On Some Feng Qi Type $h$-Integral Inequalities

Valmir Krasniqi and Armend Sh. Shabani

Department of Mathematics, University of Prishtina, Prishtinë 10000, Republic of Kosova
e-mail: vali.99@hotmail.com and armend_shabani@hotmail.com

Abstract
In this paper, several Feng Qi type $h$-integral inequalities are given by using elementary analytic methods in $h$-Calculus.

Keywords: $h$-derivative, $h$-integral, Inequalities.

2000 Mathematics Subject Classification: 33D05, 26D10, 26D15, 81P99.

1 Introduction

In [7] the following problem was posed: Under what conditions does the inequality
\[ \int_a^b [f(x)]^p dx \geq \left[ \int_a^b f(x)dx \right]^{t-1} \] holds for $t > 1$?

In [1] it has been proved the following: Let $[a, b]$ be a closed interval of $\mathbb{R}$ and let $p \geq 1$ be a real number. For any real continuous function $f$ on $[a, b]$, differentiable on $]a, b[$, such that $f(a) \geq 0$, and $f'(x) \geq p$ for all $x \in ]a, b[$, we have that
\[ \int_a^b [f(x)]^{p+2} dx \geq \frac{1}{(b-a)^{p-1}} \left[ \int_a^b f(x)dx \right]^{p+1}. \] (2)

In [2] it has been obtained the $q$-analogue of the previous result as follows. Let $p \geq 1$ be a real number and $f$ be a function defined on $[a, b]_q$ (see below for the definitions and notation), such that $f(a) \geq 0$, and $D_q f(x) \geq p$ for all $x \in (a, b)_q$. Then
\[ \int_a^b [f(x)]^{p+2} dq_x \geq \frac{1}{(b-a)^{p-1}} \left[ \int_a^b f(qx) dq_x \right]^{p+1}. \] (3)
The aim of this paper is to extend this result. We also note that in [6], Theorems 1 and 2, are valid only if $\beta \geq 2$ and not $\beta \geq 1$ as it is stated there. This paper will also provide some more sufficient conditions such that inequalities presented in [6] are valid.

2 Notations and preliminaries

For the convenience of the reader, we provide a summary of notations and definitions used in this paper. For details, one may refer to [3] and [5].

Let $h \neq 0$. A quantum derivative of a function $f$, that is characterized by an additive parameter $h$, is the $h$ derivative, denoted by $D_h f$ and given by

$$D_h f(x) = \frac{f(x+h) - f(x)}{h}.$$  \hfill (4)

One can easily verify the product and quotient rules for $h$-differentiation.

$$D_h(f(x)g(x)) = f(x)D_h g(x) + g(x+h)D_h f(x).$$ \hfill (5)

$$D_h\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)D_h f(x) - f(x)D_h g(x)}{g(x)g(x+h)}.$$ \hfill (6)

If $f'(0)$ exists, then $D_h f(0) = f'(0)$. As $h$ tends to 0, the $h$-derivative reduces to the usual derivative.

If $b - a \in h\mathbb{Z}$, the definite $h$-integral is defined by (see [5])

$$\int_a^b f(x)d_h x = \begin{cases} 
    h\left(f(a) + f(a + h) + \ldots + f(b - h)\right) & \text{if } a < b \\
    0 & \text{if } a = b \\
    -h\left(f(b) + f(b + h) + \ldots + f(a - h)\right) & \text{if } a > b
\end{cases}$$ \hfill (7)

From the above definition, one can see that the definite $h$-integral is a Riemann sum of $f$ on the interval $[a, b]$, which is partitioned into subintervals of equal width.

The following Theorem, whose proof can be found in [5], justifies (7) as an appropriate definition for the $h$-integral.

**Theorem 1** (Fundamental Theorem of $h$-calculus) If $F$ is an $h$-antiderivative of $f$ and $b - a \in h\mathbb{Z}$ then

$$\int_a^b f(x)d_h(x) = F(b) - F(a).$$ \hfill (8)
Applying Theorem 1 to $D_h(f(x)g(x))$ and using (5), one obtains the $h$-analogue of integration by parts.

$$\int_a^b f(x)dh(x)g(x) = f(b)g(b) - f(a)g(a) - \int_a^b g(x+h)dh(x)f(x). \quad (9)$$

For any function $f$ one has

$$D_h\left( \int_a^x f(t)dt \right) = f(x). \quad (10)$$

### 3 Main Results

In order to prove our main results we need the following Lemma, which is $h$-analogue of Lemma 3.1 from [2].

**Lemma 2** Let $p \geq 1$ and $g$ be a non-negative, increasing function on $[a,b]$. Then

$$pg^{p-1}(x)D_hg(x) \leq D_h[g^p(x)] \leq pg^{p-1}(x+h)D_hg(x). \quad (11)$$

**Proof.** We have

$$D_h(g^p(x)) = \frac{g^p(x+h) - g^p(x)}{h} = \frac{h}{p} \int_{g(x)}^{g(x+h)} t^{p-1}dt. \quad (12)$$

Since $g$ is non-negative increasing function we have:

$$g^{p-1}(x)(g(x+h) - g(x)) \leq \int_{g(x)}^{g(x+h)} t^{p-1}dt \leq g^{p-1}(x+h)(g(x+h) - g(x)). \quad (13)$$

Therefore by (12) and (13) one has:

$$pg^{p-1}(x)D_hg(x) \leq D_h[g^p(x)] \leq pg^{p-1}(x+h)D_hg(x).$$

**Theorem 3** If $f$ is a non-negative increasing function on $[a,b]$ and satisfies

$$f^{t-2}(x)D_hf(x) \geq (t-2)(x+2h-a)^{t-3}f^{t-2}(x+2h), \quad (14)$$

for $t \geq 3$ and $b - a \in h\mathbb{Z}$ then

$$\int_a^b f^t(x)dh(x) \geq \left( \int_a^b f(x)dh(x) \right)^{t-1}. \quad (15)$$
Proof. For \( x \in [a, b] \), let
\[
F(x) = \int_a^x f^t(u)du - \left( \int_a^x f(u)du \right)^{t-1}
\]
and \( g(x) = \int_a^x f(u)du \). By virtue of Lemma 2, it follows that
\[
D_hF(x) = f^t(x) - D_h(g^{t-1}(x))
\]
\[
\geq f^t(x) - (t-1)g^{t-2}(x+h)f(x)
\]
\[
= f(x)F_1(x),
\]
where \( F_1(x) = f^{t-1}(x) - (t-1)g^{t-2}(x+h) \). By Lemma 2 we have
\[
D_hF_1(x) = D_h(f^{t-1}(x)) - (t-1)D_h(g^{t-2}(x+h))
\]
\[
\geq (t-1)f^{t-2}(x)D_hf(x) - (t-1)(t-2)g^{t-3}(x+2h)D_hg(x+h)
\]
\[
\geq (t-1)f^{t-2}(x)D_hf(x) - (t-1)(t-2)g^{t-3}(x+2h)f(x+h).
\]
Since \( f \) is a non-negative and increasing function, then
\[
g(x+2h) = \int_a^{x+2h} f(u)du \leq (x+2h-a)f(x+2h),
\]
hence
\[
D_hF_1(x) \geq (t-1)f^{t-2}(x)D_hf(x) - (t-1)(t-2)(x+2h-a)f^{t-3}(x+2h)f(x+h)
\]
\[
\geq (t-1)f^{t-2}(x)D_hf(x) - (t-1)(t-2)(x+2h-a)f^{t-3}f^{t-2}(x+2h)
\]
\[
= (t-1)\left(f^{t-2}(x)D_hf(x) - (t-2)(x+2h-a)f^{t-3}f^{t-2}(x+2h)\right) \geq 0.
\]
We conclude that \( F_1 \) is increasing function. Hence \( F_1(x) \geq F_1(a) \geq 0 \) which means that \( D_hF(x) \geq 0 \). So \( F \) is increasing and since \( F(x) \geq F(a) = 0 \) the proof is completed.\( \blacksquare \)

**Theorem 4** Let \( p \geq 1 \). If \( f \) is a non-negative and increasing function on \([a, b]\) and satisfies
\[
f^p(x)D_hf(x) \geq \frac{p(x+2h-a)}{(b-a)^{p-1}} \cdot f^p(x+2h)
\]
then
\[
\int_a^b f^{p+2}(x)dx \geq \frac{1}{(b-a)^{p-1}} \left( \int_a^b f(x)dx \right)^{p+1}.
\]
Proof. For \( x \in [a, b] \) let
\[
F(x) = \int_a^x f^{p+2}(u) d_h t - \frac{1}{(b-a)^{p-1}} \left( \int_a^x f(u) d_h(u) \right)^{p-1}
\]
and \( g(x) = \int_a^x f(u) d_h u \). Utilizing Lemma 2 gives that
\[
D_h F(x) = f^{p+2}(x) - \frac{1}{(b-a)^{p-1}} D_h (g^{p+1}(x)) \\
\geq f^{p+2}(x) - \frac{p+1}{(b-a)^{p-1}} g^p(x + h) f(x) \\
= f(x) F_1(x),
\]
where \( F_1(x) = f^{p+1}(x) - \frac{p+1}{(b-a)^{p-1}} g^p(x + h) \). By Lemma 2 we have
\[
D_h F_1(x) = D_h(f^{p+1}(x)) - \frac{p+1}{(b-a)^{p-1}} D_h(g^p(x + h)) \\
\geq (p+1) f^p(x) D_h f(x) - \frac{p+1}{(b-a)^{p-1}} D_h g^p(x + h) \\
\geq (p+1) f^p(x) D_h f(x) - \frac{(p+1)p}{(b-a)^{p-1}} g^{p-1}(x + 2h) f(x + h).
\]
Since \( f \) is a non-negative and increasing function, then
\[
g(x + 2h) = \int_a^{x+2h} f(u) d_h t \leq (x + 2h - a) f(x + 2h),
\]
hence
\[
D_h F_1(x) \geq (p+1) f^p(x) D_h f(x) - \frac{(p+1)p(x+2h-a)^{p-1}}{(b-a)^{p-1}} f^{p-1}(x + 2h) f(x + h) \\
\geq (p+1) f^p(x) D_h f(x) - \frac{(p+1)p(x+2h-a)^{p-1}}{(b-a)^{p-1}} f^p(x + 2h) \\
= (p+1) \left( f^p(x) D_h f(x) - \frac{p(x+2h-a)^{p-1}}{(b-a)^{p-1}} f^p(x + 2h) \right) \geq 0.
\]
We conclude that \( F_1 \) is increasing function. Hence \( F_1(x) \geq F_1(a) \geq 0 \) which means that \( D_h F(x) \geq 0 \). So \( F \) is increasing and since \( F(x) \geq F(a) = 0 \) the proof is completed. 

At the end of the notes, we pose the following problem.

**Problem.** Under what conditions does the inequality
\[
\int_a^b f^t(x) d_h x \geq \left( \int_a^b x^t f(x) d_h(x) \right)^{t-1}
\]
holds for \( t \geq 1 \)?
On Some Feng Qi Type $h$-Integral Inequalities

References


