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A Mixed Problem for a Boussinesq Hyperbolic Equation With Integral Condition

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Abstract

A hyperbolic problem wich combines a classical(Dirichlet) and a non-local contraint is considered. The existence and uniqueness of strong solution are proved, we use a functionnal analysis method based on a priori estimate and on the density of the range of the operator generated by the considered problem.

Keywords: *A priori estimate, hyperbolic equation, integral condition.*

1 Introduction

The first study of evolution problems with a nonlocal condition - the so called energy specification - goes back to Cannon[5], 1963 Using an integral condition, we proved the existence and uniqueness of the solution of a mixed problem wich combine a classical (Dirichlet)and an integral condition for the equation. Problems involving local and integral condition for hyperbolic equations are investigated by the energy inequalities method in [1] , [6] , [7] , [8] , [9] , [10] , [11], and [12] . In this paper, we prove the existence and uniqueness of the solution for the mixed problem $(1) - (5)$. Our proof is based on a priori estimate and on the fact that the range of the operator generated by the considered problem is dense, in the end some open problems are given.

2 Problem Formulation

In the region $Q = (0, l) \times (0, T)$, with $l < \infty$ and $T < \infty$, we shall consider the problem

$$Lu = u_{tt} - (b(x,t)u_x)_x - \beta\frac{\partial^4 u}{\partial t^2 \partial x^2} = f(x,t), \forall(x,t) \in Q \tag{1}$$

$$l_1 u = u\,(x,0) = \varphi_1\,(x)\,, \quad x \in (0,l) \tag{2}$$

$$l_2 u = u_t\,(x,0) = \varphi_2\,(x)\,, \quad x \in (0,l) \tag{3}$$

$$u\,(0,t) = 0, \qquad t \in (0,T) \tag{4}$$

$$\int_0^l xu\,(x,t)\,dx = 0, \quad t \in (0,T) \tag{5}$$

where β is a strictly positive real number and $b(x,t)$ and its derivatives satisfy the conditions:
$C_1:\ b_0 \leq b(x,t) \leq b_1\ ,\ b_t\,(x,t) \leq b_2, b_x\,(x,t) \leq b_3$, for any $(x,t) \in \overline{Q}$,
$C_2:\ b_{tt}\,(x,t) \leq b_4\ ,\ b_{xt}\,(x,t) \leq b_5$,for any $(x,t) \in \overline{Q}$.

The functions f, φ_1 and φ_2 are known functions wich satisfy the compatibility conditions:
$\varphi_1\,(0) = \varphi_2\,(0) = \int_0^l x\varphi_1\,(x)\,dx = \int_0^l x\varphi_2\,(x)\,dx = 0.$

3 Functional Spaces

The problem (1)-(5) can be put in the following operator form: $\mathcal{L}u = \mathcal{F},\ u \in D(\mathcal{L})$, where

$$\mathcal{L}u = (Lu, l_1u, l_2u) \text{ and } \mathcal{F} = (f, \varphi_1, \varphi_2).$$

The operator $\mathcal{L}$ is considered from B to H, where B is the Banach space consisting of functions $u \in L^2(Q)$, satisfying conditions (4) and (5) with the finite norm

$$\|u\|_B^2 = \sup_{0\leq\tau\leq T} \left[\|u(.,\tau)\|_{L^2(0,l)}^2 + \|u_t(.,\tau)\|_{L^2(0,l)}^2\right] \tag{6}$$

and H is the Hilbert space $L^2(Q) \times L^2(0,l) \times L^2(0,l)$ equipped with the norm

$$\|\mathcal{F}\|_H^2 = \|f\|_{L^2(Q^\tau)}^2 + \|\varphi_1\|_{L^2(0,l)}^2 + \|\varphi_2\|_{L^2(0,l)}^2.$$

Let $D(\mathcal{L})$ denote the domain of $\mathcal{L}$ which is the set of all functions $u \in L^2(Q)$ for which $u_t, u_x, u_{tx}, u_{tt}, u_{ttx} \in L^2(Q)$ and satisfying conditions (4) and (5).

4 A priori estimate and its consequences

Theorem 1. For any function $u \in D(\mathcal{L})$ satisfyies conditions C$_1$-C$_2$ there exists a positive constant c, such that

$$\|u\|_B \leq c\,\|\mathcal{L}u\|_H\,, \tag{7}$$

Proof. We consider the scalair product in $L^2(Q^\tau)$ of the operator Lu and Mu, where $Mu = x\Im_x^* u_t - \Im_x^*(\rho u_t)$, with $Q^\tau = (0,l)\times(0,\tau)$, $0 \leq \tau \leq T$, and $\Im_x^* v = \int_x^l v(\xi,t)d\xi$, we obtain

$$\begin{aligned}(Lu, Mu)_{L^2(Q^\tau)} &= (u_{tt}, x\Im_x^* u_t)_{L^2(Q^\tau)} - ((b(x,t)\,u_x)_x\,, x\Im_x^* u_t)_{L^2(Q^\tau)} \\ &\quad -\beta\,(u_{ttxx}, x\Im_x^* u_t)_{L^2(Q^\tau)} - (u_{tt}, \Im_x^*(\rho u_t))_{L^2(Q^\tau)} \\ &\quad + ((b(x,t)\,u_x)_x\,, \Im_x^*(\rho u_t))_{L^2(Q^\tau)} + \beta\,(u_{ttxx}, \Im_x^*(\rho u_t))_{L^2(Q^\tau)}\,. \end{aligned} \tag{8}$$

Making use of conditions (2)-(5) and integrating by parts we estabilish the equalities

$$(u_{tt}, x\Im_x^* u_t)_{L^2(Q^\tau)} = \frac{1}{2}\|\Im_x^* u_t(.,\tau)\|^2_{L^2(0,l)} - \frac{1}{2}\|\Im_x^*\varphi_2\|^2_{L^2(0,l)} - (\Im_x^* u_{tt}, u_t)_{L^2_\rho(Q^\tau)}\,, \tag{9}$$

$$\begin{aligned} -((b(x,t)\,u_x)_x\,, x\Im_x^*(u_t))_{L^2(Q^\tau)} &= \frac{1}{2}\left\|\sqrt{b(.,\tau)}u(.,\tau)\right\|^2_{L^2(0,l)} \\ -\frac{1}{2}\left\|\sqrt{b(.,0)}\varphi_1\right\|^2_{L^2(0,l)} - \frac{1}{2}\left\|\sqrt{b_t(.,t)}u\right\|^2_{L^2(Q^\tau)} &- (b_x(x,t)\,u, \Im_x^* u_t)_{L^2(Q^\tau)} \\ &- (b(x,t)\,u_x, u_t)_{L^2_\rho(Q^\tau)}\,, \end{aligned} \tag{10}$$

$$-\beta\,(u_{ttxx}, x\Im_x^*(u_t))_{L^2(Q^\tau)} = \frac{\beta}{2}\|u_t(.,\tau)\|^2_{L^2(0,l)} - \frac{\beta}{2}\|\varphi_2\|^2_{L^2(0,l)} - \beta\,(u_{ttx}, u_t)_{L^2_\rho(Q^\tau)}\,. \tag{11}$$

$$-(u_{tt}, \Im_x^*(\rho u_t))_{L^2(Q^\tau)} = (\Im_x^* u_{tt}, u_t)_{L^2_\rho(Q^\tau)}\,, \tag{12}$$

$$((b(x,t)\,u_x)_x\,, \Im_x^*(\rho u_t))_{L^2(Q^\tau)} = (b(x,t)\,u_x, u_t)_{L^2_\rho(Q^\tau)}\,, \tag{13}$$

$$\beta\,(u_{ttxx}, \Im_x^*(\rho u_t))_{L^2(Q^\tau)} = \beta\,(u_{ttx}, u_t)_{L^2_\rho(Q^\tau)}\,. \tag{14}$$

Combining equalities (9)-(14) and (8) we obtain

$$\begin{aligned} &\tfrac{1}{2}\|\Im_x^* u_t(.,\tau)\|^2_{L^2(0,l)} + \tfrac{1}{2}\left\|\sqrt{b(.,t)}u(.,\tau)\right\|^2_{L^2(0,l)} + \tfrac{\beta}{2}\|u_t(.,\tau)\|^2_{L^2(0,l)} \\ &= \tfrac{1}{2}\|\Im_x^*\varphi_2\|^2_{L^2(0,l)} + \tfrac{1}{2}\left\|\sqrt{b(.,t)}\varphi_1\right\|^2_{L^2(0,l)} + \tfrac{\beta}{2}\|\varphi_2\|^2_{L^2(0,l)} + \tfrac{1}{2}\left\|\sqrt{b_t}u\right\|^2_{L^2(Q^\tau)} \\ &\quad + (b_x(x,t)\,u, \Im_x^* u_t)_{L^2(Q^\tau)} + (Lu, x\Im_x^* u_t)_{L^2(Q^\tau)} - (Lu, \Im_x^*(\rho u_t))_{L^2(Q^\tau)}\,. \end{aligned} \tag{15}$$

By applying the Cauchy inequality to the last three terms on the right-hand side of the inequality (15) and making use conditions C_1, combining with (15), we obtain

$$\begin{aligned} &\|u(.,\tau)\|^2_{L^2(0,l)} + \|u_t(.,\tau)\|^2_{L^2(0,l)} + \|\Im^*_x u_t(.,\tau)\|^2_{L^2(0,l)} \\ &\quad \le k[\|f\|^2_{L^2(Q^\tau)} + \|\varphi_1\|^2_{L^2(0,l)} + \|\varphi_2\|^2_{L^2(0,l)} \\ &\quad + \|u\|^2_{L^2(Q^\tau)} + \|u_t\|^2_{L^2(Q^\tau)} + \|\Im^*_x u_t\|^2_{L^2(Q^\tau)}], \end{aligned} \tag{16}$$

where $k = \frac{\max\left(2, b_1, \beta + l^2, b_3^2 + b_2, l^4\right)}{\min(1, b_0, \beta)}$.

Applying the Gronwall lemma to(16), and elimining the term $\|\Im^*_x u_t(.,\tau)\|^2_{L^2(0,l)}$ of the left-hand side of the inequality we obtain

$$\begin{aligned} &\|u(.,\tau)\|^2_{L^2(0,l)} + \|u_t(.,\tau)\|^2_{L^2(0,l)} \\ &\le k\exp(kT)(\|f\|^2_{L^2(Q^\tau)} + \|\varphi_1\|^2_{L^2(0,l)} + \|\varphi_2\|^2_{L^2(0,l)}). \end{aligned} \tag{17}$$

Since the left-hand side of (17) does not depend on τ, we take the supremum with τ from 0 to T, then the estimate (7) follows with $c = \sqrt{k}\exp(k\frac{T}{2})$.

5 Solvability of the problem

Proposition 1. The operator $\mathcal{L}$ acting from B to H have a closure.
Proof. (see [3])
Let $\overline{\mathcal{L}}$ be the closure of $\mathcal{L}$, $D(\overline{\mathcal{L}})$ its domain .
Definition. The solution of $\overline{\mathcal{L}}u = \mathcal{F}$ for any $u \in D(\overline{\mathcal{L}})$ is strong solution of problem(1)-(5).
We take the limit in the inequality (7), we obtain

$$\|u\|_B \le c\left\|\overline{\mathcal{L}}u\right\|_H, \forall u \in D(\overline{\mathcal{L}}). \tag{7bis}$$

From the inequality we have

Corollary 1. The strong solution of problem (1)-(5) when it exists, it's unique, and depends continuly of data f, φ_1, φ_2.

Corollary 2. The set of values $R(\overline{\mathcal{L}})$ of the operator $\overline{\mathcal{L}}$ is equal to the closure $\overline{R(\mathcal{L})}$ of $R(\mathcal{L})$.

Theorem 2. If the conditions C_1-C_2 are satisfying, then for any $\mathcal{F} = (f, \varphi_1, \varphi_2) \in H$, there exists a strong unique solution $u = \overline{\mathcal{L}}^{-1}\mathcal{F} = \overline{\mathcal{L}^{-1}}\mathcal{F}$ of the probleme (1)-(5) where the estimate $\|u\|_B \le c\|\mathcal{F}\|_H$ is satisfying, where c is a positive constant does not depends of u.

Proof. From (7 bis) we conclude that the operator $\overline{\mathcal{L}}$ acting from $D(\overline{\mathcal{L}})$ in $R(\overline{\mathcal{L}})$ have an inverse $\overline{\mathcal{L}}^{-1}$, and from corollary 2, we conclude that the range

$R(\overline{\mathcal{L}})$ of the operator $\overline{\mathcal{L}}$ is closed. Then we will be proove the density of the set $R(\mathcal{L})$ in the space H (i.e) $\overline{R(\mathcal{L})} = H$.

For this we need the following proposition

Proposition 2. If, for all functions $u \in D_0(\mathcal{L})$, where

$$D_0(\mathcal{L}) = \{u | u \in D(\mathcal{L}) : l_1 u = l_2 u = 0\},$$

and for some function $\omega \in L^2(Q)$, we have

$$(Lu, \omega)_{L^2(Q)} = 0. \tag{18}$$

then ω vanishes almost everywhere in Q.

Proof of the proposition 2. The relation (18) is given for all $u \in D_0(\mathcal{L})$, we can express it in a particular form. Let u_{tt} be a solution of

$$b(\sigma, t)\left[x\Im_x^* u_{tt} - \Im_x^*(\rho u_{tt})\right] = h(x,t), \tag{19}$$

where σ is a constant in $(0, l)$ and $h(x,t) = \int_t^T \omega(x,\tau) d\tau$.
And let u be the fonction defined by

$$u = \begin{cases} 0 & \text{si} \quad 0 \le t \le s \\ \int_s^t (t-\tau) u_{\tau\tau} d\tau & \text{si} \quad s \le t \le T \end{cases} \tag{20}$$

(19) and (20) follows u is in $D_0(L)$ and

$$\begin{aligned} \omega(x,t) = ((\Im_x)^*)^{-1} h \quad &= -\left[b(\sigma,t)\left(x\Im_x^* u_{tt} - \Im_x^*(\rho u_{tt})\right)\right]_t \\ &= \left[b(\sigma,t)\, \Im_x^*(\rho - x) u_{tt}\right]_t . \end{aligned} \tag{21}$$

To continue the proof we need the following lemma

Lemma 2. The function ω defined by (21), belongs to the space $L^2(Q)$.

Proof of lemma 2. We start with the proof of this inequality

$$\left\| \Im_x^*(\rho - x) u_{tt} \right\|^2_{L^2(0,l)} \le \frac{l^4}{12} \left\| u_{tt} \right\|^2_{L^2(0,1)} .$$

From this inequality and since the conditions C_1 are satisfied we conclude that $b_t(\sigma,t)\, \Im_x^*(\rho - x) u_{tt}$ belongs to $L^2(Q)$.

Because

$$\omega(x,t) = \left[b(\sigma,t)\, \Im_x^*(\rho - x) u_{tt}\right]_t = b_t(\sigma,t)\, \Im_x^*(\rho - x) u_{tt} + b(\sigma,t)\, \Im_x^*(\rho - x) u_{ttt},$$

then we will prove that $\quad b(\sigma,t)\, \Im_x^*(\rho - x) u_{ttt} \in L^2(Q)$.

For this we introduce the t-averaging opérators ρ_ε of the form

$$(\rho_\varepsilon f)(x,t) = \frac{1}{\varepsilon} \int_0^T \omega\left(\frac{t-s}{\varepsilon}\right) f(x,s) ds,$$

where $\omega \in C_0^\infty(0,T), \omega \geq 0,, \int_{-\infty}^{+\infty} \omega(s)ds = 1$, $\omega \equiv 0$, for $t \leq 0$ and $t \geq T$, applying the operators ρ_ε and $\frac{\partial}{\partial t}$ to the equation

$$-b(\sigma,t)\,\Im_x^*(\rho - x)u_{tt} = h(x,t),$$

we obtain

$$\begin{aligned} &\tfrac{\partial}{\partial t}\left(-b(\sigma,t)\,\Im_x^*(\rho - x)u_{tt}\right) \\ &= \tfrac{\partial}{\partial t}[-b(\sigma,t)\,\Im_x^*(\rho - x)u_{tt} + \rho_\varepsilon\left(b(\sigma,t)\,\Im_x^*(\rho - x)u_{tt}\right)] - \tfrac{\partial}{\partial t}\rho_\varepsilon h. \end{aligned}$$

Then

$$\begin{aligned} &\left\| b(\sigma,t)\,\Im_x^*(\rho - x)u_{tt} \right\|^2_{L^2(Q)} \\ &\leq 2\left\| \frac{\partial}{\partial t}\left[b(\sigma,t)\,\Im_x^*(\rho - x)u_{tt} - \rho_\varepsilon\left(b(\sigma,t)\,\Im_x^*(\rho - x)u_{tt}\right)\right] \right\|^2_{L^2(Q)} + 2\left\| \frac{\partial}{\partial t}\rho_\varepsilon h \right\|^2_{L^2(Q)}. \end{aligned}$$

Since $\rho_\varepsilon f \longrightarrow f$ when $\varepsilon \to 0$, and $\frac{\partial}{\partial t}\left(b(\sigma,t)\,\Im_x^*(\rho - x)u_{tt}\right)$ is bounded in $L^2(Q)$, then $\omega \in L^2(Q)$.

Now we return to the second proposition, we remplace ω in (18) by its representation given by (21) we have

$$\begin{aligned} \left(u_{tt}, [b(\sigma,t)\,\Im_x^*(\rho - x)u_{tt}]_t\right)_{L^2(Q)} &= \left((b(x,t)\,u_x)_x, [b(\sigma,t)\,\Im_x^*(\rho - x)u_{tt}]_t\right)_{L^2(Q)} \\ &+\beta\left(u_{ttxx}, [b(\sigma,t)\,\Im_x^*(\rho - x)u_{tt}]_t\right)_{L^2(Q)}. \end{aligned} \tag{22}$$

Making use conditions (3)-(5), and from the particular forme of u given by (19) and (20), the equality (22) can be simplified. For this integrating by parts each term of the equality on the sub-domain $Q_s = (0,l) \times (s,T)\, where 0 \leq s \leq T$

$$\begin{aligned} &\left(u_{tt}, [b(\sigma,t)\Im_x^*(\rho - x)u_{tt}]_t\right)_{L^2(Q)} = \\ &= \frac{1}{2}\left\| \sqrt{b(\sigma,s)}\Im_x^* u_{tt}(.,s) \right\|^2_{L^2(0,l)} - \frac{1}{2}\left\| \sqrt{b_t(\sigma,.)}\Im_x^* u_{tt} \right\|^2_{L^2(Q_s)} \end{aligned} \tag{23}$$

$$\begin{aligned} \left((b(x,t)u_x)_x, [b(\sigma,t)\Im_x^*(\rho - x)u_{tt}]_t\right)_{L^2(Q)} &= -\frac{1}{2}\left\| \sqrt{b(.,T)\,b(\sigma,T)}u_t(.,T) \right\|^2_{L^2(0,l)} + \\ &+\frac{1}{2}\int_{Q_s}\left[3b_t(x,t)\,b(\sigma,t) + b(x,t)\,b_t(\sigma,t)\right](u_t)^2\,dxdt \\ &-\int_0^l b_t(x,T)\,b(\sigma,T)\,u(x,T)\,u_t(x,T)\,dx \\ &+\int_{Q_s}\left[b_{tt}(x,t)\,b(\sigma,t) + b_t(x,t)\,b_t(\sigma,t)\right]uu_t dxdt \end{aligned}$$

$$+\int_{Q_s}\left[b_x(x,t)\,u_t+b_{xt}(x,t)\,u\right]b(\sigma,t)\,\Im_x^* u_{tt}dxdt \tag{23bis}$$

$$\beta\left(u_{ttxx},\left[b(\sigma,t)\,\Im_x^*(\rho-x)u_{tt}\right]_t\right)_{L^2(Q)}=$$
$$=\frac{\beta}{2}\left\|\sqrt{b_t(\sigma,.)}u_{tt}\right\|^2_{L^2(Q_s)}-\frac{\beta}{2}\left\|\sqrt{b(\sigma,s)}u_{tt}(.,s)\right\|^2_{L^2(0,l)} \tag{24}$$

Substitution of (23)-(23 bis) and (24) into (22) gives

$$\begin{aligned}\tfrac{1}{2}\left\|\sqrt{b(\sigma,s)}\Im_x^* u_{tt}(.,s)\right\|^2_{L^2(0,l)}&+\tfrac{1}{2}\left\|\sqrt{b(.,T)\,b(\sigma,T)}u_t(.,T)\right\|^2_{L^2(0,l)}\\ &+\tfrac{\beta}{2}\left\|\sqrt{b(\sigma,s)}u_{tt}(.,s)\right\|^2_{L^2(0,l)}\\ =\tfrac{1}{2}\left\|\sqrt{b_t(\sigma,s)}\Im_x^* u_{tt}\right\|^2_{L^2(Q_s)}&+\tfrac{\beta}{2}\left\|\sqrt{b_t(\sigma,.)}u_{tt}\right\|^2_{L^2(Q_s)}\\ +\tfrac{1}{2}\textstyle\int_{Q_s}&\left[3b_t(x,t)\,b(\sigma,t)+b(x,t)\,b_t(\sigma,t)\right](u_t)^2\,dxdt\\ -\textstyle\int_0^l b_t(x,T)\,b(\sigma,T)\,&u(x,T)\,u_t(x,T)\,dx\\ +\textstyle\int_{Q_s}\left[b_{tt}(x,t)\,b(\sigma,t)\right.&\left.+b_t(x,t)\,b_t(\sigma,t)\right]uu_tdxdt\\ +\textstyle\int_{Q_s}\left[b_x(x,t)\,u_t\right.&\left.+b_{xt}(x,t)\,u\right]b(\sigma,t)\,\Im_x^* u_{tt}dxdt.\end{aligned} \tag{25}$$

By applying the Cauchy inequality and Cauchy inequality with ε to estimate the last three terms on the right-hand side of the inequality (25) and making use conditions C_1-C_2, combining the estimates and (25) taking into account that $\varepsilon=\frac{b_0^2}{2b_1^2}$ we obtain

$$\begin{aligned}&\tfrac{b_0}{2}\left[\left\|\Im_x^* u_{tt}(.,s)\right\|^2_{L^2(0,l)}+\tfrac{b_0}{2}\left\|u_t(.,T)\right\|^2_{L^2(0,l)}+\beta\left\|u_{tt}(.,s)\right\|^2_{L^2(0,l)}\right]\\ &\le\left(b_1^2+\tfrac{b_2}{2}\right)\left\|\Im_x^* u_{tt}\right\|^2_{L^2(Q_s)}+\tfrac{\beta b_2}{2}\left\|u_{tt}\right\|^2_{L^2(Q_s)}+\tfrac{b_2^2+b_1^2+4b_1b_2+b_3^2}{2}\left\|u_t\right\|^2_{L^2(Q_s)}\\ &\quad+\tfrac{b_2^2+b_4^2+b_5^2}{2}\left\|u\right\|^2_{L^2(Q_s)}+\tfrac{b_1^2b_2^2}{b_0^2}\left\|u(.,T)\right\|^2_{L^2(0,l)}\end{aligned} \tag{26}$$

By virtue of the elementary inequality

$$\frac{b_1^2b_2^2}{b_0^2}\left\|u(.,T)\right\|^2_{L^2(0,l)}\le\frac{b_1^2b_2^2}{b_0^2}\left\|u\right\|^2_{L^2(Q_s)}+\frac{b_1^2b_2^2}{b_0^2}\left\|u_t\right\|^2_{L^2(Q_s)}, \tag{27}$$

where $u\in D_0(\mathcal{L})$ and satisfies the conditions (4) and (5), we estimate the last term of the right-hand side of the inequality(26), we obtain

$$\begin{aligned}&\left\|\Im_x^* u_{tt}(.,s)\right\|^2_{L^2(0,l)}+\tfrac{b_0}{2}\left\|u_t(.,T)\right\|^2_{L^2(0,l)}+\beta\left\|u_{tt}(.,s)\right\|^2_{L^2(0,l)}\\ &\le\left(\tfrac{2b_1^2+b_2}{b_0}\right)\left\|\Im_x^* u_{tt}\right\|^2_{L^2(Q_s)}+\tfrac{\beta b_2}{b_0}\left\|u_{tt}\right\|^2_{L^2(Q_s)}+\frac{\frac{2b_1^2b_2^2}{b_0^2}+2b_1b_2+(b_2+b_1)^2+b_3^2}{b_0}\left\|u_t\right\|^2_{L^2(Q_s)}\\ &\quad+\tfrac{b_0^2\left(b_2^2+b_4^2+b_5^2\right)+2b_1^2b_2^2}{b_0^3}\left\|u\right\|^2_{L^2(Q_s)}\end{aligned} \tag{28}$$

For estimate the last term of the right-hand side of the inequality (28), we will prove the inequality $\|u\|^2_{L^2(Q_s)} \leq 24T^2 \|u_t\|^2_{L^2(Q_s)}$, combining the last inequality and (28) we get

$$\begin{aligned}&\|\Im_x^* u_{tt}(.,s)\|^2_{L^2(0,l)} + \|u_{tt}(.,s)\|^2_{L^2(0,l)} + \|u_t(.,T)\|^2_{L^2(0,l)} \\ &\quad \leq k\left[\|\Im_x^* u_{tt}\|^2_{L^2(Q_s)} + \|u_{tt}\|^2_{L^2(Q_s)} + \|u_t\|^2_{L^2(Q_s)}\right], \qquad (29)\end{aligned}$$

where $k = \frac{\max\left(\beta b_2, \left(2b_1^2 + b_2\right), b_0 k(b_i,T)\right)}{b_0 \min\left(1,\beta,\frac{b_0}{2}\right)}$, and
$k(b_i,T) = \frac{2b_1^2 b_2^2 + b_0^2\left[2b_1 b_2 + (b_2+b_1)^2 + b_3^2\right] + 24T^2\left[b_0^2\left(b_2^2+b_4^2+b_5^2\right) + 2b_1^2 b_2^2\right]}{b_0^3}$.

To continue, we introduce the new function $v(x,t) = \int_t^T u_{\tau\tau} d\tau$, then $u_t(x,t) = v(x,s) - v(x,t)$, and $u_t(x,T) = v(x,s)$.
The inequality (29) it be

$$\begin{aligned}&\|\Im_x^* u_{tt}(.,s)\|^2_{L^2(0,l)} + \|u_{tt}(.,s)\|^2_{L^2(0,l)} + (1-2k(T-s))\|v(.,s)\|^2_{L^2(0,l)} \\ &\quad \leq 2k(\|\Im_x^* u_{tt}\|^2_{L^2(Q_s)} + \|u_{tt}\|^2_{L^2(Q_s)} + \|v\|^2_{L^2(Q_s)}). \qquad (30)\end{aligned}$$

If $s_0 > 0$ satisfies $(1-2k(T-s_0)) = \frac{1}{2}$, then the inequality (30) implies

$$\begin{aligned}&\|\Im_x^* u_{tt}(.,s)\|^2_{L^2(0,l)} + \|u_{tt}(.,s)\|^2_{L^2(0,l)} + \|v(.,s)\|^2_{L^2(0,l)} \\ &\quad \leq 4k(\|\Im_x^* u_{tt}\|^2_{L^2(Q_s)} + \|u_{tt}\|^2_{L^2(Q_s)} + \|v\|^2_{L^2(Q_s)}), \qquad (31)\end{aligned}$$

for all $s \in [T-s_0, T]$. We denote

$$Y(s) = \|\Im_x^* u_{tt}\|^2_{L^2(Q_s)} + \|u_{tt}\|^2_{L^2(Q_s)} + \|v\|^2_{L^2(Q_s)}. \qquad (32)$$

We get

$$Y'(s) = -\|\Im_x^* u_{tt}(.,s)\|^2_{L^2(0,l)} - \|u_{tt}(.,s)\|^2_{L^2(0,l)} - \|v(.,s)\|^2_{L^2(0,l)}.$$

Then and from (31) we obtain

$$-Y'(s) \leq 4kY(s).$$

Then $\quad -\frac{\partial}{\partial s}(Y(s)\exp(4ks)) \leq 0.$

Integrating this inequality on (s,T) and taking into account that $Y(T) = 0$, we obtain $Y(s)\exp(4ks) \leq 0$.
Then $Y(s) = 0$ for all $s \in [T-s_0, T]$. Then $\omega = 0$ almost everywhere in

Q_{T-s_0}, proceding in this way step by step, we proove that $\omega = 0$ almost everywhere in Q.

This achieves the proof of proposition. Now we return to proove the théorème. We will prove that $\overline{R(\mathcal{L})} = H$.
Since H is a Hilbert space, the equality $\overline{R(\mathcal{L})} = H$ is true, if from

$$(\mathcal{L}u, W)_H = (Lu, \omega)_{L^2(Q)} + (l_1 u, \omega_1)_{L^2(0,l)} + (l_2 u, \omega_2)_{L^2(0,l)} = 0, \tag{33}$$

where $W = (\omega, \omega_1, \omega_2) \in R(\mathcal{L})^{\perp}$, we get $\omega \equiv 0, \omega_1 \equiv 0$ and $\omega_2 \equiv 0$ in Q, for any element of $D_0(\mathcal{L})$.
From (33) we obtain $\forall u \in D_0(\mathcal{L}), (Lu, \omega)_{L^2(Q)} = 0$. Then by virtue of the second proposition, we conclude that $\omega \equiv 0$.
Then from (33), we obtain $(l_1 u, \omega_1)_{L^2(0,l)} + (l_2 u, \omega_2)_{L^2(0,l)} = 0$.

Since the quantities $l_1 u$ and $l_2 u$ can vanish independently and the ranges of the trace operators l_1 and l_2 are dense in the Hilbert space $L^2(0, l)$, then $\omega_1 = \omega_2 = 0$. Thus to conclude that $W = 0$.

6 Open Problems

We give in this section some open problems which that are treated as an extension of our present work, and has a great importance in physical applications and other domains.
We consider a nonlinear hyperbolic visco-elastic problem with a nonlocal boundary conditions.

$$u_{tt} - \beta \frac{\partial^4 u}{\partial t^2 \partial x^2} + \int_0^t g(t-s) \left(b(x,t) u_x\right)_x ds = |u|^{p-2} u, \forall (x,t) \in Q, p > 2,$$

under the conditions (2)-(5) given in our problem. We propose that

1. Establish the local and global existence of solution for the given problem.

2. When the solution u of the given problem blow up in time.

3. Show the polynomial then the exponential decay of the solution.

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