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On Subordination and Superordination of New Multiplier Transformation

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Abstract

Let $\phi_\mu(z,m,c)=\sum_{k=0}^{\infty}\frac{(\mu)_k}{k!}\frac{z^k}{(k+c)^m},\quad z\in\mathbb{C},\ |z|<1,\ c\in\mathbb{C}/\{0,-1,-2,\ldots\},\ \mu,m\in\mathbb{C}$ ***be the generalized Hurwitz–Lerch Zeta function. We consider***

$$zF(z,m,c)=z(1+c)^m\phi_\mu(z,m,c)=\sum_{k=0}^{\infty}\left(\frac{1+c}{k+c}\right)^m\frac{(\mu)_k}{k!}z^{k+1},$$

and define

$$[zF(z,m,c)]*[zF(z,m,c)]^{(-1)}=\frac{z}{(1-z)^{\lambda+1}},\ \lambda>-1,$$

where $*$ ***denotes convolution (Hadamard product). Let*** f ***be the normalized analytic function in the open unit disk*** U ***. We define a new operator*** $D_{\mu,c}^{\lambda,m}f(z)=[zF(z,m,c)]^{(-1)}*f(z)$. ***Moreover, we obtain some differential subordination and superordination results involving this operator. These results are obtained by investigating classes of admissible functions. Sandwich-type result is also studied.***

Keywords*: Hurwitz–Lerch zeta function, Hadamard product, Multiplier Transformation, Differential Subordination and Superordination*

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1 Introduction and Definitions

Let $H(U)$ denote the class of holomorphic functions in the open unit disk $U=\{z : z\in\mathbb{C}\ |z|<1\}$. For $a\in\mathbb{C}$ *and* $n\in\mathbb{N}$ we let

$$\mathrm{H}[a,n]=\{f\in H(U),\ f(z)=a+a_n z^n+a_{n+1}z^{n+1}+\dots,\ z\in U\}$$

and

$$A=\{f\in H(U),\ f(z)=z+a_2 z^2+\dots,\ z\in U\}.$$

For $f_j\in A$ given by

$$f_j(z)=z+\sum_{k=2}^{\infty}a_{k,j}z^k,\ \ (j=1,2),$$

the Hadamard product (or convolution) f_1*f_2 of f_1 and f_2 is defined by

$$(f_1*f_2)(z)=z+\sum_{k=2}^{\infty}a_{k,1}a_{k,2}z^k.$$

Let F and G be analytic functions in the unit disk U. The function F is subordinate to G, written $F\prec G$ if G is univalent, $F(0)=G(0)$ and $F(U)\subset G(U)$. In general, given two functions F and G which are analytic in U, the function F is said to be subordinate to G, if there exist a function w analytic in U with

$$w(0)=0 \text{ and } (\forall z\in U): |w(z)|<1$$

such that

$$(\forall z\in U): F(z)=G(w(z)).$$

Now let us consider the generalized Hurwitz–Lerch zeta function

$$\phi_\mu(z,m,c)=\sum_{k=0}^{\infty}\frac{(\mu)_k}{k!}\frac{z^k}{(k+c)^m}, \tag{1.1}$$
$$z\in\mathbb{C},\ |z|<1,\ c\in\mathbb{C}/\{0,-1,-2,\dots\},\ \mu,m\in\mathbb{C},$$

introduced by Goyal and Laddha [11]. Here $(x)_k$ is Pochhammer symbol (or the shifted factorial, since $(1)_k=k!$) and $(\lambda)_k$ given in terms of the Gamma functions can be written as

$$(x)_k=\frac{\Gamma(x+k)}{\Gamma(x)}=x(x+1)\dots(x+k-1)\quad \text{for } k=1,2,3,\dots\quad x\in\mathbb{C}\quad (x)_0=1.$$

Note that, the families and special cases of the Hurwitz–Lerch zeta function are studied by many authors (among them) Shy-Der Lin & Srivastava [11] and Kanemitsu et.al. [10].

Now we define the function $F(z,m,c)$ given by

$$F(z,m,c)=(1+c)^m\phi_\mu(z,s,c)=\sum_{k=0}^{\infty}\left(\frac{1+c}{k+c}\right)^m\frac{(\mu)_k}{k!}z^k$$

and

$$zF(z,m,c)=z(1+c)^m\phi_\mu(z,m,c)=\sum_{k=0}^{\infty}\left(\frac{1+c}{k+c}\right)^m\frac{(\mu)_k}{k!}z^{k+1}.$$

Thus

$$zF(z,m,c)=\sum_{k=1}^{\infty}\left(\frac{1+c}{k+c}\right)^m\frac{(\mu)_{k-1}}{(k-1)!}z^k,$$

for $z\in\mathbb{C},\ |z|<1,\ c\in\mathbb{C}/\{-1,-2,...\},\ \mu,m\in\mathbb{C}$.

Now we introduce the function $[zF(z,m,c)]^{-1}$ as the following:

$$[zF(z,m,c)]*[zF(z,m,c)]^{-1}=\frac{z}{(1-z)^{\lambda+1}},\lambda>-1,z\in\mathbb{C},\ |z|<1,$$

for $c\in\mathbb{C}/\{-1,-2,...\},\ \mu,m\in\mathbb{C}$.

Corresponding to the function $[zF(z,m,c)]^{-1}$ we define a multiplier transformation $D_{\mu,c}^{\lambda,m}$ on A and by Hadamard product for function $f\in A$, we have

$$D_{\mu,c}^{\lambda,m}=[zF(z,m,c)]^{-1}*f(z). \tag{1.2}$$

Since

$$[zF(z,m,c)]^{-1}=\sum_{k=1}^{\infty}\left(\frac{k+c}{1+c}\right)^m\frac{(\lambda+1)_{k-1}}{(\mu)_{k-1}}z^k,$$

Therefore we have

$$D_{\mu,c}^{\lambda,m} f(z) = z + \sum_{k=2}^{\infty} \left(\frac{k+c}{1+c} \right)^m \frac{(\lambda+1)_{k-1}}{(\mu)_{k-1}} a_k z^k . \tag{1.3}$$

In view of (1.3) we obtain

$$z(D_{\mu,c}^{\lambda,m} f(z))' = (\lambda+1) D_{\mu,c}^{\lambda+1,m} f(z) - \lambda D_{\mu,c}^{\lambda,m} f(z) \quad , \tag{1.4}$$

$$z(D_{\mu+1,c}^{\lambda,m} f(z))' = \mu D_{\mu,c}^{\lambda,m} f(z) - (\mu-1) D_{\mu+1,c}^{\lambda,m} f(z) \tag{1.5}$$

and also

$$z(D_{\mu,c}^{\lambda,m} f(z))' = (c+1) D_{\mu,c}^{\lambda,m+1} f(z) - c D_{\mu,c}^{\lambda,m} f(z). \tag{1.6}$$

It is clear that $D_{\mu,c}^{\lambda,m}$ are multiplier transformations. For $m \in Z$, $c \geq 1, \mu = 1 \text{and} \lambda = 0$ the operator $D_{\mu,c}^{\lambda,m}$ were studied by Cho and Srivastava [3]. For $m \in Z$, $c = 1, \mu = 1 \text{and} \lambda = 0$ the operator $D_{\mu,c}^{\lambda,m}$ were studied by Uralegaddi and Somanatha [2], for $m = -1$, $\mu = 1 \text{and} \lambda = 0$ the operator $D_{\mu,c}^{\lambda,m}$ is the integral operator studied by Owa and Srivastava [12], for any negative real number m and $\mu = 1, c = 1, \lambda = 0$ the operator $D_{\mu,c}^{\lambda,m}$ is the integral operator studied by Jung et. al. [6], for any non-negative integer number m and $\mu = 1, c = 0, \lambda = 0$ the operator $D_{\mu,c}^{\lambda,m}$ is the differential operator defined by Salagean [5], for $m = 0, \mu = 1, \lambda > -1$ the operator $D_{\mu,c}^{\lambda,m}$ is the differential operator defined by Ruscheweyh [14], for $m \in Z, \lambda = 0, \mu = 1$ the operator $D_{\mu,c}^{\lambda,m}$ are closely related to the multiplier transformations studied by Flett [17], for $\mu = 1 \text{and} \lambda > -1$ the operator $D_{\mu,c}^{\lambda,m}$ is the multiplier transformations defined by Al-Shaqsi and Darus [9], for $c = 0, \mu = 1 \text{and} \lambda > -1$ the operator $D_{\mu,c}^{\lambda,m}$ is the derivative operator given by Al-Shaqsi and Darus [8], for $c = 0,\ m, \lambda \in N_0 \ and \ \mu \in N$ the operator $D_{\mu,c}^{\lambda,m}$ is the linear operator defined by the authors [1]. In Particular, we note that $D_{1,c}^{0,0} = f(z)$ and $D_{1,0}^{0,1} = zf'(z)$.

Let $p, h \in \mathrm{H}(U)$ and let $\psi(r, s\, t\,; z): \mathbb{C}^3 \times U \to \mathbb{C}$. *If*

p and $\psi(\mathrm{p(z), zp'(z), z^2p''(z);z})$

are univalent and if p satisfies the (second-order) differential superordination

$$h(z) \prec \psi(p(z), zp'(z), z^2 p''(z); z), \quad z \in U \tag{1.7}$$

then p is called a solution of the differential superordination of (1.7).

An analytic function q is called a subordinant of the differential superodination, if $q \prec p$ for all p satisfying (1.7). A univalent subordinant $\tilde{q}$ that satisfies $q \prec \tilde{q}$ for all subordinants q of (1.7) is said to be the best subordinant. (Note that the best subordinant is unique up to a rotation of U). On the other hand, an analytic function q is said to be dominant if $p \prec q$ for all p satisfying

$$\psi(p(z), zp'(z), z^2 p''(z); z) \prec h(z), \quad z \in U \tag{1.8}$$

A univalent dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants q of (1.8) is said to be the best dominant. Recently Miller and Mocanu [16] obtained conditions on h, q and ψ for which the following implication holds:

$$h(z) \prec \psi(p(z), zp'(z), z^2 p''(z); z) \Rightarrow q(z) \prec p(z) \qquad (z \in U).$$

Denoted by Q the set of all functions q that are analytic and injective on $\overline{U} \setminus E(q)$ where

$$E(q) = \{\zeta \in \partial U : \lim_{z \to \zeta} q(z) = \infty\}$$

and are such that $q'(z) \neq 0$ *for* $\zeta \in \partial U \setminus E(q)$. *Further, let the subclass of* Q *for which* $q(0) = a$ *be denoted by* $Q(a)$ *and* $Q(1) = Q_1$.

Defnition 1.1. [15, Defnition 2.3a, p. 27]. *Let* Ω *be a set in* $\mathbb{C}$, $q \in Q$ *and* n *be a positive integer. The class of admissible functions* $\Psi_n[\Omega, q]$ *consists of those functions* $\psi: \mathbb{C}^3 \times U \to \mathbb{C}$ *that satisfy the admissibility condition* $\psi(r, s, t; z) \notin \Omega$ *whenever* $r = q(\zeta), s = k\zeta q'(\zeta)$, and

$$\Re\left\{\frac{t}{s} + 1\right\} \geq k\, \Re\left\{\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right\}, \qquad (z \in U, \zeta \in \partial U \setminus E(q), k \geq n).$$

We write $\Psi_1[\Omega, q]$ *as* $\Psi[\Omega, q]$.

Definition 1.2. [16, Definition 3, p. 817] *Let* Ω *be a set in* $\mathbb{C}$, $q \in \mathrm{H}[a,n]$ with $q'(z) \neq 0$. *The class of admissible functions* $\Psi'_{\mathrm{n}}[\Omega,q]$ *consists of those functions* $\psi : \mathbb{C}^3 \times \overline{U} \to \mathbb{C}$ *that satisfy the admissibility condition* $\psi(r,s,t;\zeta) \in \Omega$ *whenever* $r = q(z)$, $s = zq'(z)/m$, and

$$\Re\left\{\frac{t}{s}+1\right\} \leq \frac{1}{m}\Re\left\{\frac{zq''(z)}{q'(z)}+1\right\},$$

$z \in U, \zeta \in \partial U$ *and* $m \geq n \geq 1$. In particular we write $\Psi'_1[\Omega,q]$ *as* $\Psi'[\Omega,q]$.

Theorem 1.1. *[15, Theorem 2.3b, p. 28]. Let* $\psi \in \Psi_n[\Omega,q]$ *with* $q(0) = a$. *If* $p \in \mathrm{H}[a,n]$ *satisfies*

$$\psi(p(z), zp'(z), z^2 p''(z); z) \in \Omega,$$

then $p(z) \prec q(z)$.

If the behavior of q is not known on the boundary of U, Miller and Mocanu [15] introduced the following limiting procedure to prove that $p \prec q$.

Corollary 1.1. [15**,** Corollary 2.3b.1, p. 30]. *Let* $\Omega \subset \mathbb{C}$ *and* q *be univalent in* U, *with* $q(0) = a$. *Let* $\psi \in \Psi_{\mathrm{n}}[\Omega, q_\rho]$ *for some* $\rho \in (0,1)$, *where* $q_\rho(z) = q(\rho z)$. *If* $p \in \mathrm{H}[a,n]$ *and* $\psi(p(z), zp'(z), z^2 p''(z); z) \in \Omega$, *then* $p(z) \prec q(z)$.

Theorem 1.2. [16, Theorem 1, p. 818]. *Let* $\psi \in \Psi'_n[\Omega,q]$ *with* $q(0) = a$. *If* $p \in Q(a)$ and $\psi(p(z), zp'(z), z^2 p''(z); z)$ *is univalent in* U, *then*

$$\Omega \subset \left\{\psi(p(z), zp'(z), z^2 p''(z); z) : z \in U\right\}$$

Then $q(z) \prec p(z)$.

In the present paper, we shall use the method of differential subordination and Superordination introduced by Miller and Mocanu [15, Theorem 2.3b, p. 28] and [16, Theorem 1, p. 818] to derive certain properties of multiplier transformation $D_{\mu,c}^{\lambda,m} f$ and sandwich-type result is obtained.

2 Subordination Results

First, the following class of admissible functions is required in our first result.

Defnition 2.1. *Let* Ω be *a set in* $\mathbb{C}$ *and* $q(z) \in Q_1 \cap \mathrm{H}[q(0),1]$. *The class of admissible functions* $\Pi_{\mathrm{n}}[\Omega,q]$ *consists of those functions* $\pi : \mathbb{C}^3 \times U \to \mathbb{C}$ *that satisfy the admissibility condition*

$$\pi(u,v,w;z)\notin\Omega$$

whenever

$$u=q(\zeta),\qquad v=\frac{k\zeta q'(\zeta)+\mu q(\zeta)}{\mu}$$

$$\Re\left\{\frac{(\mu-1)(w-u))}{v-u}-(2\mu-1)\right\}\geq k\,\Re\left\{\frac{\zeta q''(z)}{q'(z)}+1\right\},$$

$$(z\in U,\zeta\in\partial U\setminus E(q),k\geq 1).$$

Now we will derive our first result.

Theorem 2.1. *Let* $\pi\in\Pi_n[\Omega,q]$. *If* $f\in A$ *satisfies*

$$\left\{\pi((D_{\mu+1,c}^{\lambda,m}f(z))',(D_{\mu,c}^{\lambda,m}f(z))'),(D_{\mu-1,c}^{\lambda,m}f(z))';z):z\in U\right\}\subset\Omega \tag{2.1}$$

Then

$$(D_{\mu+1,c}^{\lambda,m}f(z))'\prec q(z).$$

Proof. Define the analytic function p in U by

$$p(z)=(D_{\mu+1,c}^{\lambda,m}f(z))'. \tag{2.2}$$

In view of the relation (1.5) and from (2.2) we get

$$(D_{\mu,c}^{\lambda,m}f(z))'=\frac{zp'(z)+\mu p(z)}{\mu}. \tag{2.3}$$

Further, a simple computation shows that

$$(D_{\mu-1,c}^{\lambda,m}f(z))'=\frac{z^2p''(z)+2\mu zp'(z)+\mu(\mu-1)p(z)}{\mu(\mu-1)}. \tag{2.4}$$

Define the transformations from $\mathbb{C}^3$ to $\mathbb{C}$ by

$$u(r,s,t) = r;\ v(r,s,t) = \frac{s+\mu r}{\mu},$$
$$w(r,s,t) = \frac{t+2\mu s+\mu(\mu-1)r}{\mu(\mu-1)}. \tag{2.5}$$

Let

$$\psi(r,s,t;z) = \pi(u,v,w;z)$$
$$= \pi\left(r, \frac{s+\mu r}{\mu}, \frac{t+2\mu s+\mu(\mu-1)r}{\mu(\mu-1)}; z\right) \tag{2.6}$$

By making use of Theorem 1.1, and using equations (2.2), (2.3) and (2.4), also from (2.6), we obtain

$$\psi(p(z), zp'(z), z^2p''(z); z) = \pi((D^{\lambda,m}_{\mu+1,c}f(z))', (D^{\lambda,m}_{\mu,c}f(z))', (D^{\lambda,m}_{\mu-1,c}f(z))'; z) \tag{2.7}$$

Hence (2.1) becomes

$$\pi((D^{\lambda,m}_{\mu+1,c}f(z))', (D^{\lambda,m}_{\mu,c}f(z))', (D^{\lambda,m}_{\mu-1,c}f(z))'; z)$$
$$= \psi(p(z), zp'(z), z^2p''(z); z) \in \Omega. \tag{2.8}$$

It remains to show that the admissibility condition for $\pi \in \Pi_n[\Omega, q]$ is equivalent to the admissibility condition for ψ as given in Definition 1.1.
Note that

$$\frac{t}{s} + 1 = \frac{(\mu-1)(w-u))}{v-u} - (2\mu - 1),$$

and hence $\psi \in \Psi_n[\Omega, q]$. By Theorem 1.1, $p(z) \prec q(z)$, or $(D^{\lambda,m}_{\mu+1,c}f(z))' \prec q(z)$.

We next consider the special situation when $\Omega \neq \mathbb{C}$ is a simply connected domain. In this case $\Omega = h(U)$, where h is a conformal mapping of U onto Ω. In this case the class $\Pi_n[h(U), q]$ is written as $\Pi_n[h, q]$.

The following result is an immediate consequence of Theorem 2.1.

Theorem 2.2. Let $\pi \in \Pi_n[h,q]$ *with* $q(0)=1$. *If* $f(z) \in A$ *satisfies*

$$\pi((D^{\lambda,m}_{\mu+1,c}f(z))',(D^{\lambda,m}_{\mu,c}f(z))'),(D^{\lambda,m}_{\mu-1,c}f(z))';z) \prec h(z), \quad (2.9)$$

Then

$$(D^{\lambda,m}_{\mu+1,c}f(z))' \prec q(z).$$

By making use Corollary 1.1, we give an extension of Theorem 2.1 in the case where the behavior of q on ∂U is not known.

Corollary 2.1. *Let* $\Omega \subset \mathbb{C}$ *and let* q *be univalent in* U, $q(0)=1$. *Let* $\pi \in \Pi_n[\Omega,q_\rho]$ *for some* $\rho \in (0,1)$ *where* $q_\rho(z)=q(\rho z)$. *If* $f \in A$ *and*

$$\pi((D^{\lambda,m}_{\mu+1,c}f(z))',(D^{\lambda,m}_{\mu,c}f(z))',(D^{\lambda,m}_{\mu-1,c}f(z))';z) \in \Omega,$$

then

$$(D^{\lambda,m}_{\mu+1,c}f(z))' \prec q(z).$$

Proof. Theorem 2.1 yields $(D^{\lambda,m}_{\mu+1,c}f(z))' \prec q_\rho(z)$. The result is now deduced from $q_\rho(z) \prec q(z)$.

Theorem 2.3. *Let* h *and* q *be univalent function in* U, *with* $q(0)=1$ *and set* $q_\rho(z)=q(\rho z)$. *and* $h_\rho(z)=h(\rho z)$. *Let* $\pi:\mathbb{C}^3\times U \to \mathbb{C}$, *satisfy one of the following conditions:*

(i) $\pi \in \Pi_n[h,q_\rho]$ for some $\rho \in (0,1)$, *or*

(ii) there exists $\rho_0 \in (0,1)$, *such that* $\pi \in \Pi_n[h_\rho,q_\rho]$ *for all* $\rho \in (\rho_0,1)$.

If $f \in A$ *satisfies (2.9), then* $(D^{\lambda,m}_{\mu+1,c}f(z))' \prec q(z)$.

Proof. Following the same argument in [15, Theorem 2.3d, p. 30], we have

(i) By applying Theorem 2.1 we obtain $(D^{\lambda,m}_{\mu+1,c}f(z))' \prec q_\rho(z)$. Since $q_\rho(z) \prec q(z)$. we deduce $(D^{\lambda,m}_{\mu+1,c}f(z))' \prec q(z)$.

(ii) If we let $(D^{\lambda,m}_{\mu+1,c}f_\rho(z))' = (D^{\lambda,m}_{\mu+1,c}f(\rho z))'$, then

$$\pi((D_{\mu+1,c}^{\lambda,m}f_{\rho}(z))',(D_{\mu,c}^{\lambda,m}f_{\rho}(z))',(D_{\mu-1,c}^{\lambda,m}f_{\rho}(z))';\rho z)$$
$$=\pi((D_{\mu+1,c}^{\lambda,m}f(\rho z))',(D_{\mu,c}^{\lambda,m}f(\rho z))',(D_{\mu-1,c}^{\lambda,m}f(\rho z))';\rho z)\in h_{\rho}(U).$$

By using Theorem 2.1 and the comment associated with (2.8) with $w(z)=\rho z$ which mapping U into U, we obtain $(D_{\mu+1,c}^{\lambda,m}f_{\rho}(z))'\prec q_{\rho}(z)$, for $\rho\in(\rho_0,1)$. By letting $\rho\to 1^-$, we obtain $(D_{\mu+1,c}^{\lambda,m}f(z))'\prec q(z)$.

The next Theorem yields best dominant of the differential subordination (2.9)

Theorem 2.4. *Let h be univalent in U, and $\pi:\mathbb{C}^3\times U\to\mathbb{C}$. suppose the differential equation*

$$\pi\left(q(z),\frac{zq'(z)+\mu q(z)}{\mu},\frac{z^2q''(z)+2\mu zq'(z)+\mu(\mu-1)q(z)}{\mu(\mu-1)};z\right)=h(z) \quad (2.10)$$

has a solution q with $q(0)=1$ and one of the following conditions is satisfied:
(i) $q(z)\in Q$ and $\pi\in\Pi_n[h,q]$,
(ii) $q(z)$ is univalent in U and $\pi\in\Pi_n[h,q_{\rho}]$, for some $\rho\in(0,1)$ or
(iii) $q(z)$ is univalent in U and there exists $\rho_0\in(0,1)$ such that $\pi\in\Pi_n[h_{\rho},q_{\rho}]$, for all $\rho\in(\rho_0,1)$.
If $f(z)\in A$ satisfies (2.9), then $(D_{\mu+1,c}^{\lambda,m}f(z))'\prec q(z)$ and $q(z)$ is the best dominant.

Proof. By using same method given by [15, theorem 2.3e, p. 31], we deduce that from Theorems 2.1 and 2.2 above, q is a dominant of (2.9). Since q satisfies (2.10), it is a solution of (2.9) and therefore q will be dominated by all dominants of (2.9). Hence q will be the best dominant of (2.9).

3 Superordination and Sandwich Results

In this section the corresponding differential superordination problem is investigated and sandwich-type result is given.

Defnition 3.1. *Let Ω be a set in $\mathbb{C}$, $q(z)\in H[q(0),1]$ with $zq'(z)\neq 0$. The class of admissible functions $\Pi'_n[\Omega,q]$ consists of those functions $\pi:\mathbb{C}^3\times\overline{U}\to\mathbb{C}$ that satisfy the admissibility condition $\pi(u,v,w;\zeta)\in\Omega$ whenever*

$$u = q(z), v = \frac{zq'(z) + m(\mu+1)q(z)}{m\mu},$$

$$\Re\left\{\frac{(\mu-1)(w-u))}{v-u} - (2\mu-1)\right\} \le \frac{1}{m}\Re\left\{\frac{\zeta q''(z)}{q'(z)} + 1\right\},$$

$$(z \in U, \zeta \in \partial U, m \ge 1).$$

Theorem 3.1. *Let* $\pi \in \Pi'_n[\Omega, q]$. *If* $f \in A, (D^{\lambda,m}_{\mu+1,c} f(z))' \in Q_1$ *and*

$$\pi((D^{\lambda,m}_{\mu+1,c} f(z))', (D^{\lambda,m}_{\mu,c} f(z))'), (D^{\lambda,m}_{\mu-1,c} f(z))'; z)$$

is univalent in U, *then*

$$\Omega \subset \left\{\pi((D^{\lambda,m}_{\mu+1,c} f(z))', (D^{\lambda,m}_{\mu,c} f(z))'), (D^{\lambda,m}_{\mu-1,c} f(z))'; z) : z \in U\right\} \tag{3.1}$$

implies $q(z) \prec (D^{\lambda,m}_{\mu+1,c} f(z))'$.

Proof. Let p be defined by (2.2) and π by (2.6). Since $\pi \in \Pi'_{\rm n}[\Omega, q]$, (2.7) and (3.1) yield

$$\Omega \subset \left\{\pi(p(z), zp'(z), z^2 p''(z); z) : z \in U\right\}.$$

From (2.5), the admissibility condition for $\pi \in \Pi'_{\rm n}[\Omega, q]$, *i*s univalent to the admissibility condition for ψ as given in Definition 3.1. Hence $\psi \in \Psi'_{\rm n}[\Omega, q]$, and by Theorem 1.2, $q(z) \prec p(z)$ or $q(z) \prec (D^{\lambda,m}_{\mu+1,c} f(z))'$.

Similarly as in the previous section, we next consider the special situation when $\Omega \ne \mathbb{C}$ is a simply connected domain. In this case $\Omega = h(U)$, where h is a conformal mapping of U onto Ω. In this case the class $\Pi'_{\rm n}[h(U), q]$ is written as $\Pi'_{\rm n}[h, q]$.

The following result is an immediate consequence of Theorem 2.5.

Theorem 3.2. *Let* $q(z) \in H[q(0),1]$, $h(z)$ *be analytic in* U *and* $\pi \in \Pi'_n[h, q]$. *If* $f \in A, (D^{\lambda,m}_{\mu+1,c} f(z))' \in Q_1$ *and*

$$\pi((D^{\lambda,m}_{\mu+1,c} f(z))', (D^{\lambda,m}_{\mu,c} f(z))'), (D^{\lambda,m}_{\mu-1,c} f(z))'; z)$$

is univalent in U, *then*

$$h(z)\prec \pi((D^{\lambda,m}_{\mu+1,c}f(z))',(D^{\lambda,m}_{\mu,c}f(z))'),(D^{\lambda,m}_{\mu-1,c}f(z))';z):z\in U$$

implies $q(z)\prec (D^{\lambda,m}_{\mu+1,c}f(z))'$.

Theorem 3.3. *Let* h *be analytic in* U, *and* $\pi:\mathbb{C}^3\times\overline{U}\to\mathbb{C}$. *suppose the differential equation*

$$\pi\left(q(z),\frac{zq'(z)+\mu q(z)}{\mu},\frac{z^2q''(z)+2\mu zq'(z)+\mu(\mu-1)q(z)}{\mu(\mu-1)};z\right)=h(z)$$

has a solution $q\in Q_1$. If $\pi\in\Pi'_n[h,q]$, $f\in A,(D^{\lambda,m}_{\mu+1,c}f(z))'\in Q_1$ *and*

$$\pi((D^{\lambda,m}_{\mu+1,c}f(z))',(D^{\lambda,m}_{\mu,c}f(z))'),(D^{\lambda,m}_{\mu-1,c}f(z))';z)$$

is univalent in U, then

$$h(z)\prec \pi((D^{\lambda,m}_{\mu+1,c}f(z))',(D^{\lambda,m}_{\mu,c}f(z))'),(D^{\lambda,m}_{\mu-1,c}f(z))';z):z\in U$$

implies $q(z)\prec (D^{\lambda,m}_{\mu+1,c}f(z))'$, *and* $q(z)$ *is the best subordinant*.

Proof. The proof is similar to the proof of Theorem 2.4 and is omitted.

Combining Theorems 2.2 and 3.3, we obtain the following sandwich-type theorem.

Corollary 3.1. *Let* $h_1(z)$ *and* $q_1(z)$ *be analytic functions in* $U,h_2(z)$ *be univalent in* U, $q_2\in Q_1$, *with* $q_1(0)=q_2(0)=1$, *and* $\pi\in\Pi_n[h_2,q_2]\bigcap\Pi'_n[h_1,q_1]$. *If* $f\in A,(D^{\lambda,m}_{\mu+1,c}f(z))'\in H[q(0),1]\cap Q_1$ *and*

$$\pi((D^{\lambda,m}_{\mu+1,c}f(z))',(D^{\lambda,m}_{\mu,c}f(z))'),(D^{\lambda,m}_{\mu-1,c}f(z))';z)$$

is univalent in U, *then*

$$h_1(z)\prec \pi((D^{\lambda,m}_{\mu+1,c}f(z))',(D^{\lambda,m}_{\mu,c}f(z))'),(D^{\lambda,m}_{\mu-1,c}f(z))';z)\prec h_2(z)$$

implies $q_1(z)\prec (D^{\lambda,m}_{\mu+1,c}f(z))'\prec q_2(z)$.

4 Open Problem

The definitions, theorems and corollaries we established can be extended by using the concept of the strong differential subordination introduced in [7] by Antonino and Romaguera and studied in [4] by Oros and Oros.

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