Int. J. Open Problems Compt. Math., Vol. 2, No. 4, December 2009
ISSN 1998-6262; Copyright ©ICSRS Publication, 2009
www.i-csrs.org

Generalization of Some Inequalities for The Gamma Function

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Abstract

An inequality involving the Euler gamma function is presented. This result generalizes several recently published results by C. Alsina and M.S. Tomás, N. V . Vinh, N. P. N. Ngoc.

Mathematics Subject Classification: *33D99, 26D20, 33B15.*
Keywords: *Gamma function, Psi function, Inequality.*

1 Introduction

The Euler gamma function $\Gamma(x)$ is defined for $x > 0$ by

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt.$$

Psi or digamma function, the logarithmic derivative of the gamma function is defined by

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)},\ x > 0.$$

C.Alsina and M.S.Tomás in [1], using a geometrical method, proved the following double inequality:

$$\frac{1}{n!} \leq \frac{\Gamma(1+x)^n}{\Gamma(1+nx)} \leq 1,\ \ x \in [0,1],\ n \in N. \tag{1}$$

N. V. Vinh and N. P. N. Ngoc in [2], using Dirichlet's integral, proved the following double inequality

$$\frac{\prod_{i=1}^{n}\Gamma(1+\alpha_i)}{\Gamma(\beta+\sum_{i=1}^{n}\alpha_i)} \le \frac{\prod_{i=1}^{n}\Gamma(1+\alpha_i x)}{\Gamma\left(\beta+\left(\sum_{i=1}^{n}\alpha_i\right)x\right)} \le \frac{1}{\Gamma(\beta)} \tag{2}$$

where $x \in [0,1], \beta \ge 1, \alpha_i > 0,\ n \in N$.

2 Main results

We start with the following lemma.

Lemma 2.1 *Let* $0 < x \le y$. *Then*

$$\psi(x) \le \psi(y).$$

Proof. In [4], page 21, we have the following:

$$\psi(x) - \psi(y) = (x-y).\sum_{n=0}^{\infty}\frac{1}{(x+n)(y+n)}.$$

This proof is complete.

Theorem 2.2 *Let f be a function defined by*

$$f(x) = \frac{\prod_{i=1}^{n}\Gamma(a_i+b_i x)^{c_i}}{\Gamma\left(d+\left(\sum_{i=1}^{n}b_i\right)x\right)^{e}},\ x \ge 0 \tag{3}$$

where $c_i, e,\ i = \overline{1,n}$ are positive real numbers and $a_i,\ b_i, d,\ i = \overline{1,n}$ are real numbers such that $e \ge c_i$, $i = \overline{1,n}$, $d + \sum_{i=1}^{n} b_i x \ge a_i + b_i x > 0$, $i = \overline{1,n}$. If $\psi(d + \sum_{i=1}^{n} b_i x) > 0$ and $b_i > 0$, $i = \overline{1,n}$ then the function f is decreasing for $x \ge 0$.

Proof. Let g be a function defined by $g(x) = \ln f(x)$ for $x \in [0, \infty)$.

$$\begin{aligned} g'(x) &= \sum_{i=1}^{n} c_i b_i \frac{\Gamma'(a_i+b_i x)}{\Gamma(a_i+b_i x)} - e\left(\sum_{i=1}^{n} b_i\right)\frac{\Gamma'(d+\sum_{i=1}^{n} b_i x)}{\Gamma(d+\sum_{i=1}^{n} b_i x)} \\ &= \sum_{i=1}^{n} c_i b_i \psi(a_i+b_i x) - e\left(\sum_{i=1}^{n} b_i\right)\psi(d+\sum_{i=1}^{n} b_i x) \\ &= \sum_{i=1}^{n} b_i\left(c_i\psi(a_i+b_i x) - e\psi(d+\sum_{j=1}^{n} b_j x)\right). \end{aligned}$$

From Lemma 2.1 we have

$$\psi(a_i + b_i x) \le \psi(d + \sum_{j=1}^{n} b_j x).$$

Multiplying both sides of inequality $e \ge c_i$ with $\psi(d + \sum_{j=1}^{n} b_j x)$ we obtain

$$e\psi(d + \sum_{j=1}^{n} b_j x) \ge c_i \psi(d + \sum_{j=1}^{n} b_j x) \ge c_i \psi(a_i + b_i x).$$

Hence,

$$\sum_{i=1}^{n} b_i \left(c_i \psi(a_i + b_i x) - e\psi(d + \sum_{j=1}^{n} b_j x) \right) \le 0.$$

We have $g'(x) \le 0$. It means that g is decreasing on $[0, \infty)$. This implies that f is decreasing on $[0, \infty)$.
This proof is completed.

Corollary 2.3 *In Theorem 2.2, for $x \in [0, 1]$ the following double inequality holds*

$$\frac{\prod_{i=1}^{n} \Gamma(a_i + b_i)^{c_i}}{\Gamma\left(d + \left(\sum_{i=1}^{n} b_i\right)\right)^e} \le f(x) = \frac{\prod_{i=1}^{n} \Gamma(a_i + b_i x)^{c_i}}{\Gamma\left(d + \left(\sum_{i=1}^{n} b_i\right) x\right)^e} \le \frac{\prod_{i=1}^{n} \Gamma(a_i)^{c_i}}{\Gamma\left(d\right)^e}. \tag{4}$$

Proof. Using Theorem 2.2, we have

$$f(1) \le f(x) \le f(0).$$

Thus

$$\frac{\prod_{i=1}^{n} \Gamma(a_i + b_i)^{c_i}}{\Gamma\left(d + \left(\sum_{i=1}^{n} b_i\right)\right)^e} \le f(x) = \frac{\prod_{i=1}^{n} \Gamma(a_i + b_i x)^{c_i}}{\Gamma\left(d + \left(\sum_{i=1}^{n} b_i\right) x\right)^e} \le \frac{\prod_{i=1}^{n} \Gamma(a_i)^{c_i}}{\Gamma\left(d\right)^e}.$$

In a similar way, it is easy to prove the following theorem.

Theorem 2.4 *Let f be a function defined by*

$$f(x) = \frac{\prod_{i=1}^{n} \Gamma(a_i + b_i x)^{c_i}}{\Gamma\left(d + \left(\sum_{i=1}^{n} b_i\right) x\right)^e}, \; x \ge 0 \tag{5}$$

where $c_i, e,\ i = \overline{1,n}$ are positive real numbers and $a_i, b_i, d,\ i = \overline{1,n}$ are real numbers such that $e \ge c_i$, $i = \overline{1,n}$, $d + \sum_{i=1}^{n} b_i x \ge a_i + b_i x > 0,\ i = \overline{1,n}$. If $\psi(d + \sum_{i=1}^{n} b_i x) > 0$ and $b_i < 0,\ i = \overline{1,n}$ then the function f is increasing for $x \ge 0$.

Corollary 2.5 *In Theorem 2.4, for $x \in [0,1]$ the following double inequality holds*

$$\frac{\prod_{i=1}^{n}\Gamma(a_i)^{c_i}}{\Gamma(d)^e} \le \frac{\prod_{i=1}^{n}\Gamma(a_i+b_i x)^{c_i}}{\Gamma\left(d+\left(\sum_{i=1}^{n} b_i\right)x\right)^e} \le \frac{\prod_{i=1}^{n}\Gamma(a_i+b_i)^{c_i}}{\Gamma\left(d+\sum_{i=1}^{n} b_i\right)^e}. \tag{6}$$

Remark 2.6 *Considering (4) with $a_i = 1, c_i = 1, e = 1, b_i = 1, d = 1$, $i = \overline{1,n}$ we obtain inequality (1).*

Remark 2.7 *Considering (4) with $a_i = 1, c_i = 1, e = 1, b_i = \alpha_i, d = \beta$, $i = \overline{1,n}$ we obtain inequality (2).*

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