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Invitaion To Functional Means Some Problems

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Abstract

In this paper, we would like to present a list of open problems about functional means derived from a set of works recently published by some authors. These problems follow from the fact that the convex functional concept extends, at a new angle and in a fast way, the operator mean theory already stated in the literature.

Keywords: *Convex analysis; Operator means; Functional means.*

1 Introduction and Preliminaries

Scalar, matrix and operator means arise in several areas for solving many scientific problems. It has been proved to be a powerful tool in mathematics from the theoretical viewpoint as well as for practical purposes. In few years, an enormous amount of efforts, by many authors, has been devoted to understand the operator mean theory together with its various applications, see [3, 5, 7, 8, 9, 10, 12, 14, 15, 17, 18, 35] for instance. Afterwards, the extension of operator means from the case that the variables are positive linear bounded operators to the case that the variables are convex functionals has undergone extensive developments. For instance, the (power) arithmetic, harmonic, geometric functional means [6, 11, 23], some intermediary functional means [19, 23, 29], the logarithm of a convex functional [21], the convex logarithmic mean [25], the (Tsallis) functional relative entropy [24, 28], the extension of shorted operator

to convex functional [1, 2, 16, 32], have been recently discussed in the literature. These notions, stated at a convex functional approach, allowed the corresponding authors to obtain in a fast way the operator mean theory, together with some improvements and more new results [24, 25, 26, 27, 28, 30, 31, 32].
The purpose of the present paper is to put some open problems derived from the functional mean theory and the related works recently published in the literature. We then invite the interested readers by such area to carry out a systematic investigation as purposes for future researches.
Now, we will precise some basic notions needed throughout the following. Let H be a real or complex Hilbert space with its inner product $\langle ., . \rangle$ and its associated norm $\|.\|$. We denote by $Conv(H)$ (resp. $\Gamma_0(H)$) the cone of all convex (resp. convex lower semi-continuous) functions defined from H into $\mathbb{R} \cup \{+\infty\}$ not identically equal to $+\infty$. For $f : H \longrightarrow \mathbb{R} \cup \{+\infty\}$, its conjugate is the function $f^* : H \longrightarrow \mathbb{R} \cup \{+\infty\}$ defined by the relationship

$$\forall u^* \in H \qquad f^*(u^*) = \sup_{u \in H} \{Re \langle u^*, u\rangle - f(u)\}.$$

We recall that $f^* \in \Gamma_0(H)$ for all f and, $f^{**} = f$ if and only if $f \in \Gamma_0(H)$. We denote by $\sigma(u) = (1/2)\|u\|^2$ the unique self-conjugate functional defined on H. The notation $dom\ f$ refers to the domain of f, i.e the subset of H defined by

$$dom\ f = \{u \in H,\ f(u) < +\infty\},$$

and $\partial f(u)$ stands for the sub-differential of f at $u \in dom\ f$ given by

$$\partial f(u) = \{u^* \in H,\ \forall v \in dom\ f,\ f(v) \geq f(u) + Re \langle u^*, v - u\rangle\}.$$

For further details about background material in convex analysis, we refer the reader to [36] for instance.

2 Open Problems

As already pointed, this section will be devoted to display some open problems about functional means and its different related areas. Some conjectures will be also stated.

1. Topological interiors of $Conv(H)$ and $\Gamma_0(H)$

In previous works concerning our extensions about some notions from positive operators to convex functionals, we have observed that the topological concept appearing in a class of related results is the point-wise convergence topology. Moreover, it has been observed that $Conv(H)$ can be considered as an extension of $C^+(H)$, cone of all linear bounded positive semi-definite operators defined from H into itself. For the strong operator topology, it is well known that the interior of $C^+(H)$ is the open cone of positive definite operators. For the point-wise convergence topology, it is easy to see that $Conv(H)$ and $\Gamma_0(H)$ are not open. Therefore, one may naturally pose our first open problem.

Problem 1. *What are the interiors of* $Conv(H)$ *and* $\Gamma_0(H)$ *for the point-wise convergence topology?*

Let us denote by $S_c(H)$ (resp. $S_0(H)$) the cone of all functionals defined from H into $I\!R \cup \{+\infty\}$ which are strongly convex (resp. strongly $\Gamma_0(H)$), that is

$$S_c(H) = \{f : H \longrightarrow I\!R \cup \{+\infty\},\ \exists a > 0 \text{ such that } f - a.\sigma \in Conv(H)\},$$

$$S_0(H) = \{f : H \longrightarrow I\!R \cup \{+\infty\},\ \exists a > 0 \text{ such that } f - a.\sigma \in \Gamma_0(H)\}.$$

It is not hard to verify that $S_c(H)$ and $S_0(H)$ are two open sets for the point-wise convergence topology. We now put the following.

Conjecture A. With the above, we conjecture that

$$int\ Conv(H) = S_c(H), \quad int\ \Gamma_0(H) = S_0(H),$$

for the point-wise convergence topology.

2. Point-wise strict convexity of the conjugate operation

Let $C^{+*}(H)$ be the cone of all linear bounded positive definite operators defined from H into itself. It is well known that the map $A \longmapsto A^{-1}$ from $C^{+*}(H)$ into itself is strictly convex, see [17] for example. As consequence, the power arithmetic and harmonic means of two positive operators A and B coincide if and only if $A = B$. For functional case whose the conjugate f^* has interpreted as an extension of the inverse operator, it is also known that the map $f \longmapsto f^*$ is point-wisely convex. This implies that the power arithmetic-harmonic functional mean inequality,

$$\mathcal{H}_\alpha(f, g) := ((1-\alpha).f^* + \alpha.g^*)^* \leq (1-\alpha).f + \alpha.g := \mathcal{A}_\alpha(f, g), \quad 0 \leq \alpha \leq 1,$$

holds for all functionals f and g. We notice that, the condition $f = g$ is not necessary to have the arithmetic-harmonic equality, since for $g = f + c$, with c is a real constant, the above inequality remains an equality. So, the map $f \longmapsto f^*$ is not point-wisely strictly convex. Our second open problem may be recited as the following.

Problem 2. *What should be the necessary and sufficient conditions for which the arithmetic-harmonic functional mean inequality becomes an equality?*

Of course, the equality between $\mathcal{H}_\alpha(f,g)$ and $\mathcal{A}_\alpha(f,g)$ implies that of all intermediary functional means introduced in [19, 25] and that of several functional variables discussed in [23, 25].

3. Point-wise monotone maps with convex functional variable

In the literature, it is proved that operator monotone functions are characterized by the integral representation,

$$f(u) = a + b.u + \int_0^{+\infty} \frac{t.u}{u+t} d\nu(t),$$

where $d\nu(t)$ is any positive measure on the interval $[0,+\infty)$, a is a real and b a non-negative scalar. For positive linear operator A the above representation yields

$$f(A) = a.I + b.A + \int_0^{+\infty} t.A(t.I+A)^{-1} d\nu(t).$$

An important example of monotone map is $A \longmapsto A^\alpha$, $0 < \alpha \leq 1$, which occurs a considerable interest in the monotone operator mean theory. Another primordial example is the operator logarithm map $A \longmapsto Log\ A$ whose the interest appears in the chaotic operator mean theory. These phenomenons motivate us to put our third open problem as follows.

Problem 3. *What should be the characterization of point-wise increasing maps involving convex functional as variable?*

We put the following conjecture.

Conjecture B. Point-wise monotone increasing maps Φ with convex functional variable f are characterized by the relationship

$$\Phi(f) = a.\sigma + b.f + \int_0^{+\infty} \frac{t}{1+t}\left(\frac{1}{1+t}.\sigma + \frac{t}{1+t} f^*\right)^* d\nu(t),$$

where a, b and $d\nu(t)$ are as in the above, and $\sigma = (1/2)\|.\|^2$ is the unique self-conjugate convex functional.
Analogously to the above operator examples, the two maps $f \longmapsto f^\alpha$, $0 < \alpha \leq 1$, and $f \longmapsto \mathcal{L}(f)$, called respectively the $\alpha-$ iterate and logarithm in convex analysis, have been introduced and studied in [23], [21] respectively.

For the definition of $f \longmapsto \mathcal{L}(f)$, we can also see the section below.

4. Inverse of the logarithm functional map

In [21], the authors have introduced the functional logarithm in convex analysis which extends the logarithm of a positive definite operator. Precisely, we recall that the map $\mathcal{L} : \Gamma_\circ(H) \to \overline{I\!R}^H$ defined by

$$\forall u \in H \qquad [\mathcal{L}(f)](u) = \int_0^1 \frac{\sigma(u) - \left((1-t).\sigma + t.f\right)^*(u)}{t} dt,$$

is called the functional logarithm in the sense of convex analysis. In the quadratic case the above definition coincides with that of operator logarithm whose the reverse is the operator exponential. To determine how to obtain the reverse functional logarithm is not obvious and appears to be interesting. In another way, we put the following.

Problem 4. *What should be the analogue of exp A when the positive operator variable A is a convex functional?*

Defining the operator exponential as the inverse of the logarithm, the above open question is reduced to introduce the reverse map of the functional logarithm. In another way, it remains that to see if the functional logarithm is one to one. With this, it is equivalent to extend the chaotic partial ordering and the chaotic geometric mean from positive operators to convex functionals. When we define the operator exponential by the series sum, the above question needs a reasonable extension of A^n, $n \geq 2$, from operator to functional. This latter point will be discussed, at another view point, in the following.

5. Analogue of ABA for convex functionals

Let A and B be two given positive definite operators defined from H into itself. The operator product ABA appears in many scientific contexts, for instance:
In the algebraic Ricatti equation: find a positive operator X such that $XAX = B$. As well known, this equation has one and only one solution given by

$$X = G(A^{-1}, B) = A^{-1/2}\left(A^{1/2}BA^{1/2}\right)^{1/2}A^{-1/2},$$

geometric operator mean of A^{-1} and B.
In the monotone operator mean theory that extends the above situation,

$$AmB = A^{1/2}F\left(A^{-1/2}BA^{-1/2}\right)A^{1/2},$$

where F is a monotone continuous real function.
In the relative operator entropy,

$$S(A/B) = A^{1/2} Log \left(A^{-1/2} B A^{-1/2}\right) A^{1/2}.$$

After this, we now put the following open question.

Problem 5. *What should be the reasonable analogue of ABA when the positive variable operators A and B are convex functionals?*

Such extension includes that of A^2, hence eventually that of A^n for $n \geq 2$ and so, as already pointed, the functional exponential. Recall that a reasonable analogue of $A^{1/n}$, $n \geq 2$, from positive operator to convex functional has been discussed in the literature, and so we can see the extension of A^n as the reverse map of $f \longmapsto f^{[1/n]}$ introduced in [23], see also [6] for $n = 2$.

6. Functional limited development

The functional limited development of $(\sigma + \lambda.f)^*$ at the second order has been discussed. Precisely, the following result has been proved, [22]:
Let f $\in \Gamma_0(H)$ such that $int(domf)$ is nonempty. For all $u \in int(domf)$, there holds

$$(\sigma + \lambda f)^*(u) = \sigma(u) - \lambda f(u) + \lambda^2 \sigma(p_f(u)) + \lambda^2 (\theta_\lambda(f))(u),$$

where $\lim\limits_{\lambda \downarrow 0} \theta_\lambda(f) = 0$ in the point-wise convergence and $p_f(u) := Proj_{\partial f(u)}(0)$ is the unique point projection from 0 onto the nonempty closed convex $\partial f(u)$.
In the quadratical case, the above result yields the known formulae

$$(I + \lambda.A)^{-1} = I - \lambda.A + \lambda^2.A^2 + \lambda^2.\theta_\lambda(A),$$

where $\theta_\lambda(A)$ converges strongly to 0 as λ. Now, the following question arises naturally from the above.

Problem 6. *What is the functional limited development at order $p \geq 3$ extending the operator one,*

$$(I + \lambda.A)^{-1} = \sum_{i=0}^{\infty} (-1)^i \lambda^i A^i, \quad \lambda \longrightarrow 0?$$

7. Analogue of Hadamard product for convex functionals

The Hadamard product arises in several contexts as tool for solving many

scientific problems. There are some interaction between geometric mean and Hadamard product, see [4, 8] for instance. Given two matrices $A=(a_{ij}), B=(b_{ij})$ of the same size, their Hadamard product $A\circ B$ is the matrix of entry-wise products, i.e $A\circ B=(a_{ij}b_{ij})$. For two bounded operators A and B on a Hilbert space H, their Hadamard product is defined as follows: if U is the isometry of H into $H\otimes H$ such that $Ue_n=e_n\otimes e_n$ where (e_n) is a fixed orthonormal basis of H, then the Hadamard product $A\circ B$ for (e_n) is expressed by $A\circ B=U^*(A\otimes B)U$. This latter definition coincides with the above matrix one when $\mathbb{R}^n$ is equipped with its natural basis.
Otherwise, the tensor product $A\otimes B$ appears in various areas of operator theory. See for instance [7] where some inequalities for the shorted operator involving the tensor product $A\otimes B$ are discussed. We then put the following question.

Problem 7. *What should be the analogue of $A\otimes B$ when the operator variables A and B are convex functionals? Then, what is the Hadamard product of two convex functionals?*

8. Extension of matrix sign function

The matrix sign function was introduced by many authors as a tool for solving, for instance, the Lyapunov equation and the algebraic Riccati equations, see [13] and the related reference therein. This matrix sign extends that of real numbers in the sense that if a is a real scalar then $a=sgn(a)|a|$, where $sgn(a)=\pm 1$ is the familiar sign of a scalar. Let H be a Hilbert space and $P: H\longrightarrow H$ a linear operator having no pure imaginary eigenvalues. The operator sign of P can be defined, for example, by the Robers's integral formula

$$sgn(P)=\frac{2}{\pi}P\int_0^{+\infty}\left(t^2.I+P^2\right)^{-1}dt.$$

For numerical view point, $sgn(P)$ corresponds to the Newton method applied to the equation $X^2=I$, that is the iterative process (X_n) defined by

$$X_{n+1}=\frac{1}{2}X_n+\frac{1}{2}X_n^{-1},\ \ X_0=P,$$

converges quadratically to $sgn(P)$ for all operator P having no pure imaginary eigenvalues. The explicit formulae of $sgn(P)$ is given by $sgn(P)=P(P^2)^{-1/2}$. Now, we are in position to put the following open question.

Problem 8. *What should be the analogue of operator sign function $P\longmapsto sgn(P)$ when the operator variable P is a functional?*

3 Conclusion and Motivation

In this work, some open problems with many conjectures about functional mean have been stated. Of course, other open questions can be derived from the different works published in such area. Actually, the author is interested by two directions respectively motivated by two different approaches. In one direction, [33] concerns a generalization of the conjugate-duality operation from convex analysis to a general context in the sense to obtain an extension of ABA from the case that the variables A and B are positive linear bounded operators to the case that the variables are convex functionals. The second direction [34] turns out of to define again the duality and the mean in a general context. In fact, even for the case of operator, the mean theory has been treated with respect to the inverse-duality. For the functional approach, the chosen duality was considered with respect to the conjugate-duality in convex analysis. Precisely, we must ask the following question: what is a duality, what is a mean, in an arbitrary (topological) vector space?

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