamond \cdots \diamond M (e_n , \tfrac{t}{n \, | \beta_n^{k_l} - \beta_n |}) \end{aligned}$$

Since $\lim_{l \to \infty} \frac{t}{n \left| \beta_i^{k_l} - \beta_i \right|} = \infty$, we see that $\lim_{l \to \infty} M (e_i , \frac{t}{n \, | \beta_i^{k_l} - \beta_i |}) = 0$
$\Longrightarrow \lim_{l \to \infty} M (x_{k_l} - x , t) \leq 0 \diamond \cdots \diamond 0 = 0 \quad \forall\ t > 0$
$\Longrightarrow \lim_{l \to \infty} M (x_{k_l} - x , t) = 0 \quad \forall\ t > 0 \quad \cdots \quad (5)$

Thus, from (4) and (5) we see that

$$\lim_{l \to \infty} x_{k_l} = x \implies x \,\varepsilon\, A \quad [\text{ Since } A \text{ is closed }].$$
$$\implies A \textit{ is compact}.$$

Definition 13 *Let (U, A) and (V, B) be two IFNLS over the same field F. A mapping f from (U, A) to (V, B) is said to be* ***intuitionistic fuzzy continuous*** *(or in short IFC) at $x_0 \,\varepsilon\, U$, if for any given $\varepsilon > 0$, $\alpha \,\varepsilon\, (0, 1)$, $\exists\, \delta = \delta(\alpha, \varepsilon) > 0$, $\beta = \beta(\alpha, \varepsilon) \,\varepsilon\, (0, 1)$ such that for all $x \,\varepsilon\, U$,*

$$N_U(x - x_0, \delta) > \beta \implies N_V(f(x) - f(x_0), \varepsilon) > \alpha$$

and

$$M_U(x - x_0, \delta) < 1 - \beta \implies M_V(f(x) - f(x_0), \varepsilon) < 1 - \alpha.$$

If f is continuous at each point of U, f is said to be IFC on U.

Definition 14 *A mapping f from (U, A) to (V, B) is said to be* ***strongly intuitionistic fuzzy continuous*** *(or in short strongly IFC) at $x_0 \,\varepsilon\, U$, if for any given $\varepsilon > 0$, $\exists\, \delta = \delta(\alpha, \varepsilon) > 0$ such that for all $x \,\varepsilon\, U$,*

$$N_V(f(x) - f(x_0), \varepsilon) \geq N_U(x - x_0, \delta) \quad \textit{and}$$
$$M_V(f(x) - f(x_0), \varepsilon) < M_U(x - x_0, \delta).$$

f is said to be strongly IFC on U if f is strongly IFC at each point of U.

Definition 15 *A mapping f from (U, A) to (V, B) is said to be* ***sequentially intuitionistic fuzzy continuous*** *(or in short sequentially IFC) at $x_0 \,\varepsilon\, U$, if for any sequence $\{x_n\}_n$, $x_n \,\varepsilon\, U \;\forall\, n$, with $x_n \longrightarrow x_0$ in (U, A) implies $f(x_n) \longrightarrow f(x_0)$ in (V, B), that is,*

$$\lim_{n \to \infty} N_U(x_n - x_0, t) = 1 \;\textit{and}\; \lim_{n \to \infty} M_U(x_n - x_0, t) = 0$$
$$\implies \lim_{n \to \infty} N_V(f(x_n) - f(x_0), t) = 1 \;\textit{and}\; \lim_{n \to \infty} M_V(f(x_n) - f(x_0), t) = 0$$

If f is sequentially IFC at each point of U then f is said to be sequentially IFC on U.

Theorem 12 *Let f be a mapping from (U, A) to (V, B). If f strongly IFC then it is sequentially IFC but not conversely.*

Proof: Let $f : (U, A) \longrightarrow (V, B)$ be strongly IFC on U and $x_0 \,\varepsilon\, U$. Then for each $\varepsilon > 0$, $\exists\, \delta = \delta(x_0, \varepsilon) > 0$ such that for all $x \,\varepsilon\, U$,

$$N_V(f(x) - f(x_0), \varepsilon) \geq N_U(x - x_0, \delta) \quad \text{and}$$

$$M_V(f(x) - f(x_0), \varepsilon) < M_U(x - x_0, \delta)$$

Let $\{x_n\}_n$ be a sequence in U such that $x_n \longrightarrow x_0$, that is , for all $t > 0$,

$$\lim_{n \to \infty} N_U(x_n - x_0, t) = 1 \quad \text{and} \quad \lim_{n \to \infty} M_U(x_n - x_0, t) = 0$$

Thus, we see that

$$N_V(f(x_n) - f(x_0), \varepsilon) \geq N_U(x_n - x_0, \delta) \quad \text{and}$$
$$M_V(f(x_n) - f(x_0), \varepsilon) < M_U(x_n - x_0, \delta)$$

which implies that

$$\lim_{n \to \infty} N_V(f(x_n) - f(x_0), \varepsilon) = 1 \quad \text{and} \quad \lim_{n \to \infty} M_V(f(x_n) - f(x_0), \varepsilon) = 0$$

that is , $f(x_n) \longrightarrow f(x_0)$ in (V, B) .

To show that the sequentially IFC of f does not imply strongly IFC of f on U , consider the following example .

Example 3 *Let* $(X = \mathbb{R}, \|\cdot\|)$ *be a normed linear space where* $\|x\| = |x| \ \forall \ x \ \varepsilon \ \mathbb{R}$. *Define* $a * b = \min\{a, b\}$ *and* $a \diamond b = \max\{a, b\}$ *for all* $a, b \ \varepsilon \ [0, 1]$. *Also, define*

$$N_1, M_1, N_2, M_2 : X \times \mathbb{R}^+ \longrightarrow [0, 1] \qquad \textit{by}$$

$$N_1(x, t) = \frac{t}{t + |x|}, \quad M_1(x, t) = \frac{|x|}{t + |x|}$$

$$N_2(x, t) = \frac{t}{t + k|x|}, \quad M_1(x, t) = \frac{k|x|}{t + k|x|} \quad k > 0$$

$$\textit{Let } A = \{((x, t), N_1, M_1) : (x, t) \ \varepsilon \ X \times \mathbb{R}^+\} \textit{ and}$$
$$B = \{((x, t), N_2, M_2) : (x, t) \ \varepsilon \ X \times \mathbb{R}^+\}$$

It is easy to see that (X, A) *and* (X, B) *are IFNLS . Let us now define,* $f(x) = \frac{x^4}{1 + x^2} \quad \forall \quad x \ \varepsilon \ X$. *Let* $x_0 \ \varepsilon \ X$ *and* $\{x_n\}_n$ *be a sequence in* X *such that* $x_n \longrightarrow x_0$ *in* (X, A) *, that is , for all* $t > 0$,

$$\lim_{n \to \infty} N_1(x_n - x_0, t) = 1 \quad \textit{and} \quad \lim_{n \to \infty} M_1(x_n - x_0, t) = 0$$

that is

$$\lim_{n \to \infty} \frac{t}{t + |x_n - x_0|} = 1 \quad \textit{and} \quad \lim_{n \to \infty} \frac{|x_n - x_0|}{t + |x_n - x_0|} = 0$$
$$\Longrightarrow \quad \lim_{n \to \infty} |x_n - x_0| = 0$$

Now , for all $t > 0$

$$N_2(f(x_n) - f(x_0), t) = \frac{t}{t + k|f(x_n) - f(x_0)|}$$

$$= \frac{t}{t + k \left| \frac{x_n^4}{1 + x_n^2} - \frac{x_0^4}{1 + x_0^2} \right|}$$
$$= \frac{t(1+x_n^2)(1+x_0^2)}{t(1+x_n^2)(1+x_0^2)+k\,|x_n^4(1+x_0^2) - x_0^4(1+x_n^2)|}$$
$$= \frac{t(1+x_n^2)(1+x_0^2)}{t(1+x_n^2)(1+x_0^2)+k\,|(x_n^2 - x_0^2)(x_n^2+x_0^2)+x_n^2 x_0^2(x_n^2-x_0^2)|}$$
$\Longrightarrow \lim_{n \to \infty} N_2(f(x_n) - f(x_0), t) = 1.$

$\Longrightarrow M_2(f(x_n) - f(x_0), t) = \frac{k|(x_n^2-x_0^2)(x_n^2+x_0^2)+x_n^2x_0^2(x_n^2-x_0^2)|}{t(1+x_n^2)(1+x_0^2)+k|(x_n^2-x_0^2)(x_n^2+x_0^2)+x_n^2 x_0^2(x_n^2-x_0^2)|}$

$\Longrightarrow \lim_{n \to \infty} M_2(f(x_n) - f(x_0), t) = 0.$

Thus, f is sequentially continuous on X. From the calculation of the example [6], it follows that f is not strongly IFC.

Theorem 13 *Let f be a mapping from the IFNLS (U, A) to (V, B). Then f is IFC on U if and only if it is sequentially IFC on U.*

Proof: $\Longrightarrow$ part : Suppose f is IFC at $x_0 \,\varepsilon\, U$ and $\{x_n\}_n$ is a sequence in U such that $x_n \longrightarrow x_0$ in (U, A). Let $\varepsilon > 0$ and $\alpha \,\varepsilon\, (0, 1)$. Since f is IFC at x_0, $\exists\ \delta = \delta(\varepsilon, \alpha) > 0$ and $\exists\ \beta = \beta(\varepsilon, \alpha) \,\varepsilon\, (0, 1)$ such that for all $x \,\varepsilon\, U$,

$$N_U(x - x_0, \delta) > \beta \Longrightarrow N_V(f(x) - f(x_0), \varepsilon) > \alpha$$
$$M_U(x - x_0, \delta) < 1 - \beta \Longrightarrow M_V(f(x) - f(x_0), \varepsilon) < 1 - \alpha.$$

Since $x_n \longrightarrow x_0$ in (U, A), there exists a positive integer n_0 such that for all $n \geq n_0$

$$N_U(x_n - x_0, \delta) > \beta \text{ and } M_U(x_n - x_0, \delta) < 1 - \beta$$

$\Longrightarrow N_V(f(x_n) - f(x_0), \varepsilon) > \alpha$ and $M_V(f(x_n) - f(x_0), \varepsilon) < 1 - \alpha$

$\Longrightarrow f(x_n) \longrightarrow f(x_0)$ in (V, B), that is, f is sequentially IFC at x_0.

$\Longleftarrow$ *part* : Let f be sequentially IFC at $x_0 \,\varepsilon\, U$. If possible, we suppose that f is not IFC at x_0.

$\Longrightarrow \exists\ \varepsilon > 0$ and $\alpha \,\varepsilon\, (0, 1)$ such that for any $\delta > 0$ and $\beta \,\varepsilon\, (0, 1)$, $\exists\ y$ (depending on δ, β) such that

$$N_U(x_0 - y, \delta) > \beta \text{ but } N_V(f(x_0) - f(y), \varepsilon) \leq \alpha$$

and

$$M_U(x_0 - y, \delta) < 1 - \beta \text{ but } M_V(f(x_0) - f(y), \varepsilon) \geq 1 - \alpha.$$

Thus for $\beta = 1 - \frac{1}{n+1}$, $\delta = \frac{1}{n+1}$, $n = 1, 2, \cdots$, $\exists\ y_n$ such that $N_U(x_0 - y_n, \frac{1}{n+1}) > 1 - \frac{1}{n+1}$ but $N_V(f(x_0) - f(y_n), \varepsilon) \leq \alpha$, $M_U(x_0 - y_n, \frac{1}{n+1}) < \frac{1}{n+1}$ but $M_V(f(x_0) - f(y), \varepsilon) \geq 1 - \alpha$.

Taking $\delta > 0$, $\exists\ n_0$ such that $\frac{1}{n+1} < \delta \quad \forall\ n \geq n_0$.

$N_U(x_0 - y_n, \delta) \geq N_U(x_0 - y_n, \frac{1}{n+1}) > 1 - \frac{1}{n+1} \quad \forall\ n \geq n_0$,

$M_U(x_0 - y_n, \delta) \leq M_U(x_0 - y_n, \frac{1}{n+1}) < \frac{1}{n+1} \quad \forall\ n \geq n_0$.

$\Longrightarrow \lim_{n \to \infty} N_U(x_0 - y_n, \delta) = 1$ and $\lim_{n \to \infty} M_U(x_0 - y_n, \delta) = 0$

But, $N_V(f(x_0) - f(y_n), \varepsilon) \leq \alpha \Longrightarrow \lim_{n \to \infty} N_V(f(x_0) - f(y_n), \varepsilon) \neq 1.$

Thus, $\{f(y_n)\}_n$ does not converge to $f(x_0)$ where as $y_n \longrightarrow x_0$ in

(U, A) which is a contradiction to our assumption . Hence , f is IFC at x_0 .

Theorem 14 *Let f be a mapping from the IFNLS (U, A) to (V, B) and D be a compact subset of U . If f IFC on U then $f(D)$ is a compact subset of V .*

Proof: Let $\{y_n\}_n$ be a sequence in $f(D)$. Then for each n , $\exists\ x_n\ \varepsilon, D$ such that $f(x_n) = y_n$. Since D is compact, there exists $\{x_{n_k}\}_k$ a subsequence of $\{x_n\}_n$ and $x_0 \varepsilon D$ such that $x_{n_k} \longrightarrow x_0$ in (U, A). Since f is IFC at x_0 if for any given $\varepsilon > 0$, $\alpha\ \varepsilon\ (0, 1)$, $\exists\ \delta = \delta(\alpha, \varepsilon) > 0$, $\beta = \beta(\alpha, \varepsilon)\ \varepsilon\ (0, 1)$ such that for all $x\ \varepsilon\ U$,

$$N_U(x - x_0, \delta) > \beta \implies N_V(f(x) - f(x_0), \varepsilon) > \alpha$$

and

$$M_U(x - x_0, \delta) < 1 - \beta \implies M_V(f(x) - f(x_0), \varepsilon) < 1 - \alpha$$

Now, $x_{n_k} \longrightarrow x_0$ in (U, A) implies that $\exists\ n_0\ \varepsilon\ \mathbb{N}$ such that for all $k \geq n_0$

$$N_U(x_{n_k} - x_0, \delta) > \beta \text{ and } M_U(x_{n_k} - x_0, \delta) < 1 - \beta$$

$$\implies N_V(f(x_{n_k}) - f(x_0), \varepsilon) > \alpha\,\varepsilon) < 1 - \alpha$$

$$\text{and } M_V(f(x_{n_k}) - f(x_0), \varepsilon) < 1 - \alpha$$

$$i.e.\ N_V(y_{n_k} - f(x_0), \varepsilon) > \alpha \text{ and } M_V(y_{n_k} - f(x_0), \varepsilon) < 1 - \alpha\ \forall\, k \geq n_0$$

$\implies$ $f(D)$ is a compact subset of V .

2 Open Problem

Though there are the concepts of fuzzy inner product spaces [9, 10] but the concept of fuzzy norm could not be induced by these concepts of fuzzy inner product. So, one can develop the concept of fuzzy inner product which can induce the concept of fuzzy norm. Also, one can develop the concept of anti fuzzy inner product which can induce the concept of anti fuzzy norm [8].

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