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On Some Feng Qi Type h-Integral Inequalities

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Abstract

In this paper, several Feng Qi type h-integral inequalities are given by using elementary analytic methods in h-Calculus.

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1 Introduction

In [7] the following problem was posed: Under what conditions does the inequality

$$\int_a^b [f(x)]^t dx \geq \Big[\int_a^b f(x)dx\Big]^{t-1} \tag{1}$$

holds for $t > 1$?

In [1] it has been proved the following: Let $[a, b]$ be a closed interval of $\mathbb{R}$ and let $p \geq 1$ be a real number. For any real continuous function f on $[a, b]$, differentiable on $]a, b[$, such that $f(a) \geq 0$, and $f'(x) \geq p$ for all $x \in]a, b[$, we have that

$$\int_a^b [f(x)]^{p+2} dx \geq \frac{1}{(b-a)^{p-1}}\Big[\int_a^b f(x)dx\Big]^{p+1}. \tag{2}$$

In [2] it has been obtained the q-analogue of the previous result as follows. Let $p \geq 1$ be a real number and f be a function defined on $[a, b]_q$ (see below for the definitions and notation), such that $f(a) \geq 0$, and $D_q f(x) \geq p$ for all $x \in (a, b]_q$. Then

$$\int_a^b [f(x)]^{p+2} d_q x \geq \frac{1}{(b-a)^{p-1}}\Big[\int_a^b f(qx)d_q x\Big]^{p+1}. \tag{3}$$

The aim of this paper is to extend this result. We also note that in [6], Theorems 1 and 2, are valid only if $\beta \geq 2$ and not $\beta \geq 1$ as it is stated there. This paper will also provide some more sufficient conditions such that inequalities presented in [6] are valid.

2 Notations and preliminaries

For the convenience of the reader, we provide a summary of notations and definitions used in this paper. For details, one may refer to [3] and [5].

Let $h \neq 0$. A quantum derivative of a function f, that is characterized by an additive parameter h, is the h derivative, denoted by $D_h f$ and given by

$$D_h f(x) = \frac{f(x+h) - f(x)}{h}. \tag{4}$$

One can easily verify the product and quotient rules for h-differentiation.

$$D_h(f(x)g(x)) = f(x)D_h g(x) + g(x+h)D_h f(x). \tag{5}$$

$$D_h\Big(\frac{f(x)}{g(x)}\Big) = \frac{g(x)D_h f(x) - f(x)D_h g(x)}{g(x)g(x+h)}. \tag{6}$$

If $f'(0)$ exists, then $D_h f(0) = f'(0)$. As h tends to 0, the h-derivative reduces to the usual derivative.

If $b - a \in h\mathbb{Z}$, the definite h-integral is defined by (see [5])

$$\int_a^b f(x)d_h x = \begin{cases} h\Big(f(a) + f(a+h) + \ldots + f(b-h)\Big) & \text{if } a < b \\ 0 & \text{if } a = b \\ -h\Big(f(b) + f(b+h) + \ldots + f(a-h)\Big) & \text{if } a > b \end{cases} \tag{7}$$

From the above definition, one can see that the definite h-integral is a Riemann sum of f on the interval $[a, b]$, which is partitioned into subintervals of equal width.

The following Theorem, whose proof can be found in [5], justifies (7) as an appropriate definition for the h-integral.

Theorem 1 *(Fundamental Theorem of h-calculus) If F is an h-antiderivative of f and $b - a \in h\mathbb{Z}$ then*

$$\int_a^b f(x)d_h(x) = F(b) - F(a). \tag{8}$$

Applying Theorem 1 to $D_h(f(x)g(x))$ and using (5), one obtains the h-analogue of integration by parts.

$$\int_a^b f(x)d_hg(x) = f(b)g(b) - f(a)g(a) - \int_a^b g(x+h)d_hf(x). \tag{9}$$

For any function f one has

$$D_h\Big(\int_a^x f(t)d_ht\Big) = f(x). \tag{10}$$

3 Main Results

In order to prove our main results we need the following Lemma, which is h-analogue of Lemma 3.1 from [2].

Lemma 2 *Let $p \geq 1$ and g be a non-negative, increasing function on $[a,b]$. Then*

$$pg^{p-1}(x)D_hg(x) \leq D_h[g^p(x)] \leq pg^{p-1}(x+h)D_hg(x). \tag{11}$$

Proof. We have

$$D_h(g^p(x)) = \frac{g^p(x+h) - g^p(x)}{h} = \frac{p}{h}\int_{g(x)}^{g(x+h)} t^{p-1}dt. \tag{12}$$

Since g is non-negative increasing function we have:

$$g^{p-1}(x)(g(x+h)-g(x)) \leq \int_{g(x)}^{g(x+h)} t^{p-1}dt \leq g^{p-1}(x+h)(g(x+h)-g(x)). \tag{13}$$

Therefore by (12) and (13) one has:

$$pg^{p-1}(x)D_hg(x) \leq D_h[g^p(x)] \leq pg^{p-1}(x+h)D_hg(x).$$

■

Theorem 3 *If f is a non-negative increasing function on $[a,b]$ and satisfies*

$$f^{t-2}(x)D_hf(x) \geq (t-2)(x+2h-a)^{t-3}f^{t-2}(x+2h), \tag{14}$$

for $t \geq 3$ and $b - a \in h\mathbb{Z}$ then

$$\int_a^b f^t(x)d_hx \geq \Big(\int_a^b f(x)d_hx\Big)^{t-1}. \tag{15}$$

Proof. For $x \in [a,b]$, let

$$F(x) = \int_a^x f^t(u)d_h u - \Big(\int_a^x f(u)d_h u\Big)^{t-1}$$

and $g(x) = \displaystyle\int_a^x f(u)d_h u$. By virtue of Lemma 2, it follows that

$$\begin{aligned} D_h F(x) &= f^t(x) - D_h(g^{t-1}(x)) \\ &\geq f^t(x) - (t-1)g^{t-2}(x+h)f(x) \\ &= f(x)F_1(x), \end{aligned}$$

where $F_1(x) = f^{t-1}(x) - (t-1)g^{t-2}(x+h)$. By Lemma 2 we have

$$\begin{aligned} D_h F_1(x) &= D_h(f^{t-1}(x)) - (t-1)D_h(g^{t-2}(x+h)) \\ &\geq (t-1)f^{t-2}(x)D_h f(x) - (t-1)(t-2)g^{t-3}(x+2h)D_h g(x+h) \\ &\geq (t-1)f^{t-2}(x)D_h f(x) - (t-1)(t-2)g^{t-3}(x+2h)f(x+h). \end{aligned}$$

Since f is a non-negative and increasing function, then

$$g(x+2h) = \int_a^{x+2h} f(u)d_h t \leq (x+2h-a)f(x+2h),$$

hence

$$\begin{aligned} D_h F_1(x) &\geq (t-1)f^{t-2}(x)D_h f(x) - (t-1)(t-2)(x+2h-a)^{t-3}f^{t-3}(x+2h)f(x+h) \\ &\geq (t-1)f^{t-2}(x)D_h f(x) - (t-1)(t-2)(x+2h-a)^{t-3}f^{t-2}(x+2h) \\ &= (t-1)\Big(f^{t-2}(x)D_h f(x) - (t-2)(x+2h-a)^{t-3}f^{t-2}(x+2h)\Big) \geq 0. \end{aligned}$$

We conclude that F_1 is increasing function. Hence $F_1(x) \geq F_1(a) \geq 0$ which means that $D_h F(x) \geq 0$. So F is increasing and since $F(x) \geq F(a) = 0$ the proof is completed. ■

Theorem 4 *Let $p \geq 1$. If f is a non-negative and increasing function on $[a,b]$ and satisfies*

$$f^p(x)D_h f(x) \geq \frac{p(x+2h-a)}{(b-a)^{p-1}} \cdot f^p(x+2h) \tag{16}$$

then

$$\int_a^b f^{p+2}(x)d_h x \geq \frac{1}{(b-a)^{p-1}}\Big(\int_a^b f(x)d_h x\Big)^{p+1}. \tag{17}$$

Proof. For $x \in [a,b]$ let

$$F(x) = \int_a^x f^{p+2}(u) d_h t - \frac{1}{(b-a)^{p-1}} \Big(\int_a^x f(u) d_h(u) \Big)^{p-1}$$

and $g(x) = \displaystyle\int_a^x f(u) d_h u$. Utilizing Lemma 2 gives that

$$\begin{aligned} D_h F(x) &= f^{p+2}(x) - \frac{1}{(b-a)^{p-1}} D_h(g^{p+1}(x)) \\ &\geq f^{p+2}(x) - \frac{p+1}{(b-a)^{p-1}} g^p(x+h) f(x) \\ &= f(x) F_1(x), \end{aligned}$$

where $F_1(x) = f^{p+1}(x) - \dfrac{p+1}{(b-a)^{p-1}} g^p(x+h)$. By Lemma 2 we have

$$\begin{aligned} D_h F_1(x) &= D_h(f^{p+1}(x)) - \frac{p+1}{(b-a)^{p-1}} D_h(g^p(x+h)) \\ &\geq (p+1) f^p(x) D_h f(x) - \frac{p+1}{(b-a)^{p-1}} D_h g^p(x+h) \\ &\geq (p+1) f^p(x) D_h f(x) - \frac{(p+1)p}{(b-a)^{p-1}} g^{p-1}(x+2h) f(x+h). \end{aligned}$$

Since f is a non-negative and increasing function, then

$$g(x+2h) = \int_a^{x+2h} f(u) d_h t \leq (x+2h-a) f(x+2h),$$

hence

$$\begin{aligned} D_h F_1(x) &\geq (p+1) f^p(x) D_h f(x) - \frac{(p+1)p(x+2h-a)^{p-1}}{(b-a)^{p-1}} f^{p-1}(x+2h) f(x+h) \\ &\geq (p+1) f^p(x) D_h f(x) - \frac{(p+1)p(x+2h-a)^{p-1}}{(b-a)^{p-1}} f^p(x+2h) \\ &= (p+1) \Big(f^p(x) D_h f(x) - \frac{p(x+2h-a)^{p-1}}{(b-a)^{p-1}} f^p(x+2h) \Big) \geq 0. \end{aligned}$$

We conclude that F_1 is increasing function. Hence $F_1(x) \geq F_1(a) \geq 0$ which means that $D_h F(x) \geq 0$. So F is increasing and since $F(x) \geq F(a) = 0$ the proof is completed. ■

At the end of the notes, we pose the following problem.

Problem. Under what conditions does the inequality

$$\int_a^b f^t(x) d_h x \geq \Big(\int_a^b x^t f(x) d_h(x) \Big)^{t-1}$$

holds for $t \geq 1$?

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