

# Inequalities and Monotonicity For The Ratio of $\Gamma_p$ Functions

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### Abstract

*Let  $x > 0, y \geq 0$  be real numbers. The function  $f(x) = \frac{[\Gamma_p(x+y+1)/\Gamma_p(y+1)]^{\frac{1}{x}}}{x+y+1}$  is strictly decreasing and strictly logarithmically convex on  $(0, \infty)$ . Moreover  $\lim_{x \rightarrow 0} f(x) = \frac{e^{\psi_p(y+1)}}{y+1}$  and  $\frac{x+y+1}{x+y+2} < \frac{[\Gamma_p(x+y+1)/\Gamma_p(y+1)]^{\frac{1}{x}}}{[\Gamma_p(x+y+2)/\Gamma_p(y+1)]^{\frac{1}{x+1}}}$ .*

**Keywords:**  $\Gamma_p$  function, Inequalities.

## 1 Introduction

The Euler gamma function  $\Gamma(x)$  is defined for  $x > 0$  by

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt.$$

Euler gave another equivalent definition for the  $\Gamma(x)$  (see [5])

$$\Gamma_p(x) = \frac{p! p^x}{x(x+1) \cdot \dots \cdot (x+p)} = \frac{p^x}{x(1 + \frac{x}{1}) \cdot \dots \cdot (1 + \frac{x}{p})} \quad (1)$$

where  $\lim_{p \rightarrow \infty} \Gamma_p(x) = \Gamma(x)$ .

V. Krasniqi and A. Shabani (see [5]) defined the function

$$\psi_p(x) = \frac{\Gamma'_p(x)}{\Gamma_p(x)}. \quad (2)$$

The function  $\psi_p$ , defined by (1) has the following series representation

$$\psi_p(x) = \ln p - \sum_{k=0}^p \frac{1}{x+k} \quad (3)$$

and deriving  $n$  times the relation (3) one finds that:

$$\psi_p^{(n)}(x) = \sum_{k=0}^p \frac{(-1)^{n-1} \cdot n!}{(x+k)^{n+1}}. \quad (4)$$

In [7], H. Minc and L. Sathre proved that, if  $r$  is a positive integer and  $\phi(r) = (r!)^{\frac{1}{r}}$ , then

$$1 < \frac{\phi(r+1)}{\phi(r)} < \frac{r+1}{r}, \quad (5)$$

which can be rearranged as

$$[\Gamma(1+r)]^{\frac{1}{r}} < [\Gamma(2+r)]^{\frac{1}{r+1}} \quad (6)$$

and

$$\frac{[\Gamma(1+r)]^{\frac{1}{r}}}{r} > \frac{[\Gamma(2+r)]^{\frac{1}{r+1}}}{r+1}. \quad (7)$$

In [1, 6], H. Alzer and J. S. Martins refined the right inequality in (5) and showed that, if  $n$  is a positive integer, then for all positive real numbers  $r$ , we have

$$\frac{n}{n+1} < \left( \frac{\frac{1}{n} \sum_{i=1}^n i^r}{\frac{1}{n+1} \sum_{i=1}^{n+1} i^r} \right)^{\frac{1}{r}} < \frac{\sqrt[n]{n!}}{n^{+1} \sqrt{(n+1)!}}. \quad (8)$$

Both bounds in (8) are the best possible.

The inequalities in (5) were refined and generalized in [8, 2, 9, 11] and the following inequalities were obtained:

$$\frac{n+k+1}{n+m+k+1} < \left( \prod_{i=k+1}^{n+k} i \right)^{\frac{1}{n}} / \left( \prod_{i=k+1}^{n+m+k} i \right)^{\frac{1}{n+m}} \leq \sqrt{\frac{n+k}{n+m+k}} \quad (9)$$

where  $k$  is a nonnegative integer,  $n$  and  $m$  are natural numbers. For  $n = m = 1$ , the equality in (9) is valid.

In [8], inequalities in (9) we generalized, where Feng Qi obtained the following inequalities on the ration for the geometric means of a positive arithmetic sequence with unit difference for any nonnegative integer  $k$  and natural numbers  $n$  and  $m$ :

$$\frac{n+k+1+\alpha}{n+m+k+1+\alpha} < \frac{\left(\prod_{i=k+1}^{n+k} (i+\alpha)\right)^{\frac{1}{n}}}{\left(\prod_{i=k+1}^{n+m+k} (i+\alpha)\right)^{\frac{1}{n+m}}} \leq \sqrt{\frac{n+k+\alpha}{n+m+k+\alpha}} \quad (10)$$

where  $\alpha \in [0, 1]$  is a constant. For  $n = m = 1$ , the equality in (10) is valid.

Furthermore, for nonnegative integer  $k$  and natural numbers  $n$  and  $m$ , we have

$$\frac{a(n+k+1)+b}{a(n+m+k+1)+b} < \frac{\left(\prod_{i=k+1}^{n+k} (ai+b)\right)^{\frac{1}{n}}}{\left(\prod_{i=k+1}^{n+m+k} (ai+b)\right)^{\frac{1}{n+m}}} \leq \sqrt{\frac{a(n+k)+b}{a(n+m+k)+b}}, \quad (11)$$

where  $a$  is a positive constant and  $b$  a nonnegative integer. For  $n = m = 1$ , the equality in (11) is valid, (see [3]).

It is clear that inequalities in (11) extend those in (10).

In [4], the following monotonicity results for the Gamma function were established. The function  $[\Gamma(1 + \frac{1}{x})]^x$  decreases with  $x > 0$  and  $x[\Gamma(1 + \frac{1}{x})]^x$  increases with  $x > 0$ , which recovers the inequalities in (5) which refers to integer value of  $r$ . These are equivalent to the function  $[\Gamma(1+x)]^{\frac{1}{x}}$  being increasing and  $\frac{[\Gamma(1+x)]^{\frac{1}{x}}}{x}$  being decreasing on  $(0, \infty)$ , respectively. In addition, it was proved that the function  $x^{1-\gamma}[\Gamma(1 + \frac{1}{x})]^x$  decreases for  $0 < x < 1$ , where  $\gamma = 0.57721566 \dots$  denotes the Euler's constant, which is equivalent to  $\frac{[\Gamma(1+x)]^{\frac{1}{x}}}{x^{1-\gamma}}$  being increasing on  $(1, \infty)$ .

In [2], the following monotonicity result was obtained. The function

$$\frac{[\Gamma(x+y+1)/\Gamma(y+1)]^{\frac{1}{x}}}{x+y+1} \quad (12)$$

is decreasing for  $x \geq 1$ , for fixed  $y \geq 0$ . Then, for positive real numbers  $x$  and  $y$ , we have

$$\frac{x+y+1}{x+y+2} \leq \frac{[\Gamma(x+y+1)/\Gamma(y+1)]^{\frac{1}{x}}}{[\Gamma(x+y+2)/\Gamma(y+1)]^{\frac{1}{x+1}}}. \quad (13)$$

## 2 Main results

The following Theorem is the main result of these notes.

**Theorem 2.1** *Let  $x > 0, y \geq 0$  be real numbers. The function*

$$f(x) = \frac{[\Gamma_p(x+y+1)/\Gamma_p(y+1)]^{\frac{1}{x}}}{x+y+1} \quad (14)$$

*is strictly decreasing on  $(0, \infty)$ . Moreover*

$$\lim_{x \rightarrow 0} f(x) = \frac{e^{\psi_p(y+1)}}{y+1} \quad (15)$$

and

$$\frac{x+y+1}{x+y+2} < \frac{[\Gamma_p(x+y+1)/\Gamma_p(y+1)]^{\frac{1}{x}}}{[\Gamma_p(x+y+2)/\Gamma_p(y+1)]^{\frac{1}{x+1}}}.$$

Taking logarithm yields

$$\ln f(x) = \frac{1}{x} \left[ \ln \Gamma_p(x+y+1) - \ln \Gamma_p(y+1) \right] - \ln(x+y+1).$$

For  $x > 0$ , define

$$h(x) = x^2 \frac{f'(x)}{f(x)} = -\ln \frac{\Gamma_p(x+y+1)}{\Gamma_p(y+1)} + x\psi_p(x+y+1) - \frac{x^2}{x+y+1}.$$

Differentiation of  $h$  gives.

$$\begin{aligned}
 \frac{1}{x}h'(x) &= \psi'_p(x+y+1) - \frac{1}{x+y+1} - \frac{y+1}{(x+y+1)^2} \\
 &= \sum_{n=1}^{p+1} \frac{1}{(x+y+n)^2} - \frac{1}{x+y+1} - \frac{y+1}{(x+y+1)^2} \\
 &= \sum_{n=1}^{p+1} \frac{1}{(x+y+n)^2} - \sum_{n=1}^{\infty} \left[ \frac{1}{x+y+n} - \frac{1}{x+y+n+1} \right] - \\
 &\quad - \sum_{n=1}^{\infty} \left[ \frac{y+1}{(x+y+n)^2} - \frac{y+1}{(x+y+n+1)^2} \right] \\
 &< \sum_{n=1}^{\infty} \frac{1}{(x+y+n)^2} - \sum_{n=1}^{\infty} \left[ \frac{1}{x+y+n} - \frac{1}{x+y+n+1} \right] - \\
 &\quad - \sum_{n=1}^{\infty} \left[ \frac{y+1}{(x+y+n)^2} - \frac{y+1}{(x+y+n+1)^2} \right] \\
 &= - \sum_{n=1}^{\infty} \left[ \frac{y}{(x+y+n)^2} + \frac{1}{(x+y+n)(x+y+n+1)} - \frac{y+1}{(x+y+n+1)^2} \right] \\
 &= - \sum_{n=1}^{\infty} \frac{(2y+1)(x+y+n) + y}{(x+y+n)^2(x+y+n+1)^2} < 0.
 \end{aligned}$$

Hence, the function  $h$  is strictly decreasing and  $h(x) < h(0) = 0$ , for  $x > 0$ , which yields the desired results that  $f'(x) < 0$ , hence  $f$  is strictly decreasing on  $(0, \infty)$  and one has the following inequality

$$\frac{x+y+1}{x+y+2} < \frac{\left[ \Gamma_p(x+y+1)/\Gamma_p(y+1) \right]^{\frac{1}{x}}}{\left[ \Gamma_p(x+y+2)/\Gamma_p(y+1) \right]^{\frac{1}{x+1}}}.$$

Next, by L. Hospital rule, we conclude that

$$\lim_{x \rightarrow 0} f(x) = \frac{e^{\psi_p(y+1)}}{y+1}, y \geq 0. \quad (16)$$

**Definition 2.2** A function  $f(x)$  is logarithmically convex on the interval  $[a, b]$  if  $f > 0$  and  $\ln f(x)$  is convex on  $[a, b]$

**Theorem 2.3** The function  $f$  given by (14) is strictly logarithmically convex on  $(0, \infty)$ .

For  $x > 0$  define

$$g(x) = x^3 \frac{d^2[\ln f(x)]}{dx^2} = 2 \ln \frac{\Gamma_p(x+y+1)}{\Gamma_p(y+1)} - 2x\psi_p(x+y+1) + x^2\psi'_p(x+y+1) + \frac{x^3}{(x+y+1)^2}.$$

Differentiation of  $g$  yields

$$\begin{aligned} \frac{1}{x^2}g'(x) &= \psi''_p(x+y+1) + \frac{1}{(x+y+1)^2} + \frac{2(y+1)}{(x+y+1)^3} \\ &= -\sum_{n=1}^{p+1} \frac{2}{(x+y+n)^3} + \sum_{n=1}^{\infty} \left[ \frac{1}{(x+y+n)^2} - \frac{1}{(x+y+n+1)^2} \right] \\ &\quad + \sum_{n=1}^{\infty} \left[ \frac{2(y+1)}{(x+y+n)^3} - \frac{2(y+1)}{(x+y+n+1)^3} \right] \\ &> -\sum_{n=1}^{\infty} \frac{2}{(x+y+n)^3} + \sum_{n=1}^{\infty} \left[ \frac{1}{(x+y+n)^2} - \frac{1}{(x+y+n+1)^2} \right] \\ &\quad + \sum_{n=1}^{\infty} \left[ \frac{2(y+1)}{(x+y+n)^3} - \frac{2(y+1)}{(x+y+n+1)^3} \right]. \\ &= \sum_{n=1}^{\infty} \frac{3(2y+1)(x+y+n)^2 + (6y+1)(x+y+n) + 2y}{(x+y+n)^3(x+y+n+1)^3} > 0. \end{aligned}$$

Hence the function  $g$  is strictly increasing and  $g(x) > g(0) = 0$ , for  $x > 0$ , which yields the desired results, that is  $\frac{d^2(\ln f(x))}{dx^2} > 0$  for  $x > 0$ .

**Corollary 2.4** *Let  $y \geq 0$  be a real number. Then for all real numbers  $x > 0$*

$$\frac{\left[ \Gamma_p(x+y+1) / \Gamma_p(y+1) \right]^{\frac{1}{x}}}{x+y+1} \leq \frac{e^{\psi_p(y+1)}}{y+1}. \quad (17)$$

Since  $f$  is decreasing and

$$\lim_{x \rightarrow 0} f(x) = \frac{e^{\psi_p(y+1)}}{y+1} \quad (18)$$

we obtain

$$f(x) \leq \lim_{x \rightarrow 0} f(x) = \frac{e^{\psi_p(y+1)}}{y+1}$$

and proof is completed.

### 3 Open Problem

At the end, we pose a problem.

For positive real numbers  $x$  and  $y$ , holds

$$\frac{[\Gamma_p(x+y+1)/\Gamma_p(y+1)]^{\frac{1}{x}}}{[\Gamma_p(x+y+2)/\Gamma_p(y+1)]^{\frac{1}{x+1}}} < \sqrt{\frac{x+y}{x+y+1}}. \quad (19)$$

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