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Intuitionistic Fuzzy Continuity and Uniform Convergence

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Abstract

A few of the algebraic and topological properties of intutionistic fuzzy continuity and uniformly intutionistic fuzzy continuity are investigated. Also, the concept of uniformly intutionistic fuzzy convergence is introduced thereafter a few results on uniformly intutionistic fuzzy convergence are studied.

Keywords: *Intuitionistic fuzzy norm linear space, Intuitionistic fuzzy continuity, Cauchy sequence, Uniformly Intuitionistic fuzzy continuity, Uniformly intuitionistic fuzzy convergent.*
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1 Introduction

The concept of intuitionistic fuzzy set, as a generalisation of fuzzy sets [12] was introduced by Atanassov [1]. Intuitionistic fuzzy set is used in the process of decision making. Cheng and Moderson [4] introduced the idea of fuzzy norm on a linear space. Bag and Samanta [2] deduce the definition of fuzzy norm whose associated matric is same as the associated metric of Cheng and Moderson [4].
In this paper after an introduction of intuitionistic fuzzy norm [7] and intuitionistic fuzzy continuity [7] deduced from Bag and Samanta [2] and [3], it has been shown that the class of intuitionistic fuzzy continuous functions is

closed with respect addition, multiplication, scalar multiplication and inverse operation of multiplication. Also, the intuitionistic fuzzy continuity is being characterized by open set and a few properties of open sets are also proved in intutionistic fuzzy normed linear space. Thereafter the concept of uniformly intuitionistic fuzzy continuity is introduced and it is proved that the uniformly intuitionistic fuzzy continuity implies the intuitionistic fuzzy continuity but not the converse.
In the last section, the concept of intuitionistic fuzzy convergence and uniformly intutionistic fuzzy convergence of a sequence of functions are introduced in intutionistic fuzzy normed linear space and then it is proved that the intuitionistic fuzzy continuity of each term of a sequence of function is transmitted to the limit function under uniformly intutionistic fuzzy convergence of the sequence of functions.

2 Preliminaries

We quote some definitions and statements of a few theorems which will be needed in the sequel.

Definition 2.1 *[10]. A binary operation* $* : [0\,,\,1] \times [0\,,\,1] \longrightarrow [0\,,\,1]$ *is continuous* t *- norm if* $*$ *satisfies the following conditions* :
(i) $*$ *is commutative and associative ,*
(ii) $*$ *is continuous ,*
(iii) $a * 1 = a \quad \forall\ a\ \varepsilon\ [0\,,\,1]$,
(iv) $a * b \leq c * d$ *whenever* $a \leq c\,,\, b \leq d$ *and* $a\,,\,b\,,\,c\,,\,d\ \varepsilon\ [0\,,\,1]$.

Definition 2.2 *[10]. A binary operation* $\diamond : [0\,,\,1] \times [0\,,\,1] \longrightarrow [0\,,\,1]$ *is continuous* t*-conorm if* $\diamond$ *satisfies the following conditions* :
(i) $\diamond$ *is commutative and associative ,*
(ii) $\diamond$ *is continuous ,*
(iii) $a \diamond 0 = a \quad \forall\ a \in [0\,,\,1]$,
(iv) $a \diamond b \leq c \diamond d$ *whenever* $a \leq c\,,\, b \leq d$ *and* $a\,,\,b\,,\,c\,,\,d \in [0\,,\,1]$.

Corollary 2.3 *[11].* (a) *For any* $r_1\,,\,r_2 \in (0\,,\,1)$ *with* $r_1 > r_2$, *there exist* $r_3\,,\,r_4 \in (0\,,\,1)$ *such that* $r_1 * r_3 > r_2$ *and* $r_1 > r_4 \diamond r_2$.
(b) *For any* $r_5 \in (0\,,\,1)$ *, there exist* $r_6\,,\,r_7 \in (0\,,\,1)$ *such that* $r_6 * r_6 \geq r_5$ *and* $r_7 \diamond r_7 \leq r_5$.

Definition 2.4 *[7]. Let* $*$ *be a continuous* t*-norm ,* $\diamond$ *be a continuous* t*-conorm and* V *be a linear space over the field* $F\,(= \mathbf{R}$ *or* $\mathbf{C}\,)$. *An* ***intuitionistic fuzzy norm*** *on* V *is an object of the form* $A = \{\,((x\,,\,t)\,,\,\mu(x\,,\,t)\,,\,\nu(x\,,\,t)\,) : (x\,,\,t) \in V \times \mathbf{R}^+\,\}$, *where* μ, ν *are fuzzy sets on* $V \times \mathbf{R}^+$, μ *denotes the degree of membership and* ν *denotes the degree of non - membership* $(x, t) \in V \times \mathbf{R}^+$ *satisfying the following conditions :*

(i) $\mu(x, t) + \nu(x, t) \leq 1 \quad \forall (x, t) \in V \times \mathbf{R}^+$;

(ii) $\mu(x, t) > 0$;

(iii) $\mu(x, t) = 1$ *if and only if* $x = \theta$, θ *is null vector ;*

(iv) $\mu(cx, t) = \mu(x, \frac{t}{|c|}) \quad \forall c \in F$ *and* $c \neq 0$;

(v) $\mu(x, s) * \mu(y, t) \leq \mu(x + y, s + t)$;

(vi) $\mu(x, \cdot)$ *is non-decreasing function of* $\mathbf{R}^+$ *and* $\lim_{t \to \infty} \mu(x, t) = 1$;

(vii) $\nu(x, t) < 1$;

$(viii)$ $\nu(x, t) = 0$ *if and only if* $x = \theta$;

(ix) $\nu(cx, t) = \nu(x, \frac{t}{|c|}) \quad \forall c \in F$ *and* $c \neq 0$;

(x) $\nu(x, s) \diamond \nu(y, t) \geq \nu(x + y, s + t)$;

(xi) $\nu(x, \cdot)$ *s non-increasing function of* $\mathbf{R}^+$ *and* $\lim_{t \to \infty} \nu(x, t) = 0$.

Definition 2.5 *[7]. If A is an intuitionistic fuzzy norm on a linear space V then* (V, A) *is called an intuitionistic fuzzy normed linear space.*

For the intuitionistic fuzzy normed linear space (V, A), we further assume that μ, ν, $*$, $\diamond$ satisfy the following axioms :

(xii) $\left.\begin{smallmatrix} a \diamond a = a \\ a * a = a \end{smallmatrix}\right\}$, for all $a \in [0, 1]$.

$(xiii)$ $\mu(x, t) > 0$, for all $t > 0 \Rightarrow x = \theta$.

(xiv) $\nu(x, t) < 1$, for all $t > 0 \Rightarrow x = \theta$.

Definition 2.6 *[7]. A sequence* $\{x_n\}_n$ *in an intuitionistic fuzzy normed linear space* (V, A) *is said to* ***converge*** *to* $x \in V$ *if for given* $r > 0$, $t > 0$, $0 < r < 1$, *there exist an integer* $n_0 \in \mathbf{N}$ *such that*
$\mu(x_n - x, t) > 1 - r$ *and* $\nu(x_n - x, t) < r$ *for all* $n \geq n_0$.

Definition 2.7 *[7]. A sequence* $\{x_n\}_n$ *in an intuitionistic fuzzy normed linear space* (V, A) *is said to be* ***cauchy sequence*** *if* $\lim_{n \to \infty} \mu(x_{n+p} - x_n, t) = 1$ *and* $\lim_{n \to \infty} \nu(x_{n+p} - x_n, t) = 0$, $p = 1, 2, 3, \cdots$.

Definition 2.8 *[7]. Let,* (U, A) *and* (V, B) *be two intuitionistic fuzzy normed linear space over the same field F. A mapping f from* (U, A) *to* (V, B) *is said to be* ***intuitionistic fuzzy continuous*** *at* $x_0 \in U$, *if for any given* $\epsilon > 0$, $\alpha \in (0, 1)$, $\exists \delta = \delta(\alpha, \epsilon) > 0$, $\beta = \beta(\alpha, \epsilon) \in (0, 1)$ *such that for all* $x \in U$,

$$\mu_U(x - x_0, \delta) > 1 - \beta \Rightarrow \mu_V(f(x) - f(x_0), \epsilon) > 1 - \alpha$$

$$\nu_U(x - x_0, \delta) < \beta \Rightarrow \nu_V(f(x) - f(x_0), \epsilon) < \alpha .$$

Definition 2.9 *[7]. A mapping f from (U, A) to (V, B) is said to be* ***sequentially intuitionistic fuzzy continuous*** *at $x_0 \in U$, if for any sequence $\{x_n\}_n$, $x_n \in U$, $\forall\, n \in \mathbf{N}$ with $x_n \to x_0$ in (U, A) implies $f(x_n) \to f(x_0)$ in (V, B), that is*

$$\lim_{n \to \infty} \mu_U(x_n - x_0, t) = 1 \text{ and } \lim_{n \to \infty} \nu_U(x_n - x_0, t) = 0$$

$$\Rightarrow \lim_{n \to \infty} \mu_V(f(x_n) - f(x_0), t) = 1 \text{ and } \lim_{n \to \infty} \nu_V(f(x_n) - f(x_0), t) = 0 .$$

Theorem 2.10 *[7]. Let, f be a mapping from (U, A) to (V, B). Then f is intuitionistic fuzzy continuous on U if and only if it is sequentially intuitionistic fuzzy continuous on U.*

3 Algebra of Intuitionistic Fuzzy Continuous Functions

In this section, consider (U, A) and (V, B) be any two intuitionistic fuzzy normed linear space over the same field F.

Theorem 3.1 *If $f : (U, A) \to (V, B)$ and $g : (U, A) \to (V, B)$ are two sequentially intuitionistic fuzzy continuous functions and (U, A) and (V, B) satisfies the condition (xii) then $f + g$, kf, where $k \in F$, are also sequentially intuitionistic fuzzy continuous functions over the same field F.*

Proof : Let, $\{x_n\}_n$ be a sequence in U such that $x_n \to x$ in (U, A). Thus $\forall\, t \in \mathbf{R}$ we have

$$\lim_{n \to \infty} \mu_U(x_n - x, t) = 1 \text{ and } \lim_{n \to \infty} \nu_U(x_n - x, t) = 0 \quad \cdots \quad (1)$$

Since f and g are sequentially intuitionistic fuzzy continuous at x, (1) implies

$$\lim_{n \to \infty} \mu_V(f(x_n) - f(x), t) = 1 , \lim_{n \to \infty} \nu_V(f(x_n) - f(x), t) = 0 , \ \forall\, t \in \mathbf{R},$$

$$\lim_{n \to \infty} \mu_V(g(x_n) - g(x), t) = 1 , \lim_{n \to \infty} \nu_V(g(x_n) - g(x), t) = 0 , \ \forall\, t \in \mathbf{R}$$

Now, $\mu_V((f + g)(x_n) - (f + g)(x), t)$

$$= \mu_V(f(x_n) - f(x) + g(x_n) - g(x), t)$$

$$\geq \mu_V\left(f(x_n) - f(x), \frac{t}{2}\right) * \mu_V\left(g(x_n) - g(x), \frac{t}{2}\right)$$

Taking limit we have,
$\lim\limits_{n \to \infty} \mu_V((f+g)(x_n) - (f+g)(x), t)$

$$\geq \lim_{n\to\infty} \mu_V\left(f(x_n) - f(x), \frac{t}{2}\right) * \lim_{n\to\infty} \mu_V\left(g(x_n) - g(x), \frac{t}{2}\right) = 1 * 1 = 1.$$

Again, $\nu_V((f+g)(x_n) - (f+g)(x), t)$

$$= \nu_V(f(x_n) - f(x) + g(x_n) - g(x), t)$$

$$\leq \nu_V\left(f(x_n) - f(x), \frac{t}{2}\right) \diamond \nu_V\left(g(x_n) - g(x), \frac{t}{2}\right)$$

Taking limit we have,
$\lim\limits_{n \to \infty} \nu_V((f+g)(x_n) - (f+g)(x), t)$

$$\leq \lim_{n\to\infty} \nu_V\left(f(x_n) - f(x), \frac{t}{2}\right) \diamond \lim_{n\to\infty} \nu_V\left(g(x_n) - g(x), \frac{t}{2}\right) = 0 \diamond 0 = 0.$$

So, $f + g$ is sequentially intuitionistic fuzzy continuous.
Obviously, $k f$ is sequentially intuitionistic fuzzy continuous for every $k \in F$.

We further assume that, for an intuitionistic fuzzy normed linear space (V, A) and for $x \neq \theta$,
(xv) $\mu(x, .)$ is a continuous function of $\mathbf{R}$ and strictly increasing on the subset $\{t : 0 < \mu(x, t) < 1\}$ of $\mathbf{R}$.
(xvi) $\nu(x, .)$ is a continuous function of $\mathbf{R}$ and strictly decreasing on the subset $\{t : 0 < \nu(x, t) < 1\}$ of $\mathbf{R}$.

Theorem 3.2 *If $f : (U, A) \to (V, B)$ and $g : (U, A) \to (V, B)$ are two sequentially intuitionistic fuzzy continuous functions and (U, A) and (V, B) satisfies (xii), (xv) and (xvi) then*
(a) $f g$ is sequentially intuitionistic fuzzy continuous functions over the same field F,
(b) if $g(x) \neq 0$, $\forall x \in U$ then $\frac{f}{g}$ is sequentially intuitionistic fuzzy continuous functions over the same field F.

Proof : (a) Let $\{x_n\}_n$ be a sequence in U such that $x_n \to x$ in (U, A). Thus, for all $t \in \mathbf{R}$ we have

$$\lim_{n\to\infty} \mu_U(x_n - x, t) = 1 \quad \textit{and} \quad \lim_{n\to\infty} \nu_U(x_n - x, t) = 0 \quad \cdots \quad (2)$$

Since f and g are sequentially intuitionistic fuzzy continuous at x, from (2), we have

$$\lim_{n\to\infty} \mu_V(f(x_n) - f(x), t) = 1, \ \lim_{n\to\infty} \nu_V(f(x_n) - f(x), t) = 0, \ \forall t \in \mathbf{R},$$

$\lim_{n \to \infty} \mu_V(g(x_n) - g(x), t) = 1, \lim_{n \to \infty} \nu_V(g(x_n) - g(x), t) = 0, \forall t \in \mathbf{R}$

Now, $\mu_V((fg)(x_n) - (fg)(x_0), t)$
$= \mu_V(f(x_n)(g(x_n) - g(x_0)) + g(x_0)(f(x_n) - f(x_0)), t)$
$= \mu_V((f(x_n) - f(x_0))(g(x_n) - g(x_0)) + f(x_0)(g(x_n) - g(x_0))$
$+ g(x_0)(f(x_n) - f(x_0)), t)$
$\geq \mu_V\left((f(x_n) - f(x_0))(g(x_n) - g(x_0)), \frac{t}{3}\right) * \mu_V\left(f(x_0)(g(x_n) - g(x_0)), \frac{t}{3}\right)$
$* \mu_V\left(g(x_0)(f(x_n) - f(x_0)), \frac{t}{3}\right)$
$= \mu_V\left(f(x_n) - f(x_0), \frac{t}{3|g(x_n) - g(x_0)|}\right) * \mu_V\left(g(x_n) - g(x_0), \frac{t}{3|f(x_0)|}\right)$
$* \mu_V\left(f(x_n) - f(x_0), \frac{t}{3|g(x_0)|}\right)$

Taking limit as $n \to \infty$ we have,

$\lim_{n \to \infty} \mu_V((fg)(x_n) - (fg)(x_0), t)$
$\geq \lim_{n \to \infty} \mu_V\left(f(x_n) - f(x_0), \frac{t}{3|g(x_n) - g(x_0)|}\right) * \lim_{n \to \infty} \mu_V\left(g(x_n) - g(x_0), \frac{t}{3|f(x_0)|}\right)$
$* \lim_{n \to \infty} \mu_V\left(f(x_n) - f(x_0), \frac{t}{3|g(x_0)|}\right)$
$= \mu_V\left(f(x_n) - f(x_0), \lim_{n \to \infty} \frac{t}{3|g(x_n) - g(x_0)|}\right) * \lim_{n \to \infty} \mu_V\left(g(x_n) - g(x_0), \frac{t}{3|f(x_0)|}\right)$
$* \lim_{n \to \infty} \mu_V\left(f(x_n) - f(x_0), \frac{t}{3|g(x_0)|}\right)$, *by(vii)*
$= \mu_V(f(x_n) - f(x_0), \infty) * 1 * 1 = 1 * 1 * 1 = 1$

and

$\nu_V((fg)(x_n) - (fg)(x_0), t)$
$= \nu_V(f(x_n)(g(x_n) - g(x_0)) + g(x_0)(f(x_n) - f(x_0)), t)$
$= \nu_V((f(x_n) - f(x_0))(g(x_n) - g(x_0)) + f(x_0)(g(x_n) - g(x_0))$
$+ g(x_0)(f(x_n) - f(x_0)), t)$
$\leq \nu_V\left((f(x_n) - f(x_0))(g(x_n) - g(x_0)), \frac{t}{3}\right) \diamond \nu_V\left(f(x_0)(g(x_n) - g(x_0)), \frac{t}{3}\right)$
$\diamond \nu_V\left(g(x_0)(f(x_n) - f(x_0)), \frac{t}{3}\right)$
$= \nu_V\left(f(x_n) - f(x_0), \frac{t}{3|g(x_n) - g(x_0)|}\right) \diamond \nu_V\left(g(x_n) - g(x_0), \frac{t}{3|f(x_0)|}\right)$
$\diamond \nu_V\left(f(x_n) - f(x_0), \frac{t}{3|g(x_0)|}\right)$

Taking limit as $n \to \infty$ we have,

$\lim_{n \to \infty} \nu_V((fg)(x_n) - (fg)(x_0), t)$
$\leq \lim_{n \to \infty} \nu_V\left(f(x_n) - f(x_0), \frac{t}{3|g(x_n) - g(x_0)|}\right) \diamond \lim_{n \to \infty} \nu_V\left(g(x_n) - g(x_0), \frac{t}{3|f(x_0)|}\right)$
$\diamond \lim_{n \to \infty} \nu_V\left(f(x_n) - f(x_0), \frac{t}{3|g(x_0)|}\right)$
$= \nu_V\left(f(x_n) - f(x_0), \lim_{n \to \infty} \frac{t}{3|g(x_n) - g(x_0)|}\right) \diamond \lim_{n \to \infty} \nu_V\left(g(x_n) - g(x_0), \frac{t}{3|f(x_0)|}\right)$
$\diamond \lim_{n \to \infty} \nu_V\left(f(x_n) - f(x_0), \frac{t}{3|g(x_0)|}\right)$, *by(vii)*
$= \nu_V(f(x_n) - f(x_0), \infty) \diamond 0 \diamond 0 = 0 \diamond 0 \diamond 0 = 0$

Hence the proof.

(b) We now show that $\frac{1}{g}$ is sequentially intuitionistic fuzzy continuous at x

if $g(x) \neq 0$ for all $x \in U$. $\mu_V\left(\frac{1}{g}(x_n) - \frac{1}{g}(x_0), t\right) = \mu_V\left(\frac{g(x_n) - g(x_0)}{g(x_n)\,g(x_0)}, t\right) = \mu_V\left(\frac{1}{g(x_n)\,g(x_0)}, \frac{t}{g(x_n) - g(x_0)}\right)$

Taking limit as $n \to \infty$ we have,

$\lim_{n \to \infty} \mu_V\left(\frac{1}{g}(x_n) - \frac{1}{g}(x_0), t\right)$

$$= \mu_V\left(\frac{1}{g(x_n)\,g(x_0)}, \lim_{n \to \infty} \frac{t}{g(x_n) - g(x_0)}\right) \quad by(vii)$$

$$= \mu_V\left(\frac{1}{g(x_n)\,g(x_0)}, \infty\right) = 1.$$

Again , $\nu\left(\frac{1}{g}(x_n) - \frac{1}{g}(x_0), t\right) = \nu_V\left(\frac{g(x_n) - g(x_0)}{g(x_n)\,g(x_0)}, t\right) = \nu_V\left(\frac{1}{g(x_n)\,g(x_0)}, \frac{t}{g(x_n) - g(x_0)}\right)$

Taking limit as $n \to \infty$ we have,

$\lim_{n \to \infty} \nu_V\left(\frac{1}{g}(x_n) - \frac{1}{g}(x_0), t\right)$

$$= \nu_V\left(\frac{1}{g(x_n)\,g(x_0)}, \lim_{n \to \infty} \frac{t}{g(x_n) - g(x_0)}\right) \quad by(vii)$$

$$= \nu_V\left(\frac{1}{g(x_n)\,g(x_0)}, \infty\right) = 0.$$

Hence $\frac{1}{g}$ is sequentially intuitionistic fuzzy continuous.
The proof is complited by considering the product of f and $\frac{1}{g}$.

Note 3.3 *Let,* $(V = \mathbf{R}, \|\cdot\|)$ *be a normed linear space and define* $a * b = \min\{a, b\}$ *and* $a \diamond b = \max\{a, b\}$ *for all* $a, b \in [0, 1]$ *. For all* $t > 0$*. Define ,* $\mu(x, t) = \frac{t}{t + k\,\|x\|}$ *and* $\nu(x, t) = \frac{k\,\|x\|}{t + k\,\|x\|}$ *where* $k > 0$*. It is easy to see that* $A = \{((x, t), \mu(x, t), \nu(x, t)) : (x, t) \in V \times \mathbf{R}^+\}$ *is an intuitionistic fuzzy norm on* V*. Let* $f : \mathbf{R} \to \mathbf{R}$*.Then* f *is continuous on* $(V, \|\cdot\|)$ *if and only if it is intuitionistic fuzzy continuous on* (V, A)*.*

Proof : By example (2) of [7], $\{x_n\}_n$ is convergent in $(V, \|\cdot\|)$ if and only if $\{x_n\}_n$ is convergent in (V, A). So, f is continuous on $(V, \|\cdot\|)$
$\Longleftrightarrow$ For any sequence $\{x_n\}_n$ converging to x in $(V, \|\cdot\|)$, $\{f(x_n)\}_n$ converges to $f(x)$ in $(V, \|\cdot\|)$.
$\Longleftrightarrow$ For any sequence $\{x_n\}_n$ converging to x in (V, A), $\{f(x_n)\}_n$ converges to $f(x)$ in (V, A).
$\Longleftrightarrow$ f is continuous on (V, A).

Definition 3.4 *Let* $0 < r < 1$ *,* $t \in \mathbf{R}^+$ *and* $x \in V$*. Then the set* $B(x, r, t) = \{y \in V : \mu(x - y, t) > 1 - r, \nu(x - y, t) < r\}$ *is called an* ***open ball*** *in* (V, A) *with* x *as its center and* r *as its radious with respect to* t*.*

Definition 3.5 *A subset G of V is said to be an* ***open set*** *in (V, A) if for each $x \in G$ there exist $r_x \in (0, 1)$ and $t \in \mathbf{R}^+$ such that $B(x, r_x, t) \subseteq G$.*

Theorem 3.6 *Every open ball $B(x, r, t)$ in (V, A) is an open set in (V, A)*

Proof : Let $B(x, r, t)$ be an open ball with center at x and radious r with respect to t. Then,

$$\mu(x - y, t) > 1 - r \ \ and \ \ \nu(x - y, t) < r \qquad (3).$$

Then for every $t_0 \in (0, t)$, the relation (3) is true. So, for $t_0 \in (0, t)$,

$$\mu(x - y, t_0) > 1 - r \ \ and \ \ \nu(x - y, t_0) < r$$

Let $r_0 = \mu(x - y, t_0)$. Since, $r_0 > 1 - r$, $\exists\, s \in (0, 1)$ such that $r_0 > 1 - s > 1 - r$.
Now for given r_0 and s such that $r_0 > 1 - s$, $\exists\, r_1, r_2 \in (0, 1)$ such that $r_0 * r_1 > 1 - s$ and $(1 - r_0) \diamond (1 - r_2) < s$
Let $r_3 = max\{r_1, r_2\}$.
Then, $r_0 * r_1 \leq r_0 * r_3$ and $r_2 \leq r_3$
$\Rightarrow 1 - r_3 \leq 1 - r_2$
$\Rightarrow (1 - r_0) \diamond (1 - r_3) \leq (1 - r_0) \diamond (1 - r_2)$.
These implies that,
$1 - s < r_0 * r_1 \leq r_0 * r_3$ and $(1 - r_0) \diamond (1 - r_3) \leq (1 - r_0) \diamond (1 - r_2) < s$
i.e., $r_0 * r_3 > 1 - s$ and $(1 - r_0) \diamond (1 - r_3) < s$.
Consider the open ball $B(y, 1 - r_3, t - t_0)$.
It is sufficient to show that $B(y, 1 - r_3, t - t_0) \subset B(x, r, t)$.
Let, $z \in B(y, 1 - r_3, t - t_0)$.
Then $\mu(y - z, t - t_0) > r_3$ and $\nu(y - z, t - t_0) < 1 - r_3$.
Therefore,

$$\begin{aligned} \mu(x - z, t) &= \mu(x - y + y - z, t_0 + (t - t_0)) \\ &\geq \mu(x - y, t_0) * \mu(y - z, t - t_0) \\ &> r_0 * r_3 > 1 - s > 1 - r. \end{aligned}$$

and

$$\begin{aligned} \nu(x - z, t) &= \nu(x - y + y - z, t_0 + (t - t_0)) \\ &\leq \nu(x - y, t_0) \diamond \nu(y - z, t - t_0) \\ &< (1 - r_0) \diamond (1 - r_3) < s < r. \end{aligned}$$

Thus $z \in B(x, r, t)$ and hence $B(y, 1 - r_3, t - t_0) \subset B(x, r, t)$.

Definition 3.7 *A subset N of V is said to be a* ***neighbourhood*** *of $x \, (\in V)$ in (V, A) if there exist $r \in (0, 1)$ and $t \in \mathbf{R}^+$ such that $B(x, r, t) \subset N$.*

Theorem 3.8 *For any two intuitionistic fuzzy normed linear space (U, A) and (V, B), the following statements are equivalent:*
(i) f is intuitionistic fuzzy continuous on U.
(ii) P is open in $(V, B) \Rightarrow f^{-1}(P)$ is open in (U, A).
(iii) For each $x \in U$, N is a neighbourhood of $f(x)$ in $(V, B) \Rightarrow f^{-1}(N)$ is a neighbourhood of x in (U, A).

Proof : $(i) \Rightarrow (ii)$: Suppose f is intuitionistic fuzzy continuous on U and P is open in (V, B). If $f^{-1}(P) = \phi$, then their is nothing to prove. Let, $f^{-1}(P) \neq \phi$ and $x_0 \in f^{-1}(P)$. Then $f(x_0) \in P$. So, there exist $\epsilon(> 0)$ and $\alpha \in (0, 1)$ such that $B(f(x_0), \alpha, \epsilon) \subset P$. Since f is intuitionistic fuzzy continuous on U, there exist $\delta(> 0)$ and $\beta \in (0, 1)$ such that for all $x \in U$,

$$\mu_U(x - x_0, \delta) > 1 - \beta \Rightarrow \mu_V(f(x) - f(x_0), \epsilon) > 1 - \alpha$$

$$\nu_U(x - x_0, \delta) < \beta \Rightarrow \nu_V(f(x) - f(x_0), \epsilon) < \alpha$$

i.e., $x \in B(x_0, \beta, \delta) \Rightarrow f(x) \in B(f(x_0), \alpha, \epsilon) \subset P$
$\Rightarrow B(x_0, \beta, \delta) \subset f^{-1}(P)$
$\Rightarrow f^{-1}(P)$ is open in (U, A).
$(ii) \Rightarrow (i)$: Let, $\epsilon(> 0)$ and $\alpha \in (0, 1)$ and $x_0 \in U$. Then $B(f(x_0), \alpha, \epsilon)$ is open in (V, B).
$\Rightarrow f^{-1}(B(f(x_0), \alpha, \epsilon))$ is open in (U, A) containing x_0.
$\Rightarrow \exists\, \delta > 0$ and $\beta \in (0, 1)$ such that $B(x_0, \beta, \delta) \subset f^{-1}(B(f(x_0), \alpha, \epsilon))$.
$\Rightarrow f(B(x_0, \beta, \delta)) \subset B(f(x_0), \alpha, \epsilon)$.
$\Rightarrow f$ is intuitionistic fuzzy continuous on U.
$(ii) \Rightarrow (iii)$: Let, $x \in U$ and N be a neighbourhood of $f(x)$ in (V, B). Therefore, there exist $r \in (0, 1)$ and $t > 0$ such that $B((f(x), r, t)) \subset N \Rightarrow x \in f^{-1}(B((f(x), r, t)) \subset f^{-1}(N)$.
Again, $x \in f^{-1}(B((f(x), r, t))$ and $f^{-1}(B((f(x), r, t))$ is open in (U, A). So, there exist $r_1 \in (0, 1)$ and $t_1 > 0$ such that
$B(x, r_1, t_1) \subset f^{-1}(B((f(x), r, t)) \subset f^{-1}(N)$
This shows that $f^{-1}(N)$ is a neighbourhood of x in (U, A).
$(iii) \Rightarrow (ii)$: Let, P be open in (V, B) and $x \in f^{-1}(P)$. Then $f(x) \in P$ and therefore there exist $\epsilon(> 0)$ and $\alpha \in (0, 1)$ such that $B(f(x), \alpha, \epsilon) \subset P$
$\Rightarrow P$ is a neighbourhood of $f(x)$ in (V, B)
$\Rightarrow f^{-1}(P)$ is a neighbourhood of x in (U, A)
$\Rightarrow \exists\, \delta(> 0)$ and $\beta \in (0, 1)$ such that $B(x, \beta, \delta) \subset f^{-1}(P)$.
$\Rightarrow f^{-1}(P)$ is open in (U, A).

Definition 3.9 *$f : U \to V$ is said to be **uniformly intuitionistic fuzzy continuous** on U if for any given $\epsilon > 0, \alpha \in (0,1)$ $\exists$ $\delta = \delta(\alpha, \epsilon) > 0$, $\beta = \beta(\alpha, \epsilon) > 0$ such that for any two points $x_1, x_2 \in U$,*

$$\mu_U(x_1 - x_2, \delta) > 1 - \beta \quad and \quad \nu_U(x_1 - x_2, \delta) < \beta$$

$$\Rightarrow \mu_V(f(x_1) - f(x_2), \epsilon) > 1 - \alpha \quad and \quad \nu_V(f(x_1) - f(x_2), \epsilon) < \alpha$$

Theorem 3.10 *Let f be uniformly intuitionistic fuzzy continuous on U. If $\{x_n\}_n$ is a cauchy sequence in (U, A), then $\{f(x_n)\}_n$ is a cauchy sequence in (V, B).*

Proof : f is uniformly intuitionistic fuzzy continuous on U.
$\Rightarrow$ For any given $\epsilon > 0$, $\alpha \in (0,1)$ $\exists$ $\delta = \delta(\alpha, \epsilon) > 0$, $\beta = \beta(\alpha, \epsilon) > 0$ such that for any two points $x', x'' \in U$, $\mu_U(x' - x'', \delta) > 1 - \beta$ and $\nu_U(x' - x'', \delta) < \beta$
$\Rightarrow$ $\mu_V(f(x') - f(x''), \epsilon) > 1 - \alpha$ and $\nu_V(f(x') - f(x'', \epsilon)) < \alpha$ $\cdots$ (4)Since $\{x_n\}_n$ is a cauchy sequence, for $\delta > 0$ and $\beta \in (0, 1)$ there exist a natural number k such that $\mu_U(x_n - x_m, \delta) > 1 - \beta$ and $\nu_U(x_n - x_m, \delta) < \beta$ $\forall$ $m, n \geq k \Rightarrow \mu_U(f(x_n) - f(x_m), \epsilon) > 1 - \alpha$ *and* $\nu_U(f(x_n) - f(x_m), \epsilon) < \alpha, \forall m, n \geq k$ (by (4)) $\Rightarrow$ $\{f(x_n)\}_n$ is a cauchy sequence in (V, B)

Theorem 3.11 *If $f : U \to V$ is uniformly intuitionistic fuzzy continuous on U then f is intuitionistic fuzzy continuous on U but not the converse.*

Proof : Obvious.

To show the converse result does not hold, consider the following example.

Example 3.12 *Let $(X = \mathbf{R}, \| \cdot \|)$ be a normed linear space, where $\| x \| = | x |$, $\forall$ $x \in \mathbf{R}$. Define $a * b = min\{a, b\}$ and $a \diamond b = max\{a, b\}$ $\forall$ $a, b \in [0,1]$. Also, define*

$$\mu_1, \nu_1, \mu_2, \nu_2 : X \times \mathbf{R} \to [0,1] \quad by$$

$$\mu_1 = \frac{t}{t + |x|}, \nu_1 = \frac{|x|}{t + |x|}, \mu_2 = \frac{t}{t + k|x|}, \nu_2 = \frac{k|x|}{t + k|x|},$$

Let $A = \{((x, t), \mu_1, \nu_1) : (x, t) \in X \times \mathbf{R}\}$ and $B = \{((x, t), \mu_2, \nu_2) : (x, t) \in X \times \mathbf{R}\}$ be two intuitionistic fuzzy norm on X.
Let us define $f(x) = \frac{1}{x}$ $\forall$ $x \in (0,1)$. First we show that f is intuitionistic

fuzzy continuous on $(0, 1)$. *Let* $x_0 \in (0, 1)$ *and* $\{x_n\}_n$ *be a sequence in* $(0, 1)$ *such that* $x_n \to x_0$ *in* (X, A). *i.e., for all* $t > 0$,

$$\lim_{n \to \infty} \mu_1(x_n - x_0, t) = 1 \quad and \quad \lim_{n \to \infty} \nu_1(x_n - x_0, t) = 0$$

$$\Rightarrow \lim_{n \to \infty} \frac{t}{t + |x_n - x_0|} = 1 \quad and \quad \lim_{n \to \infty} \frac{|x_n - x_0|}{t + |x_n - x_0|} = 0$$

$$\Rightarrow \lim_{n \to \infty} |x_n - x_0| = 0$$

Again, for all $t > 0$,

$$\mu_2(f(x_n) - f(x_0), t) = \frac{t}{t + k\,|f(x_n) - f(x_0)|}, = \frac{t\,x_n\,x_0}{t\,x_n\,x_0 + k\,|x_n - x_0|}$$

$$\Rightarrow \lim_{n \to \infty} \mu_2(f(x_n) - f(x_0), t) = 1$$

and

$$\nu_2(f(x_n) - f(x_0), t) = \frac{k\,|f(x_n) - f(x_0)|}{t + k\,|f(x_n) - f(x_0)|} = \frac{k\,|x_n - x_0|}{t\,x_n\,x_0 + k\,|x_n - x_0|}$$

$$\Rightarrow \lim_{n \to \infty} \nu_2(f(x_n) - f(x_0), t) = 0$$

Thus f *is sequentially intuitionistic fuzzy continuous on* $(0, 1)$ *and hence intuitionistic fuzzy continuous on* $(0, 1)$. *We now show that* f *is not uniformly intuitionistic fuzzy continuous on* $(0, 1)$. *By example* 2 *of [7], we see that* $\{x_n\}_n$ *is a cauchy sequence in* $(X, \|\cdot\|)$ *if and only if* $\{x_n\}_n$ *is a cauchy sequence in* (X, A) *or* (X, B).
Let, $x_n = \frac{1}{n+1}$ $\forall\, n \in \mathbf{N}$. *So,* $\{f(x_n)\}_n$ *is a not a cauchy sequence in* $(X, \|\cdot\|)$ *and hence not a cauchy sequence* (X, B).
Consequently, f *is not uniformly intuitionistic fuzzy continuous on* $(0, 1)$.

4 Uniformly Intuitionistic Fuzzy Convergence

Definition 4.1 *Let* $f_n : (U, A) \longrightarrow (V, B)$ *be a sequence of functions. The sequence* $\{f_n\}_n$ *is said to be* ***pointwise intuitionistic fuzzy convergent*** *on* U *with respect to* A *if for each* $x \in U$, *the sequence* $\{f_n(x)\}_n$ *is convergent with respect to* B.

Let the sequence $\{f_n\}_n$ be pointwise intuitionistic fuzzy convergent on U and let $c \in U$. Then the sequence $\{f_n(c)\}_n$ is intuitionistic fuzzy convergent on (V, B). Let $f_n(c) \longrightarrow y_c$ in (V, B). Then y_c is unique. Let us now define $f : (U, A) \longrightarrow (V, B)$ by $f(x) = y_x\ \forall\, x \in U$, where

$f_n(x) \longrightarrow y_x$ in (V, B). Then f is said to be the intuitionistic fuzzy limit function of the sequence $\{f_n\}_n$ on U and it is written as $f_n \longrightarrow f$ on (U, A).

Example 4.2 *Let* $a * b = min\{a, b\}$, $a \diamond b = max\{a, b\}$ *for all* $a, b \in [0, 1]$. *Define* $\mu(x, t) = \frac{t}{t + |x|}$ *and* $\nu(x, t) = \frac{|x|}{t + |x|}$.
Let $U = (-1, 1)$, $V = \mathbf{R}, \mu = \mu_U = \mu_V$, $\nu = \nu_U = \nu_V$ *and* $f_n : (U, A) \to (V, B)$ *be defined by* $f_n(x) = x^n \ \forall x \in U$. *Also, let* $O(x) = 0 \ \forall x \in U$. *Therefore,*

$$\mu(f_n(x) - 0, t) = \frac{t}{t + |x|^n} \longrightarrow 1 \quad as \quad n \to \infty,$$

$$\nu(f_n(x) - 0, t) = \frac{|x|^n}{t + |x|^n} = 1 - \frac{t}{t + |x|^n} \to 0 \quad as \quad n \to \infty$$

$\Rightarrow$ $\{f_n\}_n$ *is pointwise intuitionistic fuzzy convergent to* O *on* (U, A).

Example 4.3 *Let* $a * b = min\{a, b\}$, $a \diamond b = max\{a, b\}$ *for all* $a, b \in [0, 1]$. *Let* $U = \{x \in \mathbf{R} : x \geq 0\}$, $V = \mathbf{R}$, $\mu = \mu_U = \mu_V$, $\nu = \nu_U = \nu_V$ *where*

$$\mu(x, t) = \frac{t}{t + |x|} \quad and \quad \nu(x, t) = \frac{|x|}{t + |x|}.$$

$$Consider,\ g_n(x) = \frac{n}{x + n} \ \forall x \in U \quad and \quad g(x) = 1 \ \forall x \in U.$$

$$Therefore,\quad g_n(x) - g(x) = \frac{n}{x + n} - 1 = -\frac{x}{x + n}$$

$$\mu(g_n(x) - g(x), t) = \mu(-\frac{x}{x + n}, t)$$

$$= \frac{t}{t + |-\frac{x}{x+n}|} = \frac{t}{t + \frac{x}{x+n}} \to 1 \ as \ n \to \infty$$

and

$$\nu(g_n(x) - g(x), t) = \frac{\frac{x}{x+n}}{t + \frac{x}{x+n}} = \frac{x}{x + t(x + n)} \to 0 \ as \ n \to \infty$$

Thus, we see that $g_n(x) \to g(x) \ \forall x \in U$ *with respect to* B.

Definition 4.4 *Let, $f_n : (U, A) \longrightarrow (V, B)$ be a sequence of functions. The sequence $\{f_n\}_n$ is said to be* ***uniformly intuitionistic fuzzy convergent*** *on U to a function f with respect to A, if given $0 < r < 1$, $t > 0$ there exist a positive integer $n_0 = n_0(r, t)$ such that $\forall\, x \in U$ and $\forall\, n \geq n_0$,*

$$\mu(f_n(x) - f(x), t) > 1 - r \;,\; \nu(f_n(x) - f(x), t) < r$$

Example 4.5 *In the example (4.1), we have seen that $f_n \longrightarrow O$ with respect to A. Let us show that this convergence is not uniform on $(0, 1)$ but converges uniformly on $[0, a]$ where $0 < a < 1$, with respect to A.*
Let $c \in (0, 1)$, $r \in (0, 1)$ and $t > 0$. Then,

$$\mu(f_n(c) - O(c), t) > 1 - r \;\; and \;\; \nu(f_n(c) - O(c), t) < r$$

$$\Rightarrow \frac{t}{t + c^n} > 1 - r \;\; and \;\; \frac{c^n}{t + c^n} < r$$

$$\Rightarrow c^n < \frac{r\,t}{(1 - r)} \;\; \Rightarrow \frac{1}{c^n} > \frac{(1 - r)}{r\,t}$$

$$\Rightarrow n > \frac{\log\left(\frac{(1-r)}{r\,t}\right)}{\log\left(\frac{1}{c}\right)}$$

Let $k = \left[\frac{\log\left(\frac{1-r}{r\,t}\right)}{\log\left(\frac{1}{c}\right)} \right] + 1$
Then, for each $x \in (0, 1)$ and given $r \in (0, 1)$ and $t > 0$,

$$\mu(f_n(x) - O(x), t) > 1 - r \;\; and \;\; \nu(f_n(x) - O(x), t) < r \quad \forall\, n \geq k$$

where, $k = \left[\frac{\log\left(\frac{1-r}{r\,t}\right)}{\log\left(\frac{1}{x}\right)} \right] + 1$, *which shows that k depends on r, t as well as on x. Also, we see that as $x \rightarrow 1 \Rightarrow k \rightarrow \infty$.*
$\Rightarrow$ *$\{f_n\}_n$ is not uniformly intuitionistic fuzzy convergent on $(0, 1)$ with respect to A.*
Let $a \in (0, 1)$. In $[0, a]$, the greatest value of $\left[\frac{\log\left(\frac{1-r}{r\,t}\right)}{\log\left(\frac{1}{x}\right)} \right]$ *is* $\left[\frac{\log\left(\frac{1-r}{r\,t}\right)}{\log\left(\frac{1}{a}\right)} \right]$.
So, let $n_0 = \left[\frac{\log\left(\frac{1-r}{r\,t}\right)}{\log\left(\frac{1}{a}\right)} \right] + 1$.
Therefore, for all $x \in [0, a]$, given $r \in (0, 1)$ and $t > 0$, there exist a natural number $n_0 = n_0(r, t)$ such that

$$\mu(f_n(x) - O(x), t) > 1 - r \;\; and \;\; \nu(f_n(x) - O(x), t) < r \quad \forall\, n \geq n_0$$

$\Rightarrow$ *$\{f_n\}_n$ is uniformly intuitionistic fuzzy convergent on $[0, a]$ with respect to A, where $a \in (0, 1)$.*

Result 4.6 *Let* $(U, \|\cdot\|_1)$ *and* $(V, \|\cdot\|_2)$ *be two normed linear space over the field* $\mathbf{K} = \mathbf{R}$ *or* $\mathbf{C}$, $f_n : U \longrightarrow V \; \forall \, n \in \mathbf{N}$, $a * b = \min\{a, b\}$, $a \diamond b = \max\{a, b\} \; \forall \, a, b \in [0, 1]$. *For all* $t > 0$, *define*

$$\mu_U(x, t) = \frac{t}{t + k\,\|x\|_1} \quad , \quad \nu_U(x, t) = \frac{k\,\|x\|_1}{t + k\,\|x\|_1} \; ,$$

$$\mu_V(x, t) = \frac{t}{t + k\,\|x\|_2} \quad , \quad \nu_V(x, t) = \frac{k\,\|x\|_2}{t + k\,\|x\|_2} \; ,$$

where $k > 0$. *Let*

$$A = \left\{ ((x, t), \mu_U(x, t), \nu_U(x, t)) \; : \; (x, t) \in U \times \mathbf{R}^+ \right\} ,$$

$$B = \left\{ ((x, t), \mu_V(x, t), \nu_V(x, t)) \; : \; (x, t) \in U \times \mathbf{R}^+ \right\}$$

Then (U, A) *and* (V, B) *are intuitionistic fuzzy normed linear space. Following the example* (2) *of [7] , it can shown that* $\{f_n\}$ *is uniformly intuitionistic fuzzy convergent on* U *with respect to* A *if and only if* $\{f_n\}$ *is uniformly convergent with respect to* $\|\cdot\|_1$.

Theorem 4.7 *Let* $f_n : (U, A) \longrightarrow (V, B)$, $\forall \, n \in \mathbf{N}$ *be a sequence of functions. Then the sequence* $\{f_n\}_n$ *is uniformly intuitionistic fuzzy convergent on* (U, A) *if and only if for any given* $r \in (0, 1)$ *and* $t > 0$ *there exist a natural number* $k = k(r, t)$ *such that* $\forall \, x \in U$,

$$\mu(f_{n+p}(x) - f_n(x), t) > 1 - r \; , \; \nu(f_{n+p}(x) - f_n(x), t) < r \; ,$$

$$\forall \, n \geq k \text{ and } p = 1, 2, 3, \cdots$$

Proof : $\Rightarrow$ **part :** Let, $\{f_n\}_n$ be uniformly intuitionistic fuzzy convergent on (U, A) and f be its limit function. Then for any given $r \in (0, 1)$ and $t > 0$ there exist a natural number $n_0 = n_0(r, t)$ such that for all $x \in U$, and $\forall \, n \geq n_0$,

$$\mu\left(f_n(x) - f(x), \frac{t}{2}\right) > 1 - r \; , \; \nu\left(f_n(x) - f(x), \frac{t}{2}\right) < r$$

$\Rightarrow$ For all $n \geq n_0$ and $p = 1, 2, 3, \cdots$ and $x \in U$,

$$\mu\left(f_{n+p}(x) - f(x), \frac{t}{2}\right) > 1 - r \; , \; \nu\left(f_{n+p}(x) - f(x), \frac{t}{2}\right) < r$$

Now, for all $x \in U$ and $p = 1, 2, 3, \cdots$, , we see that

$$\mu(f_{n+p}(x) - f_n(x), t)$$

$$
\begin{aligned}
&= \mu\left(f_{n+p}(x) - f(x) + f(x) - f_n(x)\,,\, \frac{t}{2} + \frac{t}{2}\right)\\
&\geq \mu\left(f_{n+p}(x) - f(x)\,,\, \frac{t}{2}\right) * \mu\left(f(x) - f_n(x)\,,\, \frac{t}{2}\right)\\
&= \mu\left(f_{n+p}(x) - f(x)\,,\, \frac{t}{2}\right) * \mu\left(f_n(x) - f(x)\,,\, \frac{t}{2}\right)\\
&> (1-r) * (1-r) = (1-r) \quad \forall\, n \geq n_0
\end{aligned}
$$

and

$$
\begin{aligned}
&\nu\,(f_{n+p}(x) - f_n(x)\,,\, t)\\
&\quad = \nu\left(f_{n+p}(x) - f(x) + f(x) - f_n(x)\,,\, \frac{t}{2} + \frac{t}{2}\right)\\
&\quad \leq \nu\left(f_{n+p}(x) - f(x)\,,\, \frac{t}{2}\right) \diamond \nu\left(f(x) - f_n(x)\,,\, \frac{t}{2}\right)\\
&\quad = \nu\left(f_{n+p}(x) - f(x)\,,\, \frac{t}{2}\right) \diamond \nu\left(f_n(x) - f(x)\,,\, \frac{t}{2}\right)\\
&\quad < r \diamond r = r \quad \forall\, n \geq n_0
\end{aligned}
$$

Hence the $\Rightarrow$ part.
$\Leftarrow$ **part :** In this part, we suppose that for any given $r \in (0\,,1)$ and $t > 0$ there exist a natural number $n_0 = n_0(r\,,\,t)$ such that for all $x \in U$ and $\forall\, n \geq n_0$

$$\mu\,(f_{n+p}(x) - f_n(x)\,,\,t) > 1 - r\,,\ \nu\,(f_{n+p}(x) - f_n(x)\,,\,t) < r.$$

Let $x_0 \in U$. Then for $\forall\, n \geq n_0$ we see that,

$$\mu\,(f_{n+p}(x_0) - f_n(x_0)\,,\,t) > 1 - r\,,\ \nu\,(f_{n+p}(x_0) - f_n(x_0)\,,\,t) < r.$$

$\Rightarrow$ $\{f_n(x_0)\}_n$ is an intuitionistic fuzzy cauchy sequence in $(V\,,\,B)$.
$\Rightarrow$ $\{f_n(x_0)\}_n$ is an intuitionistic fuzzy convergent in $(V\,,\,B)$.
$\Rightarrow$ $\{f_n\}_n$ is pointwise intuitionistic fuzzy convergent on $(U\,,\,A)$.
Let f be the intuitionistic fuzzy limit function of $\{f_n\}_n$ on $(U\,,\,A)$. Let $r \in (0\,,1)$ and $t > 0$. Then by the given condition, there exist a natural number $n_0 = n_0(r\,,t)$ such that for all $x \in U$ and $p = 1\,,2\,,3\,,\cdots$ and $\forall\, n \geq n_0$

$$\mu\left(f_{n+p}(x) - f_n(x)\,,\, \frac{t}{2}\right) > 1 - r\,,\ \nu\left(f_{n+p}(x) - f_n(x)\,,\, \frac{t}{2}\right) < r.$$

Again since $f_n \to f$ as $n \to \infty$ on $(U\,,\,A)$, we see that $f_{n+p} \to f$ as $n \to \infty$ on $(U\,,\,A)$, which implies that for all $n \geq n_0$ and for all $x \in U$,

$$\mu\left(f_{n+p}(x) - f(x)\,,\, \frac{t}{2}\right) > 1 - r\,,\ \nu\left(f_{n+p}(x) - f(x)\,,\, \frac{t}{2}\right) < r$$

Now, for all $x \in U$ we see that

$$\begin{aligned}
&\mu\,(f_n(x) - f(x)\,,\,t) \\
&\quad = \mu\left(f_n(x) - f_{n+p}(x) + f_{n+p}(x) - f(x)\,,\,\frac{t}{2} + \frac{t}{2}\right) \\
&\quad \geq \mu\left(f_n(x) - f_{n+p}(x)\,,\,\frac{t}{2}\right) * \mu\left(f_{n+p}(x) - f(x)\,,\,\frac{t}{2}\right) \\
&\quad > (1-r) * (1-r) = (1-r)\ , \quad \forall\, n \geq n_0
\end{aligned}$$

and

$$\begin{aligned}
&\nu\,(f_n(x) - f(x)\,,\,t) \\
&\quad = \nu\left(f_n(x) - f_{n+p}(x) + f_{n+p}(x) - f(x)\,,\,\frac{t}{2} + \frac{t}{2}\right) \\
&\quad \leq \nu\left(f_n(x) - f_{n+p}(x)\,,\,\frac{t}{2}\right) \diamond \nu\left(f_{n+p}(x) - f(x)\,,\,\frac{t}{2}\right) \\
&\quad < r \diamond r = r\ , \quad \forall\, n \geq n_0
\end{aligned}$$

$\Rightarrow$ $\{f_n\}_n$ is uniformly intuitionistic fuzzy convergent on (U, A).

Equivalent Statement: Let $f_n : (U, A) \longrightarrow (V, B)$, $\forall\, n \in \mathbf{N}$ be a sequence of functions. Then the sequence $\{f_n\}_n$ is uniformly intuitionistic fuzzy convergent on (U, A) if and only if for any given $r \in (0, 1)$ and $t > 0$ there exist a natural number $n_0 = n_0(r, t)$ such that $\forall\, x \in U$,

$$\mu(f_n(x) - f_m(x)\,,\,t) > 1 - r\ ,\ \nu(f_n(x) - f_m(x)\,,\,t) < r\ ,\ \forall\, n, m \geq n_0.$$

Example 4.8 *In the example* (4.3), *though we have seen that* $\{f_n\}_n$ *is uniformly intuitionistic fuzzy convergent on* $[0, a]$, *where* $a \in (0, 1)$ *and* $f_n(x) = x^n$, *again, we will verify it by using the above theorem . Let* $r \in (0, 1)$ *and* $t > 0$. *Again let,* $m, n \in \mathbf{N}$ *such that* $m < n$. *Now ,*

$$\begin{aligned}
&\mu(f_n(x) - f_m(x)\,,\,t) > 1 - r\ ,\ \ \nu(f_n(x) - f_m(x)\,,\,t) < r \\
&\quad \Rightarrow\ \mu(x^n - x^m\,,\,t) > 1 - r\ ,\ \ \nu(x^n - x^m\,,\,t) < r \\
&\quad \Rightarrow |\, x^n - x^m \,| < \frac{r\,t}{(1-r)}\ .
\end{aligned}$$

Since, $\sup\limits_{x \in [0, a]} |x^n - x^m| = 2\,a^m$, $m < n$ *we have ,* $2\,a^m < \frac{r\,t}{(1-r)}$, *which implies that* $m > \left[\frac{\log\, 2\left(\frac{1-r}{r\,t}\right)}{\log\left(\frac{1}{a}\right)}\right]$ *Let ,* $k = \left[\frac{\log\, 2\left(\frac{1-r}{r\,t}\right)}{\log\left(\frac{1}{a}\right)}\right] + 1$. *Thus, we see that for given* $r \in (0, 1)$ *and* $t > 0$, *there exist a natural* *number* $k = k(r, t)$ *such that* $\forall\, x \in [0, a]$, $a \in (0, 1)$ *and* $\forall\, n > m \geq k$

$$\mu(f_n(x) - f_m(x), t) > 1 - r \;,\; \nu(f_n(x) - f_m(x), t) < r \,.$$

This completes the verification .

Theorem 4.9 ***(Uniform Limit Theorem)***: *Let* (U, A) *and* (V, B) *be two intuitionistic fuzzy normed linear space satisfying the condition* (*xii*) *. Also, let* $f_n : (U, A) \longrightarrow (V, B)$, $\forall\, n \in \mathbf{N}$ *and* f_n *be intuitionistic fuzzy continuous on* (U, A) *. If* $\{f_n\}_n$ *be uniformly intuitionistic fuzzy convergent on* (U, A) *to a function* f *then* f *is intuitionistic fuzzy continuous on* (U, A) *.*

Proof : Let $\{f_n\}_n$ be uniformly intuitionistic fuzzy convergent to the function f on (U, A) . Then for any given $r \in (0, 1)$ and $t > 0$, there exists a natural number $k = k(r, t)$ such that for all $x \in U$ and for all $n \geq k$,

$$\mu_V\left(f_n(x) - f(x), \frac{t}{3}\right) > 1 - r \;,\; \nu_V\left(f_n(x) - f(x), \frac{t}{3}\right) < r$$

Thus, for all $x \in U$,

$$\mu_V\left(f_k(x) - f(x), \frac{t}{3}\right) > 1 - r \;,\; \nu_V\left(f_k(x) - f(x), \frac{t}{3}\right) < r$$

Let x_0 be an arbitrary but fixed point of U . Then we have

$$\mu_V\left(f_k(x_0) - f(x_0), \frac{t}{3}\right) > 1 - r \;,\; \nu_V\left(f_k(x_0) - f(x_0), \frac{t}{3}\right) < r$$

Since each f_n is intuitionistic fuzzy continuous on U , f_k is intuitionistic fuzzy continuous at x_0 . So, for any given $r \in (0, 1)$ and $t > 0$, there exist $\delta = \delta\left(r, \frac{t}{3}\right) > 0$, $\beta = \beta\left(r, \frac{t}{3}\right) \in (0, 1)$ such that

$$\mu_U(x - x_0, \delta) > 1 - \beta \;\Rightarrow\; \mu_V\left(f_k(x) - f_k(x_0), \frac{t}{3}\right) > 1 - r \,,$$

$$\nu_U(x - x_0, \delta) < \beta \;\Rightarrow\; \nu_V\left(f_k(x) - f_k(x_0), \frac{t}{3}\right) < r$$

Thus, we see that for $\mu_U(x - x_0, \delta) > 1 - \beta$,

$\mu_V(f(x) - f(x_0), t) = \mu_V(f(x) - f_k(x) + f_k(x) - f_k(x_0) + f_k(x_0) - f(x_0), t)$
$\geq \mu_V\left(f(x) - f_k(x), \frac{t}{3}\right) * \mu_V\left(f_k(x) - f_k(x_0), \frac{t}{3}\right)$
$* \mu_V\left(f_k(x_0) - f(x_0), \frac{t}{3}\right)$

$> (1-r) * (1-r) * (1-r) = 1-r$

Thus, we have

$$\mu_U(x - x_0, \delta) > 1-\beta \Rightarrow \mu_V(f(x) - f(x_0), t) > 1-r \quad \cdots \quad (5)$$

Again , for $\nu_U(x - x_0, \delta) < \beta$,

$\nu_V(f(x) - f(x_0), t) = \nu_V(f(x) - f_k(x) + f_k(x) - f_k(x_0) + f_k(x_0) - f(x_0), t)$

$\leq \nu_V\left(f(x) - f_k(x), \frac{t}{3}\right) \diamond \nu_V\left(f_k(x) - f_k(x_0), \frac{t}{3}\right) \diamond \nu_V\left(f_k(x_0) - f(x_0), \frac{t}{3}\right)$

$< r \diamond r \diamond r = r.$

Hence, we have

$$\nu_U(x - x_0, \delta) < \beta \Rightarrow \nu_V(f(x) - f(x_0), t) < r \quad \cdots \quad (6)$$

Thus , from (5) and (6) it follows that f is intuitionistic fuzzy continuous on (U, A) .

Note 4.10 *The converse of the above theorem is not necessarily true. For example, we consider the sequence of functions of example 4.3. It is obvious that each* f_n *is sequentially intuitionistic fuzzy continuous on* $(0, 1)$ *and hence is intuitionistic fuzzy continuous on* $(0, 1)$. *Also, the limit function* f *is intuitionistic fuzzy continuous on* $(0, 1)$, *but the intuitionistic fuzzy convergence is not uniformly intuitionistic fuzzy convergent on* $(0, 1)$

5 Open Problem

One can develop the concept of differentiation and Riemann integration in an intuitionistic fuzzy normed linear space and then verify whether the term by term differentiation and integration are valid or not for a sequence of function in an intuitionistic fuzzy normed linear space.

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