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On The Poset Of Subhypergroups Of A Hypergroup

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Abstract

This paper deals with the set of subhypergroups of a hypergroup, partially ordered by set inclusion. We present first three significant examples of hypergroups, for which we determine explicitly this set. We conclude that in general it is not a lattice, in contrast with the set of substructures of classical algebraic structures (semigroups, monoids, groups, rings, modules, vector spaces, algebras). Some further research directions and a list of open problems concerning to this topic are also indicated.

Keywords: *hypergroups, subhypergroups, join spaces, posets, lattices, chains.*

MSC (2000): *Primary* 20N20; *Secondary* 06B99.

1 Introduction

The notion of hypergroup was introduced by F. Marty in 1934 at the 8th Congress of Scandinavian Mathematicians and plays a central role in the theory of algebraic hyperstructures. Since then this theory has enjoyed a rapid development. Its general aspects, the connections with classical algebraic structures and various applications (in geometry, topology, combinatorics, theory of binary relations, theory of fuzzy and rough sets, probability theory, cryptography and codes theory, automata theory, ... and so on) have been investigated (see [5]).

Many mathematicians have contributed to the study of hypergroups (for a detailed list, see the web page of AHA[1]). In all their papers we found few information about the poset $(Sub(H), \subseteq)$ of all subhypergroups of a hypergroup H. We also observed that no classical operation (such as the intersection, the union, the product/sum) is well-defined for subhypergroups. These remarks constitute the starting point for our paper. Its main goal is to study the subhypergroups of some particular hypergroups H and to draw several conclusions on the poset $(Sub(H), \subseteq)$. In the final section we propose some problems with respect to this subject.

Most of our notation is standard and will usually not be repeated here. Elementary concepts and results on hypergroups can be found in [4] and [5]. For lattice (respectively subgroup lattice) notions we refer the reader to [2] and [6] (respectively to [8] and [9]).

First of all, we shall recall some basic definitions of hypergroup theory.

Let H be a nonempty set and denote by $\mathcal{P}^*(H)$ the set of all nonempty subsets of H. A *hyperoperation* on H is a map

$$\circ : H \times H \longrightarrow \mathcal{P}^*(H).$$

A nonempty set H endowed with a hyperoperation $\circ$ is said to be a *hypergroupoid*. The image of the pair $(a, b) \in H \times H$ is usually denoted by $a \circ b$ and called the *hyperproduct* of a and b. If A and B are nonempty subsets of H, then we put

$$A \circ B = \bigcup_{a \in A,\ b \in B} a \circ b$$

and call it the *hyperproduct* of A and B. Also, for any $a, b \in H$, we put

$$a/b = \{x \in H \mid a \in x \circ b\}.$$

The hypergroupoid $(H, \circ)$ is called a *semihypergroup* if the hyperoperation $\circ$ is associative, that is

$$a \circ (b \circ c) = (a \circ b) \circ c, \text{ for all } (a, b, c) \in H^3.$$

If $(H, \circ)$ satisfies the *reproducibility law*

$$a \circ H = H \circ a = H, \text{ for all } a \in H,$$

then we say that it is a *quasihypergroup*. A *hypergroup* is a hypergroupoid which is both a semihypergroup and a quasihypergroup.

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[1]AHA (Algebraic Hyperstructures and Applications) – A scientific group at Democritus University of Thrace, School of Education, Greece, aha.eled.duth.gr

A particular class of hypergroups consists of the so-called *join spaces*, that is the commutative hypergroups $(H, \circ)$ such that, for any $(a, b, c, d) \in H^4$, we have

$$a/b \cap c/d \neq \emptyset \Longrightarrow a \circ d \cap b \circ c \neq \emptyset.$$

Next, let us assume that $(H, \circ)$ is a hypergroup. Then a nonempty subset K of H is called a *subhypergroup* of H if $K \circ K \subseteq K$ and K is a hypergroup under the hyperoperation $\circ$. In other words, K is a subhypergroup of H if and only if the following conditions are satisfied:

a) $a \circ b \subseteq K$, for all $a, b \in K$;

b) $a \circ K = K \circ a = K$, for all $a \in K$.

Clearly, H itself is a subhypergroup of H. Any other subhypergroup of H will be called a *proper subhypergroup*. Recall also that by $Sub(H)$ we denote the set consisting of all subhypergroups of H. Moreover, we remark that it is partially ordered under the set inclusion.

We end this section by presenting three well-known examples of hypergroups, which will be used in our paper.

Example 1. Let H be a nonempty set and define

$$a \circ_1 b = \{a, b\}, \text{ for all } a, b \in H.$$

Then $(H, \circ_1)$ is a join space.

Example 2. Let G be a group. For all $a, b \in G$, we define

$$a \circ_2 b = \langle a, b \rangle, \text{ the subgroup of } G \text{ generated by } a \text{ and } b.$$

Then $(G, \circ_2)$ is a commutative hypergroup.

Example 3. Let $(L, \wedge, \vee)$ be a complete lattice and, for every $a \in L$, denote by $F(a)$ the principal filter of L generated by a (that is, the set $\{x \in L \mid a \leq x\}$). Then L is a join space under the hyperoperation

$$a \circ_3 b = F(a \wedge b), \text{ for all } a, b \in L.$$

2 Main results

Theorem 2.1. *For the join space $(H, \circ_1)$ given by Example 1, the following equality holds*

$$Sub(H) = \mathcal{P}^*(H),$$

that is, every nonempty subset of H is a subhypergroup of H.

Proof. Let K be a nonempty subset of H. Then, for any $a, b \in K$, one obtains $a \circ_1 b = \{a, b\} \subseteq K$. Given an element $a \in K$, we obviously have $a \circ_1 K \subseteq K$. If b is an arbitrary element of K, then there is $x = b \in K$ such that $b \in a \circ_1 x$. This shows that $a \circ K = K$, completing the proof. ∎

The above theorem leads to the following simple remarks.

Remarks.

1) Suppose that H in Theorem 2.1 possesses at least two elements and take $a, b \in H$ with $a \neq b$. Then both $S_1 = \{a\}$ and $S_2 = \{b\}$ are subhypergroups of H, but $\inf\{S_1, S_2\}$ does not exist (it would be the empty set, which is not a subhypergroup of H). We infer that in general the poset of subhypergroups of a hypergroup is not a lattice.

2) The above join space H, with $\mid H \mid \geq 2$, satisfies the following property: the union of any two its subhypergroups is also a subhypergroup, even if the poset $(Sub(H), \subseteq)$ is not fully ordered. It is well-known that such a property does not hold for classical algebraic structures.

Theorem 2.2. *For the commutative hypergroup $(G, \circ_2)$ given by Example 2, the following equality holds*

$$Sub(G) = L(G),$$

that is, the subhypergroups of G coincide with the subgroups of G. In particular, $Sub(G)$ is a lattice.

Proof. Let $K \in Sub(G)$. Then K is nonempty and we have $a \circ_2 b = \langle a, b \rangle \subseteq K$, for all $a, b \in K$. Since $ab^{-1} \in \langle a, b \rangle$, one obtains $ab^{-1} \in K$ and therefore K is a subgroup of G.

Conversely, let $K \in L(G)$. Then it is clear that $a \circ_2 b = \langle a, b \rangle \subseteq K$. For any $a \in K$, we obviously have $a \circ_2 K \subseteq K$. If b is an arbitrary element of K, then there is $x = b \in K$ such that $b \in a \circ_2 x = \langle a, x \rangle$, proving that $a \circ K = K$. Hence K is a subhypergroup of G. ∎

It is well-known that there exist large classes of (finite) non-isomorphic groups whose lattices of subgroups are isomorphic (for example, see [8] and [9]). By using Theorem 2.2, we infer that a similar result also holds for hypergroups. We observe that the group operation of G can be seen as a hyperoperation

$$a \circ_2' b = \{ab\}, \text{ for all } a, b \in G,$$

and $(G, \circ_2')$ is also a commutative hypergroup. Obviously, the subhypergroups of $(G, \circ_2')$ coincide with the subgroups of G and therefore with the subhypergroups of $(G, \circ_2)$.

Corollary 2.3. *Let G be a group. Then there exist at least two distinct hyperoperations $\circ_2$ and $\circ_2'$ on G such that $(G, \circ_2)$ and $(G, \circ_2')$ are hypergroups with the same lattice of subhypergroups.*

Clearly, this result is a little more powerful as the similar ones in the case of groups. Another two immediate remarks are the following.

Remarks.

1) For the hypergroup $(G, \circ_2)$, the intersection of an arbitrary family of subhypergroups is also a subhypergroup (in contrast with the hypergroup in Theorem 2.1, where this property fails).

2) The hyperproduct $K_1 \circ_2' K_2$ of two subhypergroups K_1 and K_2 of $(G, \circ_2')$ is in fact the product of the subgroups K_1 and K_2 of the group G, and in general this is not a subgroup of G. Thus, the hyperproduct of two (or many) subhypergroups of a hypergroup is not necessarily a subhypergroup.

Theorem 2.4. *For the join space $(L, \circ_3)$ given by Example 3, the following equality holds*

$$Sub(L) = F(L) = \{F(a) \mid a \in L\},$$

that is, the subhypergroups of L coincide with the principal filters of L. In particular, $Sub(L)$ is a lattice anti-isomorphic to L.

Proof. First of all, we shall prove that $F(L)$ is contained in $Sub(L)$. Let $a \in L$. Then, for any $x, y \in F(a)$, we have $a \leq x$ and $a \leq y$. These lead to $a \leq x \wedge y$. Now, take an element $z \in F(x \wedge y)$. Then $x \wedge y \leq z$ and so $a \leq z$, that is $z \in F(a)$. This shows that $x \circ_3 y = F(x \wedge y) \subseteq F(a)$. Suppose next that x is an arbitrary element of $F(a)$. One obtains

$$x \circ_3 F(a) = \bigcup_{y \in F(a)} x \circ_3 y = \bigcup_{y \in F(a)} F(x \wedge y) = \bigcup_{a \leq y} F(x \wedge y).$$

Let $z \in x \circ_3 F(a)$. Then there is $y \in L$ with $a \leq y$ such that $z \in F(x \wedge y)$. On the other hand, we also know that $a \leq x$, implying $a \leq x \wedge y$. It results $a \leq z$, i.e. $z \in F(a)$, proving that $x \circ_3 F(a) \subseteq F(a)$. Obviously, for any $b \in F(a)$, there is $y = b \in F(a)$ such that $b \in x \circ_3 y$. In other words, we have $F(a) \subseteq x \circ_3 F(a)$, and therefore $x \circ_3 F(a) = F(a)$. Hence $F(a)$ is a subhypergroup of L.

Conversely, let $S \in Sub(L)$. Then $a \circ_3 b = F(a \wedge b) \subseteq S$, for all $a, b \in S$. Thus, given an element $x \in L$, we have $x \in S$ whenever $a \wedge b \leq x$ for some $a, b \in S$. In particular, one obtains that S is closed under the operation $\wedge$. Let $A = (a_i)_{i \in I}$ be the set of minimal elements of S with respect to the order on L. If $\mid I \mid \geq 2$, then we can choose two distinct elements of A, say a_{i_1} and a_{i_2} (note that in this case the initial element of the complete lattice L, usually denoted by 0, does not belong to A). It follows that $a_{i_1} \wedge a_{i_2} \in S$, a contradiction. In this way, A contains a unique element, say a_0. Now, it is clear that $S = F(a_0)$, i.e. $S \in F(L)$. We also remark that $S = L$ if and only if $a_0 = 0$.

Moreover, we can easily see that in the poset $(Sub(L), \subseteq)$ every two elements $F(a)$ and $F(b)$ $(a, b \in L)$ have an infimum and a supremum, namely

$$\inf\{F(a), F(b)\} = F(a \vee b) \text{ and } \sup\{F(a), F(b)\} = F(a \wedge b).$$

These equalities show that in fact $(Sub(L), \subseteq)$ is a lattice and the map F establishes a lattice anti-homomorphism from L to $Sub(L)$. On the other hand, F is obviously a bijection. Hence the lattices L and $Sub(L)$ are anti-isomorphic, which completes the proof. ∎

The above theorem has many consequences. One of them, probably the most interesting, is indicated in the next corollary.

Corollary 2.5. *Any complete lattice can be seen as the lattice of subhypergroups of a certain hypergroup. In particular, any finite lattice and any chain are the lattices of subhypergroups of certain hypergroups.*

Note that Theorem 2.4 also gives an explicit method to construct hypergroups having a prescribed (complete) lattice of subhypergroups. We can easily infer the following remarks, too.

Remarks.

1) Given a lattice L, the union of two (or many) principal filters of L is not necessarily a principal filter. So, the union of two (or many) subhypergroups of the join space $(L, \circ_3)$ is not necessarily a subhypergroup.

2) For the join space $(L, \circ_3)$, the hyperproduct of subhypergroups is an operation on $Sub(L)$. More precisely, for any $a, b \in L$, we have

$$F(a) \circ_3 F(b) = F(a \wedge b).$$

Indeed, if $c \in F(a) \circ_3 F(b)$, then there are $x \in F(a)$ and $y \in F(b)$ such that $c \in F(x \wedge y)$. Since $a \leq x$ and $b \leq y$, one obtains $a \wedge b \leq x \wedge y \leq c$, that is $c \in F(a \wedge b)$. Conversely, for $c \in F(a \wedge b)$ there are $x \in F(a)$ and $y \in F(b)$, namely $x = a$ and $y = b$, such that $c \in x \circ_3 y$.

3 Conclusions and further research

For almost all classical algebraic structures, the set of substructures (such as the submonoids of a monoid, the subgroups of a group, the ideals of a ring, the submodules of a module, the subspaces of a vector space, ..., and so on), partially ordered by inclusion, forms a lattice. This plays an essential role in studying these structures. Consequently, it seems to be interesting to extend this study to algebraic hyperstructures and, in particular, to hypergroups.

As show our previous results, the poset $(Sub(H), \subseteq)$ consisting of all subhypergroups of a hypergroup H is not a lattice in general. This is caused mainly by the fact that the intersection fails to be an operation on $Sub(H)$ (see Theorem 2.1). On the other hand, we remarked that there exist many situations in which $Sub(H)$ becomes a lattice. Moreover, it can coincide with the lattice of subgroups of any group (as follows from Theorem 2.2) or can be isomorphic with any complete lattice (as follows from Theorem 2.4).

The above remarks show that a detailed study of the posets of subhypergroups of hypergroups, focused especially on the particular case when they are lattices, can constitute an interesting topic in hypergroup theory. Also, it is clear that all important problems in subgroup lattice theory (see [8] and [9]) can be investigated for hypergroups (and for other algebraic hyperstructures). These will surely be the subject of some further research.

A list containing several open problems on this topic will be presented in our final section.

4 Open problems

Problem 4.1. Describe the poset $(Sub(H), \subseteq)$ for other remarkable classes of hypergroups/join spaces H and determine $|Sub(H)|$. When $|Sub(H)|= 1$ (i.e. H has no proper subhypergroup)?

Problem 4.2. Find some necessary and sufficient conditions on a hypergroup H such that $(Sub(H), \subseteq)$ is a lattice. Study when this lattice is of a special type: modular, distributive, complemented, pseudocomplemented, Boolean algebra, Heyting algebra, ..., and so on. In particular, characterize the hypergroups H for which $(Sub(H), \subseteq)$ is a chain.

Problem 4.3. Study the hypergroups $(H, \circ)$ satisfying the property that the hyperproduct of subhypergroups becomes an operation on $Sub(H)$, that is $K_1 \circ K_2 \in Sub(H)$ for all $K_1, K_2 \in Sub(H)$.

Problem 4.4. Let $(L, \wedge, \vee)$ be a lattice. Find all maps $F : L \longrightarrow \mathcal{P}(L)$ such that, by defining on L the hyperoperation

$$a \circ b = F(a \wedge b),$$

$(L, \circ)$ is a hypergroup/join space.

Problem 4.5. Let H_1 and H_2 be two hypergroups. Study the homomorphisms between the posets $(Sub(H_1), \subseteq)$ and $(Sub(H_2), \subseteq)$. What can be said about H_1 and H_2 if $(Sub(H_1), \subseteq)$ and $(Sub(H_2), \subseteq)$ are isomorphic?

Problem 4.6. For a hypergroup/join space H, many types of subhypergroups (as closed subhypergroups, ultraclosed subhypergroups, invertible subhypergroups, invariant subhypergroups, complete parts, ..., and so on) have been identified and investigated. Obviously, each of them determines a subposet of $(Sub(H), \subseteq)$. When these subposets are lattices? If $Sub(H)$ is itself a lattice, study whether they are closed under the binary operations of $Sub(H)$ (i.e. they form sublattices of $Sub(H)$).

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