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Normalsize Some Fixed Point Theorems in Intuitionistic Fuzzy n-Normed Linear spaces

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Abstract

In this paper the concepts of sectional intuitionistic fuzzy continuous mapping, l-intuitionistic fuzzy compact sets and asymptotic intuitionistic fuzzy normal structure are introduced. Schauder type and Baillon and Schoneberg type fixed point theorems are also established in an intuitionistic fuzzy n-normed linear space.

Keywords: *Sectional intuitionistic fuzzy continuous mapping, l-intuitionistic fuzzy compact set, fuzzy normal structure, fixed point theorem.*
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1 Introduction and Preliminaries

Inspired by the theory of n-normed linear space [9, 10, 11, 13, 15] and fuzzy normed linear space [1, 2, 3, 6, 7, 8, 14] the notions of fuzzy n-normed linear space [16] and intuitionistic fuzzy n-normed linear space [17] have been introduced. Bag and Samanta [4] have developed the concepts of l-fuzzy compact sets and asymptotic fuzzy normal structure and proved some fixed point theorems in fuzzy normed linear spaces. We have also introduced the concept of intuitionistic fuzzy compact linear operators between intuitionistic fuzzy n-normed linear spaces and studied some of their properties [19]. Iqbal H. Jebril and Ra'ed Hatamleh [12] have introduced the concept of random n-normed linear space as a further generalization of n-normed space introduced by Gunawan and Mashadi [11].

Our aim in this paper is to introduce the notions of sectional intuitionistic fuzzy continuous mapping, l-intuitionistic fuzzy compact sets and asymptotic intuitionistic fuzzy normal structure in an intuitionistic fuzzy n-normed linear space and to prove Schauder type and Baillon and Schoneberg type fixed point theorems [5] in an intuitionistic fuzzy n-normed linear space.

Definition 1.1 [16]. Let X be a linear space over the field $\mathbb{F}$ of real numbers. A fuzzy subset N of $X^n \times R$ is called a fuzzy n-norm on X if and only if:

(N1) For all $t \in \mathrm{R}$ with $t \leq 0$, $N(x_1, x_2, ..., x_n, t) = 0$.

(N2) For all $t \in \mathrm{R}$ with $t > 0$, $N(x_1, x_2, ..., x_n, t) = 1$ if and only if $x_1, x_2, ..., x_n$ are linearly dependent.

(N3) $N(x_1, x_2, ..., x_n, t)$ is invariant under any permutation of $x_1, x_2, ..., x_n$.

(N4) For all $t \in \mathrm{R}$ with $t > 0$, $N(x_1, x_2, ..., cx_n, t) = N(x_1, x_2, ..., x_n, \frac{t}{|c|})$, if $c \neq 0, c \in \mathbb{F}$.

(N5) For all $s, t \in R$

$$N(x_1, x_2, ..., x_n + x_n', s + t) \geq \min\left\{ N(x_1, x_2, ..., x_n, s),\ N(x_1, x_2, ..., x_n', t)\right\}.$$

(N6) $N(x_1, x_2, ..., x_n, t)$ is a non-decreasing function of $t \in R$ and $\lim_{t\to\infty} N(x_1, x_2, ..., x_n, t) = 1$.

The pair (X, N) is called a fuzzy n-normed linear space or in short f-n-NLS.

Theorem 1.2 [16]. Let (X, N) be a f-n-NLS. Assume the condition that (N7) $N(x_1, x_2, ..., x_n, t) > 0$ for all $t > 0$ implies $x_1, x_2, ..., x_n$ are linearly dependent. Define $||x_1, x_2, ..., x_n||_\alpha = \inf\{t : N(x_1, x_2, ..., x_n, t) \geq \alpha\}$, $\alpha \in (0,1)$. Then $\{\, ||\bullet, ..., \bullet||_\alpha : \alpha \in (0,1)\,\}$ is an ascending family of n-norms on X. We call these n-norms as α-n-norms on X corresponding to the fuzzy n-norm on X.

Definition 1.3 [17]. An intuitionistic fuzzy n-normed linear space or in short i-f-n-NLS is an object of the form

$A = \{(X, N(x_1, x_2, ...x_n, t), M(x_1, x_2, ..., x_n, t))/(x_1, x_2, ..., x_n) \in X^n\}$

where X is a linear space over the field $\mathbb{F}$, $*$ is a continuous t-norm, $\diamond$ is a continuous t-co -norm and N, M are fuzzy sets on $X^n \times (0, \infty)$; N denotes the degree of membership and M denotes the degree of non-membership of $(x_1, x_2, ..., x_n, t) \in X^n \times (0, \infty)$ satisfying the following conditions:

(1) $N(x_1, x_2, ..., x_n, t) + M(x_1, x_2, ..., x_n, t) \leq 1$.

(2) $N(x_1, x_2, ..., x_n, t) > 0$.

(3) $N(x_1, x_2, ..., x_n, t) = 1$ if and only if $x_1, x_2, ..., x_n$ are linearly dependent.

(4) $N(x_1, x_2, ..., x_n, t)$ is invariant under any permutation of $x_1, x_2, ..., x_n$.

(5) $N(x_1, x_2, ..., cx_n, t) = N(x_1, x_2, ..., x_n, \frac{t}{|c|})$ if $c \neq 0, c \in \mathbb{F}$.

(6) $N(x_1, x_2, ..., x_n, s) * N(x_1, x_2, ..., x_n', t) \leq N(x_1, x_2, ..., x_n + x_n', s + t)$.

(7) $N(x_1, x_2, ..., x_n, t) : (0, \infty) \to [0, 1]$ is continuous in t.

(8) $M(x_1, x_2, ..., x_n, t) > 0$.

(9) $M(x_1, x_2, ..., x_n, t) = 0$ if and only if $x_1, x_2, ..., x_n$ are linearly dependent.

(10) $M(x_1, x_2, ..., x_n, t)$ is invariant under any permutation of $x_1, x_2, ..., x_n$.

(11) $M(x_1, x_2, ..., cx_n, t) = M(x_1, x_2, ..., x_n, \frac{t}{|c|})$, if $c \neq 0$, $c \in \mathbb{F}$.

(12) $M(x_1, x_2, ..., x_n, s) \diamond M(x_1, x_2, ..., x_n', t) \geq M(x_1, x_2, ..., x_n + x_n', s + t)$.

(13) $M(x_1, x_2, ..., x_n, t) : (0, \infty) \to [0, 1]$ is continuous in t.

Remark 1.4. For convenience we denote an intuitionistic fuzzy n-normed linear space by $A = (X, N, M, *, \diamond)$.

Definition 1.5 [17]. Let A be an i-f-n-NLS. Assume the condition that
(14) For all $t > 0, N(x_1, x_2, ..., x_n, t) > 0$ implies $x_1, x_2, ..., x_n$ are linearly dependent. Define $||x_1, x_2, ..., x_n||_\alpha = \inf\{t > 0 : N(x_1, x_2, ..., x_n, t) \geq \alpha$ and $M(x_1, x_2, ..., x_n, t) \leq 1 - \alpha\}, \alpha \in (0, 1)$. Then $\{||\bullet, ..., \bullet||_\alpha : \alpha \in (0, 1)\}$ is an ascending family of n-norms on X. We call these n-norms as α-n-norms on X corresponding to the i-f-n-NLS A.

Definition 1.6 [3]. Let (X, N) be a fuzzy normed linear space satisfying (N7). Let X^*_α be the first dual space of $X, 0 < \alpha < 1$. A sequence $\{x_n\}$ in X is said to be l-fuzzy weakly convergent and converges to x_0 if for all $f \in X^*_\alpha$, $f(x_n) \to f(x_0)$ as $n \to \infty$.

Definition 1.7 [3]. A subset B of X is said to be l-fuzzy weakly compact if every sequence $\{x_n\}$ in B contains a subsequence converging l-fuzzy weakly to a point in B.

Definition 1.8 [4]. Let (X, N) be a fuzzy normed linear space satisfying (N7) and $K \subset X$. The intersection of all l-fuzzy closed sets containing K is called l-fuzzy closure of K and is denoted by $\overline{K}$.

Remark 1.9 [4]. If (X, N) is a fuzzy normed linear space and $K \subset X$, then $\overline{K}$ is closed in $(X, ||\bullet, ..., \bullet||_\alpha)$ and $\overline{K}^\alpha \subset \overline{K}$ $\forall \alpha \in (0, 1)$, where $\overline{K}^\alpha$ denotes the closure of K in $(X, ||\bullet, ..., \bullet||_\alpha)$.

2 Schauder Fixed Point Theorem

Definition 2.1. Let A be an i-f-n-NLS. X is said to be level intuitionistic fuzzy bounded (l-i-f-b) if for any $\alpha \in (0, 1)$ $\exists$ $t(\alpha) > 0$ such that $N(x_1, x_2, ..., x_n, t(\alpha)) \geq \alpha$ and $M(x_1, x_2, ..., x_n, t(\alpha)) \leq 1 - \alpha$ for all $(x_1, x_2, ..., x_n) \in X^n$.

Theorem 2.2. Let A be an i-f-n-NLS. Then X is l-intuitionistic fuzzy bounded if and only if X is bounded with respect to $||\bullet, ..., \bullet||_\alpha$ for all $\alpha \in (0, 1)$ where $||\bullet, ..., \bullet||_\alpha$ denotes the α-n-norm of X.

Proof. First suppose that X is l-i-f-b. Then for each $\alpha \in (0, 1)$ $\exists$ $t(\alpha) > 0$ such that $N(x_1, x_2, ..., x_n, t(\alpha)) \geq \alpha$ and $M(x_1, x_2, ..., x_n, t(\alpha)) \leq 1 - \alpha$ for all $(x_1, x_2, ..., x_n) \in X^n$. (2.1)
Now from the definition we have

$$||x_1, x_2, ..., x_n||_\alpha = \inf\{t > 0 : N(x_1, x_2, ..., x_n, t) \geq \alpha \text{ and } M(x_1, x_2, ..., x_n, t) \leq 1 - \alpha\}, \alpha \in (0, 1). \quad (2.2)$$

From (2.1) we have, $||x_1, x_2, ..., x_n||_\alpha \leq t(\alpha)$ for all $(x_1, x_2, ..., x_n) \in X^n$
$\Rightarrow X$ is bounded with respect to $||\bullet, ..., \bullet||_\alpha$.

Conversely, suppose that X is bounded with respect to $||\bullet, ..., \bullet||_\alpha; 0 < \alpha < 1$. Then for each $\alpha \in (0, 1) \exists t(\alpha)$ such that $||x_1, x_2, ..., x_n||_\alpha \leq t(\alpha)$ for all $(x_1, x_2, ..., x_n) \in X^n$.

That is, $||x_1, x_2, ..., x_n||_\alpha \leq t(\alpha) < t(\alpha) + 1$ for all $(x_1, x_2, ..., x_n) \in X^n$.
By (2.2)$N(x_1, x_2...., x_n, t(\alpha) + 1) \geq \alpha$ and $M(x_1, x_2, ..., x_n, t(\alpha) + 1) \leq 1 - \alpha$ for all $(x_1, x_2, ..., x_n) \in X^n \Rightarrow X$ is l-i-f-b. □

Definition 2.3. Let A be an i-f-n-NLS. A subset B of X is said to be l-intuitionistic fuzzy closed (l-i-f-closed) if for each $\alpha \in (0,1)$ and for any sequence $\{x_n\}$ in B and $x \in X$, $\lim_{n\to\infty} N(x_1, x_2, ..., x_{n-1}, x_n - x, t) \geq \alpha$ and $\lim_{n\to\infty} M(x_1, x_2, ..., x_{n-1}, x_n - x, t) \leq 1 - \alpha$ for all $t > 0 \Rightarrow x \in B$.

Theorem 2.4. Let A be an i-f-n-NLS and $B \subset X$. Then B is l-i-f-closed if and only if B is closed with respect to $||\bullet, ..., \bullet||_\alpha$ for each $\alpha \in (0,1)$.

Proof. First suppose that B is l-i-f-closed. Take $\alpha_0 \in (0,1)$. Let $\{x_n\}$ be a sequence in B such that $\lim_{n\to\infty} ||x_1, x_2, ..., x_{n-1}, x_n - x||_{\alpha_0} = 0$. Then for a given $\epsilon > 0$, $\exists$ a positive integer $N(\epsilon)$ such that $||x_1, x_2, ..., x_{n-1}, x_n - x||_{\alpha_0} < \epsilon$, for all $n \geq N(\epsilon)$.

$\Rightarrow N(x_1, x_2, ..., x_{n-1}, x_n - x, \epsilon) \geq \alpha_0$ for all $n \geq N(\epsilon)$.

$\Rightarrow \lim_{n\to\infty} N(x_1, x_2, ..., x_{n-1}, x_n - x, \epsilon) \geq \alpha_0$ for all $t > 0$,

since ϵ is arbitrary and by condition (1) of Definition 1.3 it follows that $\lim_{n\to\infty} M(x_1, x_2, ..., x_{n-1}, x_n - x, \epsilon) \leq 1 - \alpha_0$

$\Rightarrow x \in B.$

$\Rightarrow B$ is closed with respect to $||\bullet, ..., \bullet||_{\alpha_0}$.

Since $0 < \alpha_0 < 1$ is arbitrary, it follows that B is closed with respect to $||\bullet, ..., \bullet||_\alpha, 0 < \alpha < 1$.

Converse follows by conditions (1),(7) and (13) of Definition 1.3. □

Definition 2.5. Let A be an i-f-n-NLS. A subset B of X is said to be l-intuitionistic fuzzy compact if for any sequence $\{x_n\}$ and for each $\alpha \in (0,1)$, $\exists$ a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $x \in B$ both depending on $\{x_n\}$ and α such that $\lim_{k\to\infty} N(x_{n_1}, x_{n_2}, ..., x_{n_{k-1}}, x_{n_k} - x, t) \geq \alpha$ and $\lim_{k\to\infty} M(x_{n_1}, x_{n_2}, ..., x_{n_{k-1}}, x_{n_k} - x, t) \leq 1 - \alpha \,\forall\, t > 0.$

Lemma 2.6. Let A be an i-f-n-NLS satisfying (14). Then a subset B of X is l-intuitionistic fuzzy compact if and only if B is compact with respect to $||\bullet, ..., \bullet||_\alpha$ for each $\alpha \in (0,1)$.

Proof. First suppose that B is l-intuitionistic fuzzy compact.
Take $\alpha_0 \in (0,1)$. Let $\{x_n\}$ be a sequence in B. Thus $\exists$ a subsequence $\{x_{n_k}\}$ and x in B (both depending on α_0) such that $\lim_{k\to\infty} N(x_{n_1}, x_{n_2}, ..., x_{n_{k-1}}, x_{n_k} - x, t) \geq \alpha_0$ and

$$\lim_{k\to\infty} M(x_{n_1}, x_{n_2}, ..., x_{n_{k-1}}, x_{n_k} - x, t) \leq 1 - \alpha_0 \,\forall\, t > 0.$$

$\Rightarrow$ for a given $\epsilon > 0$ with $\alpha_0 - \epsilon > 0$ and for a $t > 0$ $\exists$ a positive integer $K(\epsilon, t)$ such that $N(x_{n_1}, x_{n_2}, ..., x_{n_{k-1}}, x_{n_k} - x, t) \geq \alpha_0 - \epsilon$ and $M(x_{n_1}, x_{n_2}, ..., x_{n_{k-1}}, x_{n_k} - x, t) \leq 1 - \alpha_0 + \epsilon \,\forall\, k \geq K(\epsilon, t)$

$\Rightarrow \left\|x_{n_1}, x_{n_2}, ..., x_{n_{k-1}}, x_{n_k} - x\right\|_{\alpha_0 - \epsilon} \leq t \,\forall\, k \geq K(\epsilon, t)$

$\Rightarrow \lim_{k\to\infty} \left\|x_{n_1}, x_{n_2}, ..., x_{n_{k-1}}, x_{n_k} - x\right\|_{\alpha_0 - \epsilon} = 0$

$\Rightarrow B$ is compact with respect to $||\bullet, ..., \bullet||_{\alpha_0 - \epsilon}$.

Since $\alpha_0 \in (0,1)$ and $\epsilon > 0$ are arbitrary it follows that B is compact with respect to $||\bullet, ..., \bullet||_\alpha$ for each $\alpha \in (0,1)$.
Converse follows from Theorem 2.2 and 2.4. □

In what follows $A = (X, N_1, M_1, *, \diamond)$ and $B = (Y, N_2, M_2, *', \diamond')$ will denote two intuitionistic fuzzy n-normed linear spaces, where X and Y are linear spaces over the same real field.

Definition 2.7. Let A and B be i-f-n-NLS. A mapping $T : A \to B$ is said to be sectional intuitionistic fuzzy continuous at $x_0 = (x_{01}, x_{02}, ..., x_{0n}) \in X^n$, if $\exists\, \alpha \in (0,1)$ such that for each $\epsilon > 0, \exists\, \delta > 0$ such that,
$N_1(x - x_0, \delta) \geq \alpha$ and $M_1(x - x_0, \delta) \leq 1 - \alpha$
$\Rightarrow N_2(T(x) - T(x_0), \epsilon) \geq \alpha$ and $M_2(T(x) - T(x_0), \epsilon) \leq 1 - \alpha$, for every $x = (x_1, x_2, ..., x_n) \in X^n$.

Lemma 2.8. Let A and B be i-f-n-NLS satisfying (14). Then a mapping $T : A \to B$ is sectional intuitionistic fuzzy continuous if and only if $T : (X, ||\bullet, ..., \bullet||^1_\alpha) \to (Y, ||\bullet, ..., \bullet||^2_\alpha)$ is continuous for some $\alpha \in (0,1)$.

Proof. First suppose that $T : A \to B$ is sectional intuitionistic fuzzy continuous. Thus $\forall\ y \in X^n$, $\exists\, \alpha_0$ (say) $\in (0,1)$ such that for each $\epsilon > 0$, $\exists\ \delta > 0$ and $N_1(x - y, \delta) \geq \alpha_0$ and $M_1(x - y, \delta) \leq 1 - \alpha_0$
$\Rightarrow N_2(T(x) - T(y), \epsilon) \geq \alpha_0$ and $M_2(T(x) - T(y), \epsilon) \leq 1 - \alpha_0\ \forall\, x \in X^n$.
Choose η_0 such that $\delta_1 = \delta - \eta_0 > 0$. Let $||x - y||^1_{\alpha_0} \leq \delta - \eta_0 = \delta_1$.
Then $||x - y||^1_{\alpha_0} \leq \delta - \eta_0 < \delta$
$\Rightarrow N_1(x - y, \delta) \geq \alpha_0$ and $M_1(x - y, \delta) \leq 1 - \alpha_0$
$\Rightarrow N_2(T(x) - T(y), \epsilon) \geq \alpha_0$ and $M_2(T(x) - T(y), \epsilon) \leq 1 - \alpha_0$
$\Rightarrow ||T(x) - T(y)||^2_{\alpha_0} \leq \epsilon$.
Thus T is continuous with respect to $||\bullet, ..., \bullet||^1_{\alpha_0}$ and $||\bullet, ..., \bullet||^2_{\alpha_0}$.
Next let T be continuous with respect to $||\bullet, ..., \bullet||^1_{\alpha_0}$ and $||\bullet, ..., \bullet||^2_{\alpha_0}$. Thus $\forall y \in X^n$ and $\epsilon > 0 \exists \delta > 0$ such that $||x - y||^1_{\alpha_0} \leq \delta \Rightarrow ||T(x) - T(y)||^2_{\alpha_0} \leq \frac{\epsilon}{2}$.
Let $N_1(x - y, \delta) \geq \alpha_0$ and $M_1(x - y, \delta) \leq 1 - \alpha_0$. Then $||x - y||^1_{\alpha_0} \leq \delta$
$\Rightarrow ||T(x) - T(y)||^2_{\alpha_0} \leq \frac{\epsilon}{2} < \epsilon$
$\Rightarrow N_2(T(x) - T(y), \epsilon) \geq \alpha_0$ and $M_2(T(x) - T(y), \epsilon) \leq 1 - \alpha_0$.
Therefore $N_1(x - y, \delta) \geq \alpha_0$ and $M_1(x - y, \delta) \leq 1 - \alpha_0$
$\Rightarrow N_2(T(x) - T(y), \epsilon) \geq \alpha_0$ and $M_2(T(x) - T(y), \epsilon) \leq 1 - \alpha_0$.
This completes the proof. □

Theorem 2.9. (Schauder fixed point theorem). Let K be a non-empty convex, l-intuitionistic fuzzy compact subset of an i-f-n-NLS A satisfying (14) and $T : K \to K$ be sectional intuitionistic fuzzy continuous. Then T has a fixed point.

Proof. Since A satisfies (14), $(X, ||\bullet, ..., \bullet||_\alpha)$ is an α-n-normed linear space. As K is a l-intuitionistic fuzzy compact subset of X, K is a compact subset of $(X, ||\bullet, ..., \bullet||_\alpha)$ for each $\alpha \in (0,1)$ by Lemma 2.6.
Again since $T : K \to K$ is sectional intuitionistic fuzzy continuous $\exists\, \alpha_0$ (say) $\in (0,1)$ such that $T : K \to K$ is continuous with respect to $(X, ||\bullet, ..., \bullet||_{\alpha_0})$ by Lemma 2.8.

Hence it follows that K is a non-empty, convex and compact subset of the n-normed linear space $(X, ||\bullet, ..., \bullet||_{\alpha_0})$ and $T : K \to K$ is a continuous mapping. So by Schauder fixed point theorem [18] it follows that T has a fixed point. □

We also give another version of Schauder fixed point theorem in i-f-n-NLS.

Theorem 2.10. Let K be a non-empty, l-i-f-closed, convex subset of an i-f-n-NLS A satisfying (14) and let $T : K \to K$ be sectional intuitionistic fuzzy continuous with $\overline{T}(K)$ being l-intuitionistic fuzzy compact.Then T has a fixed point in K.

Proof. Since A satisfies (14) it follows from Theorem 2.6 of [17] that $(X, ||\bullet, ..., \bullet||_{\alpha})$ is an α-n-NLS. Again since K is l-i-f-closed, K is closed with respect to $||\bullet, ..., \bullet||_{\alpha}$ for each $\alpha \in (0, 1)$ (by Theorem 2.4). Now $T : K \to K$ is sectional intuitionistic fuzzy continuous.
Thus $\exists\, \alpha_0$ (say) $\in (0, 1)$ such that $T : K \to K$ is continuous with respect to $||\bullet, ..., \bullet||_{\alpha_0}$.
Also since $\overline{T}(K)$ is l-intuitionistic fuzzy compact by Lemma 2.6 $\overline{T}(K)$ is compact with respect to $||\bullet, ..., \bullet||_{\alpha}$ for each $\alpha \in (0, 1)$.
In particular $\overline{T}(K)$ is compact in $(X, ||\bullet, ..., \bullet||_{\alpha_0})$.
Also from Remark 1.9 $\overline{T}(K)$ is closed in $(X, ||\bullet, ..., \bullet||_{\alpha_0})$ and $\overline{T}(K)^{\alpha_0} \subset \overline{T}(K)$ where $\overline{T}(K)^{\alpha_0}$ is the closure of $T(K)$ in $(X, ||\bullet, ..., \bullet||_{\alpha_0})$.
So $\overline{T}(K)^{\alpha_0}$ is compact in $(X, ||\bullet, ..., \bullet||_{\alpha_0})$.

Thus K is a non-empty closed, convex subset of a n-normed linear space $(X, ||\bullet, ..., \bullet||_{\alpha_0})$ and $T : K \to K$ is continuous with $\overline{T}(K)^{\alpha_0}$ compact.

Therefore by Schauder fixed point theorem [18] it follows that T has a fixed point. □

3 Asymptotic Intuitionistic Fuzzy Normal Structure

In this section we introduce the idea of asymptotic intuitionistic fuzzy normal structure and establish a Baillon and Schoneberg type fixed point theorem in i-f-n-NLS.

Definition 3.1. Let A be an i-f-n-NLS and $K^n (\neq \phi) \subset X^n$,
$K^n = \underbrace{K \times ... \times K}_{n}$, $K \subset X$. We define the intuitionistic fuzzy diameter of K^n as

$$\sup_{\alpha\in(0,1)} \{\inf\{t > 0 : N(u_1 - v_1, u_2 - v_2, ..., u_n - v_n, t) \geq \alpha,$$
$$M(u_1 - v_1, u_2 - v_2, ...u_n - v_n, t) \leq 1 - \alpha \text{ for all } u = (u_1, u_2, ..., u_n),$$
$$v = (v_1, v_2, ..., v_n) \in K^n\}\} \text{(or) simply}$$
$$\sup_{\alpha\in(0,1)} \{\inf\{t > 0 : N(u - v, t) \geq \alpha, M(u - v, t) \leq 1 - \alpha, \text{ for all } u, v \in K^n\}\}$$

and denote it as f-$\delta(K^n)$.

Remark 3.2. Let A be an i-f-n-NLS satisfying (14) and $K^n \subset X^n$. Let $x = (x_1, x_2, ..., x_n), y = (y_1, y_2, ..., y_n) \in K^n$. Then $\sup\{||x - y|| : x, y \in K^n\} = \sup_{\alpha\in(0,1)} ||x_1 - y_1, x_2 - y_2, ..., x_n - y_n||_\alpha$ is denoted as $\alpha\text{-}\delta(K^n)$.

Theorem 3.3. Let A be an i-f-n-NLS satisfying (14) and $K^n \subset X^n$. Then $f\text{-}\delta(K^n) = \sup_{\alpha\in(0,1)} \{\alpha\text{-}\delta(K^n)\}$.

Proof. If K is singleton then clearly $\alpha\text{-}\delta(K^n) = f\text{-}\delta(K^n) = 0$ for all $\alpha \in (0,1)$. Assume that K is not singleton. Now $k > f\text{-}\delta(K^n)$

$$\Rightarrow k > \sup_{\alpha\in(0,1)} \{\inf\{t > 0 : N(u - v, t) \geq \alpha, M(u - v, t) \leq 1 - \alpha \text{ for all } u, v \in K^n\}\}$$
$$\Rightarrow N(u-v,k) \geq \alpha, M(u-v,k) \leq 1-\alpha \text{ for all } u, v \in K^n,\ \alpha \in (0,1)$$
$$\Rightarrow N(u - v, k) = 1, M(u - v, k) = 0 \text{ for all } u, v \in K^n$$
$$\Rightarrow \|u - v\|_\alpha \leq k \text{ for all } u, v \in K^n \text{ and for all } \alpha \in (0,1)$$
$$\Rightarrow \sup\{\|u - v\|_\alpha : u, v \in K^n\} \leq k \text{ for all } \alpha \in (0,1)$$
$$\Rightarrow \alpha\text{-}\delta(K^n) \leq k \text{ for all } \alpha \in (0,1)$$
$$\Rightarrow f\text{-}\delta(K^n) \geq \alpha\text{-}\delta(K^n) \text{ for all } \alpha \in (0,1)$$
$$\Rightarrow f\text{-}\delta(K^n) \geq \sup_{\alpha\in(0,1)} \{\alpha\text{-}\delta(K^n)\} \tag{3.1}$$

Also $f\text{-}\delta(K^n) > k$

$$\Rightarrow \sup_{\alpha\in(0,1)} \{\inf\{t > 0 : N(u - v, t) \geq \alpha, M(u - v, t) \leq 1 - \alpha \text{ for all } u, v \in K^n\}\} > k$$
$$\Rightarrow \exists\alpha_0 \in (0,1) \text{ such that } \inf\{t > 0 : N(u - v, t) \geq \alpha_0, M(u - v, t) \leq 1 - \alpha_0, \text{ for all } u, v \in K^n\} > k$$
$$\Rightarrow \exists u_0, v_0 \in K^n \text{ such that } N(u_0 - v_0, k) < \alpha_0, M(u_0 - v_0, k) > 1 - \alpha_0 \tag{3.2}$$

Now by (3.2)
$\|u_0 - v_0\|_{\alpha_0} = \inf\{t > 0 : N(u_0 - v_0, t) \geq \alpha_0, M(u_0 - v_0, t) \leq 1 - \alpha_0\} \geq k$.
So $\alpha_0\text{-}\delta(K^n) = \sup\{\|u - v\|_{\alpha_0} : u, v \in (K^n)\} \geq \|u_0 - v_0\|_{\alpha_0} \geq k$.
Thus $\sup_{\alpha\in(0,1)} \{\alpha\text{-}\delta(K^n)\} \geq \alpha_0\text{-}\delta(K^n) \geq k$

$$\Rightarrow \sup_{\alpha\in(0,1)} \{\alpha\text{-}\delta(K^n)\} \geq k$$
$$\Rightarrow \sup_{\alpha\in(0,1)} \{\alpha\text{-}\delta(K^n)\} \geq f\text{-}\delta(K^n) \tag{3.3}$$

From (3.1) and (3.3) we get $f\text{-}\delta(K^n) = \sup_{\alpha\in(0,1)} \{\alpha\text{-}\delta(K^n)\}$. □

Definition 3.4. Let A be an i-f-n-NLS satisfying (14). A mapping $T : A \to A$ is said to be intuitionistic fuzzy non-expansive mapping if $N(T(x) - T(y), t) \geq N(x - y, t)$ and $M(T(x) - T(y), t) \geq M(x - y, t)$ $\forall\, x, y \in X^n, \forall\, t \in R$.

Definition 3.5. Let A be an i-f-n-NLS satisfying (14). A convex subset $K^n \subset X^n$ where $K^n = \underbrace{K \times ... \times K}_{n}$ is said to have asymptotic intuitionistic fuzzy normal structure if for any l-intuitionistic fuzzy bounded convex subset

H^n of K^n and each sequence $\{x_n\}$ in H^n for which
$\lim\limits_{n\to\infty} N(x_1, x_2, ..., x_n - x_{n+1}, t) = 1$ and
$\lim\limits_{n\to\infty} M(x_1, x_2, ..., x_n - x_{n+1}, t) = 0 \,\forall\, t > 0$, $\exists\, x_0 \in H^n$ such that

$$\sup_{\alpha\in(0,1)} \{\sup_{r>0}\{\inf_{k\geq r}\{\inf\{t > 0 : N(x_1, x_2, ..., x_k - x_0, t) \geq \alpha \text{ and } M(x_1, x_2, ..., x_k - x_0, t) \leq 1 - \alpha\}\}\}\} < \text{f}\text{-}\delta(H^n)$$

where f - $\delta(H^n)$ denotes the intuitionistic fuzzy diameter of H^n.

Proposition 3.6. Let A be an i-f-n-NLS satisfying (14) and $\{x_n\}$ be a sequence in X. Then $\lim\limits_{n\to\infty} N(x_1, x_2, ..., x_n - x, t) = 1$ and $\lim\limits_{n\to\infty} M(x_1, x_2, ..., x_n - x, t) = 0 \,\forall\, t > 0$ if and only if $\lim\limits_{n\to\infty} ||x_1, x_2, ..., x_n - x||_\alpha = 0 \,\forall\, \alpha \in (0, 1)$ where $||\bullet, ..., \bullet||_\alpha$ denotes the corresponding α-n-norm of N.

Proof. Let $\{x_n\}$ be a sequence in X such that $x_n \to x$ (w.r.t N). Thus $\lim\limits_{n\to\infty} N(x_1, x_2, ..., x_n - x, t) = 1$ and $\lim\limits_{n\to\infty} M(x_1, x_2, ..., x_n - x, t) = 0 \,\forall\, t > 0$.
Choose $0 < \alpha < 1$. Then $\exists$ a positive integer $n_0(\alpha, t)$ such that

$$N(x_1, x_2, ..., x_n - x, t) \geq \alpha \text{ and } M(x_1, x_2, ..., x_n - x, t) \leq 1 - \alpha \,\forall\, n \geq n_0(\alpha, t) \tag{3.4}$$

By definition

$$||x_1, x_2, ..., x_n||_\alpha = \inf\{t > 0 : N(x_1, x_2, ..., x_n, t) \geq \alpha \text{ and } M(x_1, x_2, ..., x_n, t) \leq 1-\alpha\} \tag{3.5}$$

Thus from (3.4) it follows that $||x_1, x_2, ..., x_n - x||_\alpha \leq t \,\forall\, n \geq n_0(\alpha, t)$.
Since $t > 0$ is arbitrary it follows that $||x_1, x_2, ..., x_n - x||_\alpha \to 0$ as $n \to \infty \,\forall\, \alpha \in (0, 1)$.

Conversely suppose that, $||x_1, x_2, ..., x_n - x||_\alpha \to 0$ as $n \to \infty \,\forall\, \alpha \in (0, 1)$. Then for $\alpha \in (0, 1)$ and $\epsilon > 0, \exists$ a positive integer $n_0(\alpha, \epsilon)$ such that $||x_1, x_2, ..., x_n - x||_\alpha < \epsilon \,\forall\, n \geq n_0(\alpha, \epsilon)$.
From (3.5) we have

$$\epsilon > ||x_1, x_2, ..., x_n - x||_\alpha = \inf\{t > 0 : N(x_1, x_2, ..., x_n - x, t) \geq \alpha \text{ and } M(x_1, x_2, ..., x_n - x, t) \leq 1 - \alpha\} \,\forall\, n \geq n_0(\alpha, \epsilon).$$

$\Rightarrow \exists\, t_n > 0, \epsilon > t_n > 0$ such that $N(x_1, x_2, ..., x_n - x, t_n) \geq \alpha$ and $M(x_1, x_2, ..., x_n - x, t_n) \leq 1 - \alpha \,\forall\, n \geq n_0(\alpha, \epsilon)$.
$\Rightarrow N(x_1, x_2, ..., x_n - x, \epsilon) \geq \alpha$ and $M(x_1, x_2, ..., x_n - x, \epsilon) \leq 1 - \alpha \,\forall\, n \geq n_0(\alpha, \epsilon)$.
$\Rightarrow \lim\limits_{n\to\infty} N(x_1, x_2, ..., x_n - x, \epsilon) \geq \alpha$ and $\lim\limits_{n\to\infty} M(x_1, x_2, ..., x_n - x, \epsilon) \leq 1 - \alpha$
$\Rightarrow \lim\limits_{n\to\infty} N(x_1, x_2, ..., x_n - x, \epsilon) = 1$ and $\lim\limits_{n\to\infty} M(x_1, x_2, ..., x_n - x, \epsilon) = 0$, since $\alpha \in (0, 1)$ is arbitrary
$\Rightarrow x_n \to x$ (w.r.t N). This completes the proof. □

Lemma 3.7. If a convex subset K^n of an i-f-n-NLS A satisfying (14) has asymptotic intuitionistic fuzzy normal structure then $\exists \alpha_0$ (say) $\in (0, 1)$ such that K^n has asymptotic normal structure with respect to $||\bullet, ..., \bullet||_{\alpha_0}$ where $||\bullet, ..., \bullet||_\alpha$ is an α-n-norm $(0 < \alpha < 1)$ corresponding to the i-f-n-NLS A.

Proof. Since A is an i-f-n-NLS it follows that $(X, ||\bullet, ..., \bullet||_\alpha)$ is a n-normed linear space for each $\alpha \in (0,1)$.

Suppose K^n has asymptotic intuitionistic fuzzy normal structure. Let H^n be an l-intuitionistic fuzzy bounded convex subset of K^n and $\{x_n\}$ be a sequence in H^n such that $\lim_{n\to\infty} N(x_1, x_2, ..., x_n - x, t) = 1$ and

$$\lim_{n\to\infty} M(x_1, x_2, ..., x_n - x, t) = 0\,\forall\, t > 0\,.$$

We know that, H^n is l-intuitionistic fuzzy bounded if and only if H^n is bounded with respect to $||\bullet, ..., \bullet||_\alpha$ for each $\alpha \in (0,1)$ by Theorem 2.2.
Hence $\lim_{n\to\infty} N(x_1, x_2, ..., x_n - x, t) = 1$ and

$\lim_{n\to\infty} M(x_1, x_2, ..., x_n - x, t) = 0\,\forall\, t > 0$ if and only if $||x_1, x_2, ..., x_n - x||_\alpha \to 0$ as $n \to \infty\,\forall\,\alpha \in (0,1)$ by Proposition 3.6.
Since K^n has asymptotic intuitionistic fuzzy normal structure $\exists\, x_0 \in H^n$ such that

$$\sup_{\alpha\in(0,1)}\{\sup_{r>0}\{\inf_{k\geq r}\{\inf\{t > 0 : N(x_1, x_2, ...x_k - x_0, t) \geq \alpha \text{ and } M(x_1, x_2, ...x_k - x_0, t) \leq 1 - \alpha\}\}\}\} < \text{f - }\delta(H^n)\,.$$

Choose $s > 0$ such that

$$\sup_{\alpha\in(0,1)}\{\sup_{s>0}\{\inf_{k\geq s}\{\inf\{t > 0 : N(x_1, x_2, ...x_k - x_0, t) \geq \alpha \text{ and } M(x_1, x_2, ...x_k - x_0, t) \leq 1 - \alpha\}\}\}\} < r < \text{f - }\delta(H^n) \quad (3.6)$$

Now

$$\sup_{\alpha\in(0,1)}\{\sup_{s>0}\{\inf_{k\geq s}\{\inf\{t > 0 : N(x_1, x_2, ...x_k - x_0, t) \geq \alpha \text{ and } M(x_1, x_2, ...x_k - x_0, t) \leq 1 - \alpha\}\}\}\} < r$$

$$\Rightarrow \sup_{s>0}\{\inf_{k\geq s}\{||x_1, x_2, ..., x_k - x_0||_\alpha\}\} < r\,\forall\,\alpha \in (0,1)$$

$$\Rightarrow \lim_{k\to\infty} ||x_1, x_2, ..., x_k - x_0||_\alpha < r\,\forall\,\alpha \in (0,1)$$

Now from (3.6) we have

$$\lim_{k\to\infty} ||x_1, x_2, ..., x_k - x_0||_\alpha < \text{f - }\delta(H^n)\,\forall\,\alpha \in (0,1)$$

$$\Rightarrow \lim_{k\to\infty} ||x_1, x_2, ..., x_k - x_0||_\alpha < \sup_{\alpha\in(0,1)}\{\alpha - \delta(H^n)\}\,\forall\,\alpha \in (0,1)$$

$\Rightarrow\ \exists\,\alpha_0$ say $\in (0,1)$ such that

$$\lim_{k\to\infty} ||x_1, x_2, ..., x_k - x_0||_\alpha < \alpha_0 - \delta(H^n)\,\forall\,\alpha \in (0,1)$$

(ie) $\lim_{k\to\infty} ||x_1, x_2, ..., x_k - x_0||_\alpha < \alpha_0 - \delta(H^n)$

This implies that K^n has asymptotic normal structure with respect to $||\bullet, ..., \bullet||_{\alpha_0}$. □

Theorem 3.8. (Baillon and Schoneberg type fixed point theorem) Let A be an i-f-n-NLS satisfying (14). Let K^n be a non-empty l-intuitionistic weakly compact convex subset of X^n having asymptotic intuitionistic fuzzy normal structure and $T : K^n \to K^n$ be an intuitionistic fuzzy non-expansive mapping. Then T has a fixed point.

Proof. Follows directly from Lemma 3.7. □

4 Open Problem

We suggest the following open problems for further research.
1. Introduce the concept of random fuzzy n-normed linear space and study its properties.
2. Construct α-n-norms in random fuzzy n-normed linear space and study its properties.
3. Establish the conditions under which fixed point theorems hold in random fuzzy n-normed linear space.

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