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Inequalities and Monotonicity For The Ratio of Γ_p Functions

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Abstract

Let $x > 0, y \geq 0$ ***be real numbers. The function*** $f(x) = \frac{[\Gamma_p(x+y+1)/\Gamma_p(y+1)]^{\frac{1}{x}}}{x+y+1}$ ***is strictly decreasing and strictly logarithmically convex on*** $(0, \infty)$. ***Moreover*** $\lim_{x\to 0} f(x) = \frac{e^{\psi_p(y+1)}}{y+1}$ ***and*** $\frac{x+y+1}{x+y+2} < \frac{\left[\Gamma_p(x+y+1)/\Gamma_p(y+1)\right]^{\frac{1}{x}}}{\left[\Gamma_p(x+y+2)/\Gamma_p(y+1)\right]^{\frac{1}{x+1}}}$.

Keywords: Γ_p *function, Inequalities.*

1 Introduction

The Euler gamma function $\Gamma(x)$ is defined for $x > 0$ by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt.$$

Euler gave another equivalent definition for the $\Gamma(x)$ (see [5])

$$\Gamma_p(x) = \frac{p! p^x}{x(x+1) \cdot \ldots \cdot (x+p)} = \frac{p^x}{x(1+\frac{x}{1}) \cdot \ldots \cdot (1+\frac{x}{p})} \tag{1}$$

where $\lim_{p\to\infty} \Gamma_p(x) = \Gamma(x)$.

V. Krasniqi and A. Shabani (see [5]) defined the function

$$\psi_p(x) = \frac{\Gamma_p'(x)}{\Gamma_p(x)}. \tag{2}$$

The function ψ_p, defined by (1) has the following series representation

$$\psi_p(x) = \ln p - \sum_{k=0}^{p} \frac{1}{x+k} \tag{3}$$

and deriving n times the relation (3) one finds that:

$$\psi_p^{(n)}(x) = \sum_{k=0}^{p} \frac{(-1)^{n-1} \cdot n!}{(x+k)^{n+1}}. \tag{4}$$

In [7], H. Minc and L. Sathre proved that, if r is a positive integer and $\phi(r) = (r!)^{\frac{1}{r}}$, then

$$1 < \frac{\phi(r+1)}{\phi(r)} < \frac{r+1}{r}, \tag{5}$$

which can be rearranged as

$$[\Gamma(1+r)]^{\frac{1}{r}} < [\Gamma(2+r)]^{\frac{1}{r+1}} \tag{6}$$

and

$$\frac{[\Gamma(1+r)]^{\frac{1}{r}}}{r} > \frac{[\Gamma(2+r)]^{\frac{1}{r+1}}}{r+1}. \tag{7}$$

In [1, 6], H. Alzer and J. S. Martins refined the right inequality in (5) and showed that, if n is a positive integer, then for all positive real numbers r, we have

$$\frac{n}{n+1} < \left(\frac{\frac{1}{n}\sum_{i=1}^{n} i^r}{\frac{1}{n+1}\sum_{i=1}^{n+1} i^r} \right)^{\frac{1}{r}} < \frac{\sqrt[n]{n!}}{\sqrt[n+1]{(n+1)!}}. \tag{8}$$

Both bounds in (8) are the best possible.

The inequalities in (5) were refined and generalized in [8, 2, 9, 11] and the following inequalities were obtained:

$$\frac{n+k+1}{n+m+k+1} < \left(\prod_{i=k+1}^{n+k} i \right)^{\frac{1}{n}} \Big/ \left(\prod_{i=k+1}^{n+m+k} i \right)^{\frac{1}{n+m}} \leq \sqrt{\frac{n+k}{n+m+k}} \tag{9}$$

where k is a nonnegative integer, n and m are natural numbers. For $n = m = 1$, the equality in (9) is valid.

In [8], inequalities in (9) we generalized, where Feng Qi obtained the following inequalities on the ration for the geometric means of a positive arithmetic sequence with unit difference for any nonnegative integer k and natural numbers n and m:

$$\frac{n+k+1+\alpha}{n+m+k+1+\alpha} < \frac{\left(\prod_{i=k+1}^{n+k}(i+\alpha)\right)^{\frac{1}{n}}}{\left(\prod_{i=k+1}^{n+m+k}(i+\alpha)\right)^{\frac{1}{n+m}}} \leq \sqrt{\frac{n+k+\alpha}{n+m+k+\alpha}} \tag{10}$$

where $\alpha \in [0,1]$ is a constant. For $n = m = 1$, the equality in (10) is valid.

Furthermore, for nonnegative integer k and natural numbers n and m, we have

$$\frac{a(n+k+1)+b}{a(n+m+k+1)+b} < \frac{\left(\prod_{i=k+1}^{n+k}(ai+b)\right)^{\frac{1}{n}}}{\left(\prod_{i=k+1}^{n+m+k}(ai+b)\right)^{\frac{1}{n+m}}} \leq \sqrt{\frac{a(n+k)+b}{a(n+m+k)+b}}, \tag{11}$$

where a is a positive constant and b a nonnegative integer. For $n = m = 1$, the equality in (11) is valid, (see [3]).

It is clear that inequalities in (11) extend those in (10).

In [4], the following monotonicity results for the Gamma function were established. The function $[\Gamma(1+\frac{1}{x})]^x$ decreases with $x > 0$ and $x[\Gamma(1+\frac{1}{x})]^x$ increases with $x > 0$, which recovers the inequalities in (5) which refers to integer value of r. These are equivalent to the function $[\Gamma(1+x)]^{\frac{1}{x}}$ being increasing and $\frac{[\Gamma(1+x)]^{\frac{1}{x}}}{x}$ being decreasing on $(0,\infty)$, respectively. In addition, it was proved that the function $x^{1-\gamma}[\Gamma(1+\frac{1}{x})]^x$ decreases for $0 < x < 1$, where $\gamma = 0.57721566\cdots$ denotes the Euler's constant, which is equivalent to $\frac{[\Gamma(1+x)]^{\frac{1}{x}}}{x^{1-\gamma}}$ being increasing on $(1,\infty)$.

In [2], the following monotonicity result was obtained. The function

$$\frac{[\Gamma(x+y+1)/\Gamma(y+1)]^{\frac{1}{x}}}{x+y+1} \tag{12}$$

is decreasing for $x \geq 1$, for fixed $y \geq 0$. Then, for positive real numbers x and y, we have

$$\frac{x+y+1}{x+y+2} \leq \frac{[\Gamma(x+y+1)/\Gamma(y+1)]^{\frac{1}{x}}}{[\Gamma(x+y+2)/\Gamma(y+1)]^{\frac{1}{x+1}}}. \tag{13}$$

2 Main results

The following Theorem is the main result of these notes.

Theorem 2.1 *Let $x > 0, y \geq 0$ be real numbers. The function*

$$f(x) = \frac{[\Gamma_p(x+y+1)/\Gamma_p(y+1)]^{\frac{1}{x}}}{x+y+1} \tag{14}$$

is strictly decreasing on $(0, \infty)$. Moreover

$$\lim_{x \to 0} f(x) = \frac{e^{\psi_p(y+1)}}{y+1} \tag{15}$$

and

$$\frac{x+y+1}{x+y+2} < \frac{\Big[\Gamma_p(x+y+1)/\Gamma_p(y+1)\Big]^{\frac{1}{x}}}{\Big[\Gamma_p(x+y+2)/\Gamma_p(y+1)\Big]^{\frac{1}{x+1}}}.$$

Taking logarithm yields

$$\ln f(x) = \frac{1}{x}\Big[\ln \Gamma_p(x+y+1) - \ln \Gamma_p(y+1)\Big] - \ln(x+y+1).$$

For $x > 0$, define

$$h(x) = x^2 \frac{f'(x)}{f(x)} = -\ln \frac{\Gamma_p(x+y+1)}{\Gamma_p(y+1)} + x\psi_p(x+y+1) - \frac{x^2}{x+y+1}.$$

Differentiation of h gives.

$$\frac{1}{x}h^{'}(x) = \psi_p^{'}(x+y+1) - \frac{1}{x+y+1} - \frac{y+1}{(x+y+1)^2}$$
$$= \sum_{n=1}^{p+1} \frac{1}{(x+y+n)^2} - \frac{1}{x+y+1} - \frac{y+1}{(x+y+1)^2}$$
$$= \sum_{n=1}^{p+1} \frac{1}{(x+y+n)^2} - \sum_{n=1}^{\infty}\left[\frac{1}{x+y+n} - \frac{1}{x+y+n+1}\right] -$$
$$- \sum_{n=1}^{\infty}\left[\frac{y+1}{(x+y+n)^2} - \frac{y+1}{(x+y+n+1)^2}\right]$$
$$< \sum_{n=1}^{\infty} \frac{1}{(x+y+n)^2} - \sum_{n=1}^{\infty}\left[\frac{1}{x+y+n} - \frac{1}{x+y+n+1}\right] -$$
$$- \sum_{n=1}^{\infty}\left[\frac{y+1}{(x+y+n)^2} - \frac{y+1}{(x+y+n+1)^2}\right]$$
$$= -\sum_{n=1}^{\infty}\left[\frac{y}{(x+y+n)^2} + \frac{1}{(x+y+n)(x+y+n+1)} - \frac{y+1}{(x+y+n+1)^2}\right]$$
$$= -\sum_{n=1}^{\infty} \frac{(2y+1)(x+y+n)+y}{(x+y+n)^2(x+y+n+1)^2} < 0.$$

Hence, the function h is strictly decreasing and $h(x) < h(0) = 0$, for $x > 0$, which yields the desired results that $f^{'}(x) < 0$, hence f is strictly decreasing on $(0, \infty)$ and one has the following inequality

$$\frac{x+y+1}{x+y+2} < \frac{\left[\Gamma_p(x+y+1)/\Gamma_p(y+1)\right]^{\frac{1}{x}}}{\left[\Gamma_p(x+y+2)/\Gamma_p(y+1)\right]^{\frac{1}{x+1}}}.$$

Next, by L. Hospital rule, we conclude that

$$\lim_{x\to 0} f(x) = \frac{e^{\psi_p(y+1)}}{y+1}, y \geq 0. \tag{16}$$

Definition 2.2 *A function $f(x)$ is logarithmically convex on the interval $[a, b]$ if $f > 0$ and $\ln f(x)$ is convex on $[a, b]$*

Theorem 2.3 *The function f given by (14) is strictly logarithmically convex on $(0, \infty)$.*

For $x > 0$ define

$$g(x) = x^3 \frac{d^2[\ln f(x)]}{dx^2} = 2\ln \frac{\Gamma_p(x+y+1)}{\Gamma_p(y+1)} - 2x\psi_p(x+y+1) + \\ + x^2\psi_p^{'}(x+y+1) + \frac{x^3}{(x+y+1)^2}.$$

Differentiation of g yields

$$\begin{aligned}\frac{1}{x^2} g^{'}(x) &= \psi_p^{''}(x+y+1) + \frac{1}{(x+y+1)^2} + \frac{2(y+1)}{(x+y+1)^3} \\ &= -\sum_{n=1}^{p+1} \frac{2}{(x+y+n)^3} + \sum_{n=1}^{\infty} \Big[\frac{1}{(x+y+n)^2} - \frac{1}{(x+y+n+1)^2}\Big] \\ &+ \sum_{n=1}^{\infty} \Big[\frac{2(y+1)}{(x+y+n)^3} - \frac{2(y+1)}{(x+y+n+1)^3}\Big] \\ &> -\sum_{n=1}^{\infty} \frac{2}{(x+y+n)^3} + \sum_{n=1}^{\infty} \Big[\frac{1}{(x+y+n)^2} - \frac{1}{(x+y+n+1)^2}\Big] \\ &+ \sum_{n=1}^{\infty} \Big[\frac{2(y+1)}{(x+y+n)^3} - \frac{2(y+1)}{(x+y+n+1)^3}\Big].\end{aligned}$$

$$= \sum_{n=1}^{\infty} \frac{3(2y+1)(x+y+n)^2 + (6y+1)(x+y+n) + 2y}{(x+y+n)^3(x+y+n+1)^3} > 0.$$

Hence the function g is strictly increasing and $g(x) > g(0) = 0$, for $x > 0$, which yields the desired results, that is $\frac{d^2(\ln f(x))}{dx^2} > 0$ for $x > 0$.

Corollary 2.4 *Let $y \geq 0$ be a real number. Then for all real numbers $x > 0$*

$$\frac{\Big[\Gamma_p(x+y+1)\Big/\Gamma_p(y+1)\Big]^{\frac{1}{x}}}{x+y+1} \leq \frac{e^{\psi_p(y+1)}}{y+1}. \tag{17}$$

Since f is decreasing and

$$\lim_{x\to 0} f(x) = \frac{e^{\psi_p(y+1)}}{y+1} \tag{18}$$

we obtain

$$f(x) \leq \lim_{x\to 0} f(x) = \frac{e^{\psi_p(y+1)}}{y+1}$$

and proof is completed.

3 Open Problem

At the end, we pose a problem.

For positive real numbers x and y, holds

$$\frac{[\Gamma_p(x+y+1)/\Gamma_p(y+1)]^{\frac{1}{x}}}{[\Gamma_p(x+y+2)/\Gamma_p(y+1)]^{\frac{1}{x+1}}} < \sqrt{\frac{x+y}{x+y+1}}. \tag{19}$$

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