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On Fixed Point Theorems in Fuzzy Metric Spaces

C. T. Aage and J. N. Salunke

School of Mathematical Sciences, N. M. U., Jalgaon,India.
e-mail:caage17@gmail.com
School of Mathematical Sciences, N. M. U., Jalgaon, India.
e-mail:drjnsalunke@gmail.com

Abstract

This paper presents some common fixed point theorems for occasionally weakly compatible mappings in fuzzy metric spaces.

Keywords: *Occasionally weakly compatible mappings, fuzzy metric space.*

1 Introduction

Fuzzy set was defined by Zadeh [27]. Kramosil and Michalek [15] introduced fuzzy metric space, George and Veermani [7] modified the notion of fuzzy metric spaces with the help of continuous t-norms. Many researchers have obtained common fixed point theorems for mappings satisfying different types of commutativity conditions. Vasuki [26] proved fixed point theorems for R-weakly commutating mappings. Pant [19, 20, 21] introduced the new concept reciprocally continuous mappings and established some common fixed point theorems. Balasubramaniam et al.[5], have shown that Rhoades [23] open problem on the existence of contractive definition which generates a fixed point but does not force the mappings to be continuous at the fixed point, posses an affirmative answer. Pant and Jha [21] obtained some anologus results proved by Balasubramaniam et al. Recent literature on fixed point in fuzzy metric space can be viewed in [1, 2, 3, 10, 17, 25].

This paper presents some common fixed point theorems for more general commutative condition i.e. occasionally weakly compatible mappings in fuzzy metric space.

2 Preliminary Notes

Definition 2.1 *[27] A fuzzy set A in X is a function with domain X and values in* $[0, 1]$.

Definition 2.2 *[24] A binary operation* $* : [0, 1] \times [0, 1] \to [0, 1]$ *is a continuous t-norms if * is satisfying conditions:*
(i) $*$ *is an commutative and associative;*
(ii) $*$ *is continuous;*
(iii) $a * 1 = a$ *for all* $a \in [0, 1]$;
(iv) $a * b \leq c * d$ *whenever* $a \leq c$ *and* $b \leq d$, *and* $a, b, c, d \in [0, 1]$.

Definition 2.3 *[7] A 3-tuple* $(X, M, *)$ *is said to be a fuzzy metric space if* X *is an arbitrary set,* $*$ *is a continuous t-norm and* M *is a fuzzy set on* $X^2 \times (0, \infty)$ *satisfying the following conditions, for all* $x, y, z \in X, s, t > 0$,
$(f1)$ $M(x, y, t) > 0$;
$(f2)$ $M(x, y, t) = 1$ *if and only if* $x = y$
$(f3)$ $M(x, y, t) = M(y, x, t)$;
$(f4)$ $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$;
$(f5)$ $M(x, y, \cdot) : (0, \infty) \to (0, 1]$ *is continuous.*

Then M is called a fuzzy metric on X. Then $M(x, y, t)$ denotes the degree of nearness between x and y with respect to t.

Example 2.4 *(Induced fuzzy metric [7]) Let* (X, d) *be a metric space. Denote* $a * b = ab$ *for all* $a, b \in [0, 1]$ *and let* M_d *be fuzzy sets on* $X^2 \times (0, \infty)$ *defined as follows:*

$$M_d(x, y, t) = \frac{t}{t + d(x, y)}.$$

Then $(X, M_d, *)$ *is a fuzzy metric space. We call this fuzzy metric induced by a metric d as the standard intuitionistic fuzzy metric.*

Definition 2.5 *[7]: Let* $(X, M, *)$ *be a fuzzy metric space. Then*
(a) *a sequence* $\{x_n\}$ *in* X *is said to converges to* x *in* X *if for each* $\epsilon > 0$ *and each* $t > 0$, *there exists* $n_0 \in N$ *such that* $M(x_n, x, t) > 1 - \epsilon$ *for all* $n \geq n_0$.
(b) *a sequence* $\{x_n\}$ *in* X *is said to be Cauchy if for each* $\epsilon > 0$ *and each* $t > 0$, *there exists* $n_0 \in N$ *such that* $M(x_n, x_m, t) > 1 - \epsilon$ *for all* $n, m \geq n_0$.
(c) *A fuzzy metric space in which every Cauchy sequence is convergent is said to be complete.*

Definition 2.6 *[26] A pair of self-mappings* (f, g) *of a fuzzy metric space* $(X, M, *)$ *is said to be*
(i) *weakly commuting if* $M(fgx, gfx, t) \geq M(fx, gx, t)$ *for all* $x \in X$ *and* $t > 0$.
(ii) *R-weakly commuting if there exists some* $R > 0$ *such that* $M(fgx, gfx, t) \geq M(fx, gx, t/R)$ *for all* $x \in X$ *and* $t > 0$.

Definition 2.7 *[11] Two self mappings f and g of a fuzzy metric space $(X, M, *)$ are called compatible if* $\lim_{n\to\infty} M(fgx_n, gfx_n, t) = 1$ *whenever* $\{x_n\}$ *is a sequence in X such that* $lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = x$ *for some x in X.*

Definition 2.8 *[5]: Two self maps f and g of a fuzzy metric space $(X, M, *)$ are called reciprocally continuous on X if* $lim_{n\to\infty} fgx_n = fx$ *and* $\lim_{n\to\infty} gfx_n = gx$ *whenever* $\{x_n\}$ *is a sequence in X such that* $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = x$ *for some x in X.*

Lemma 2.9 *Let $(X, M, *)$ be a fuzzy metric space. If there exists $q \in (0, 1)$ such that $M(x, y, qt) \geq M(x, y, t)$ for all $x, y \in X$ and $t > 0$, then $x = y$.*

Definition 2.10 *Let X be a set, f, g selfmaps of X. A point x in X is called a coincidence point of f and g iff $fx = gx$. We shall call $w = fx = gx$ a point of coincidence of f and g.*

Definition 2.11 *[12] A pair of maps S and T is called weakly compatible pair if they commute at coincidence points.*

The concept occasionally weakly compatible is introduced by M. Al-Thagafi and Naseer Shahzad [4]. It is stated as follows.

Definition 2.12 *Two self maps f and g of a set X are occasionally weakly compatible (owc) iff there is a point x in X which is a coincidence point of f and g at which f and g commute.*

A. Al-Thagafi and Naseer Shahzad [4] shown that occasionally weakly is weakly compatible but converse is not true.

Example 2.13 *[4] Let R be the usual metric space. Define $S, T : R \to R$ by $Sx = 2x$ and $Tx = x^2$ for all $x \in R$. Then $Sx = Tx$ for $x = 0, 2$ but $ST0 = TS0$, and $ST2 \neq TS2$. S and T are occasionally weakly compatible self maps but not weakly compatible*

Lemma 2.14 *[13] Let X be a set, f, g owc self maps of X. If f and g have a unique point of coincidence, $w = fx = gx$, then w is the unique common fixed point of f and g.*

3 Main Results

Theorem 3.1 *Let $(X, M, *)$ be a complete fuzzy metric space and let A, B, S and T be self-mappings of X. Let the pairs $\{A, S\}$ and $\{B, T\}$ be owc. If there exists $q \in (0, 1)$ such that*

$$M(Ax, By, qt) \geq \min\{M(Sx, Ty, t), M(Sx, Ax, t), M(By, Ty, t), M(Ax, Ty, t), M(By, Sx, t)\} \quad (1)$$

for all $x, y \in X$ and for all $t > 0$, then there exists a unique point $w \in X$ such that $Aw = Sw = w$ and a unique point $z \in X$ such that $Bz = Tz = z$. Moreover, $z = w$, so that there is a unique common fixed point of A, B, S and T.

Proof: Let the pairs $\{A, S\}$ and $\{B, T\}$ be owc, so there are points $x, y \in X$ such that $Ax = Sx$ and $By = Ty$. We claim that $Ax = By$. If not, by inequality (1)

$$\begin{aligned} M(Ax, By, qt) &\geq \min\{M(Sx, Ty, t), M(Sx, Ax, t), M(By, Ty, t), \\ &\quad M(Ax, Ty, t), M(By, Sx, t)\} \\ &= \min\{M(Ax, By, t), M(Ax, Ax, t), M(By, By, t), \\ &\quad M(Ax, By, t), M(By, Ax, t)\} \\ &= M(Ax, By, t). \end{aligned}$$

Therefore $Ax = By$, i.e. $Ax = Sx = By = Ty$. Suppose that there is a another point z such that $Az = Sz$ then by (1) we have $Az = Sz = By = Ty$, so $Ax = Az$ and $w = Ax = Sx$ is the unique point of coincidence of A and S. By Lemma 2.14 w is the only common fixed point of A and S. Similarly there is a unique point $z \in X$ such that $z = Bz = Tz$.

Assume that $w \neq z$. We have

$$\begin{aligned} M(w, z, qt) &= M(Aw, Bz, qt) \\ &\geq \min\{M(Sw, Tz, t), M(Sw, Az, t), M(Bz, Tz, t), \\ &\quad M(Aw, Tz, t), M(Bz, Sw, t)\} \\ &= \min\{M(w, z, t), M(w, z, t), M(z, z, t), \\ &\quad M(w, z, t), M(z, w, t)\} \\ &= M(w, z, t) \end{aligned}$$

Therefore we have $z = w$ by Lemma 2.14 and z is a common fixed point of A, B, S and T. The uniqueness of the fixed point holds from (1).

Theorem 3.2 *let $(X, M, *)$ be a complete fuzzy metric space and let A, B, S and T be self-mappings of X. Let the pairs $\{A, S\}$ and $\{B, T\}$ be owc. If there exists $q \in (0, 1)$ such that*

$$\begin{aligned} M(Ax, By, qt) \geq \phi(\min\{&M(Sx, Ty, t), M(Sx, Ax, t), M(By, Ty, t), \\ &M(Ax, Ty, t), M(By, Sx, t)\}) \end{aligned} \tag{2}$$

for all $x, y \in X$ and $\phi : [0, 1] \to [0, 1]$ such that $\phi(t) > t$ for all $0 < t < 1$, then there exists a unique common fixed point of A, B, S and T.

Proof: The proof follows from Theorem 3.1.

Theorem 3.3 *let* $(X, M, *)$ *be a complete fuzzy metric space and let* A, B, S *and* T *be self-mappings of* X. *Let the pairs* $\{A, S\}$ *and* $\{B, T\}$ *be owc. If If there exists* $q \in (0, 1)$ *such that*

$$\begin{aligned} M(Ax, By, qt) \geq \phi(&M(Sx, Ty, t), M(Sx, Ax, t), M(By, Ty, t), \\ &M(Ax, Ty, t), M(By, Sx, t)) \end{aligned} \tag{3}$$

for all $x, y \in X$ *and* $\phi : [0, 1]^5 \to [0, 1]$ *such that* $\phi(t, 1, 1, t, t) > t$ *for all* $0 < t < 1$, *then there exists a unique common fixed point of* A, B, S *and* T.

Proof: Let the pairs $\{A, S\}$ and $\{B, T\}$ are owc, there are points point $x, y \in X$ such that $Ax = Sx$ and $By = Ty$. We claim that $Ax = By$. By inequality (3) we have

$$\begin{aligned} M(Ax, By, qt) &\geq \phi(M(Sx, Ty, t), M(Sx, Ax, t), M(By, Ty, t), \\ &\quad M(Ax, Ty, t), M(By, Sx, t)) \\ &= \phi(M(Ax, By, t), M(Ax, Ax, t), M(By, By, t), \\ &\quad M(Ax, By, t), M(By, Ax, t)) \\ &= \phi(M(Ax, By, t), 1, 1, M(Ax, By, t), M(By, Ax, t)) \\ &> M(Ax, By, t). \end{aligned}$$

a contradiction, therefore $Ax = By$, i.e. $Ax = Sx = By = Ty$. Suppose that there is a another point z such that $Az = Sz$ then by (3) we have $Az = Sz = By = Ty$, so $Ax = Az$ and $w = Ax = Tx$ is the unique point of coincidence of A and T. By Lemma 2.14 w is a unique common fixed point of A and S. Similarly there is a unique point $z \in X$ such that $z = Bz = Tz$. Thus z is a common fixed point of A, B, S and T. The uniqueness of the fixed point holds from (3).

Theorem 3.4 *let* $(X, M, *)$ *be a complete fuzzy metric space and let* A, B, S *and* T *be self-mappings of* X. *Let the pairs* $\{A, S\}$ *and* $\{B, T\}$ *are owc. If there exists a point* $q \in (0, 1)$ *for all* $x, y \in X$ *and* $t > 0$

$$\begin{aligned} M(Ax, By, qt) \geq\ &M(Sx, Ty, t) * M(Ax, Sx, t) * M(By, Ty, t) \\ &* M(Ax, Ty, t), \end{aligned} \tag{4}$$

then there exists a unique common fixed point of A, B, S *and* T.

Proof: Let the pairs $\{A, S\}$ and $\{B, T\}$ are owc, there are points $x, y \in X$ such that $Ax = Sx$ and $By = Ty$. We claim that $Ax = By$. By inequality (4) we have

$$\begin{aligned}M(Ax,By,qt) &\geq M(Sx,Ty,t)*M(Ax,Sx,t)*M(By,Ty,t)\\ &\quad *M(Ax,Ty,t)\\ &= M(Ax,By,t)*M(Ax,Ax,t)*M(By,By,t)\\ &\quad *M(Ax,By,t)\\ &\geq M(Ax,By,t)*1*1*M(Ax,By,t)\\ &\geq M(Ax,By,t)\end{aligned}$$

Thus we have $Ax = By$, i.e. $Ax = Sx = By = Ty$. Suppose that there is a another point z such that $Az = Sz$ then by (4) we have $Az = Sz = By = Ty$, so $Ax = Az$ and $w = Ax = Sx$ is the unique point of coincidence of A and S. Similarly there is a unique point $z \in X$ such that $z = Bz = Tz$. Thus w is a common fixed point of A, B, S and T.

Corollary 3.5 *let $(X, M, *)$ be a complete fuzzy metric space and let A, B, S and T be self-mappings of X. Let the pairs $\{A, S\}$ and $\{B, T\}$ are owc. If there exists a point $q \in (0, 1)$ for all $x, y \in X$ and $t > 0$*

$$\begin{aligned}M(Ax,By,qt) &\geq M(Sx,Ty,t)*M(Ax,Sx,t)*M(By,Ty,t)\\ &\quad *M(By,Sx,2t)*M(Ax,Ty,t), \end{aligned} \tag{5}$$

then there exists a unique common fixed point of A, B, S and T.

Proof: We have

$$\begin{aligned}M(Ax,By,qt) &\geq M(Sx,Ty,t)*M(Ax,Sx,t)*M(By,Ty,t)\\ &\quad *M(By,Sx,2t)*M(Ax,Ty,t)\\ &\geq M(Sx,Ty,t)*M(Ax,Sx,t)*M(By,Ty,t)\\ &\quad *M(Sx,Ty,t)*M(Ty,By,t)*M(Ax,Ty,t)\\ &\geq M(Sx,Ty,t)*M(Ax,Sx,t)*M(By,Ty,t)\\ &\quad *M(Ax,Ty,t)\end{aligned}$$

and therefore from Theorem 3.4, A, B, S and T have a common fixed point.

Corollary 3.6 *let $(X, M, *)$ be a complete fuzzy metric space and let A, B, S and T be self-mappings of X. Let the pairs $\{A, S\}$ and $\{B, T\}$ are owc. If there exists a point $q \in (0, 1)$ for all $x, y \in X$ and $t > 0$*

$$M(Ax,By,qt) \geq M(Sx,Ty,t), \tag{6}$$

then there exists a unique common fixed point of A, B, S and T.

Proof: The Proof follows from Corollary 3.5

Theorem 3.7 *let $(X, M, *)$ be a complete fuzzy metric space. Then continuous self mappings S and T of X have a common fixed point in X if and only if there exists a self mapping A of X such that the following conditions are satisfied*
(i) $AX \subset TX \cap SX$
(ii) the pairs $\{A, S\}$ and $\{A, T\}$ are weakly compatible,
(iii) there exists a point $q \in (0, 1)$ such that for every $x, y \in X$ and $t > 0$

$$M(Ax, Ay, qt) \geq M(Sx, Ty, t) * M(Ax, Sx, t) * M(Ay, Ty, t) * M(Ax, Ty, t) \tag{7}$$

Then A, S and T have a unique common fixed point.

Proof: Since compatible implies owc, the result follows from 3.4

Theorem 3.8 *let $(X, M, *)$ be a complete fuzzy metric space and let A and B be self-mappings of X. Let the A and B are owc. If there exists a point $q \in (0, 1)$ for all $x, y \in X$ and $t > 0$*

$$M(Sx, Sy, qt) \geq \alpha M(Ax, Ay, t) + \beta \min\{M(Ax, Ay, t), M(Sx, Ax, t), M(Sy, Ay, t)\} \tag{8}$$

for all $x, y \in X$, where $\alpha, \beta, > 0, \alpha + \beta > 1$. Then A and S have a unique common fixed point.

Proof: Let the pairs $\{A, S\}$ be owc, so there is a point $x \in X$ such that $Ax = Sx$. Suppose that there exist another point $y \in X$ for which $Ay = Sy$. We claim that $Sx = Sy$. By inequality (8) we have

$$\begin{aligned} M(Sx, Sy, qt) &\geq \alpha M(Ax, Ay, t) + \beta \min\{M(Ax, Ay, t), \\ &\quad M(Sx, Ax, t), M(Sy, Ay, t)\} \\ &= \alpha M(Sx, Sy, t) + \beta \min\{M(Sx, Sy, t), \\ &\quad M(Sx, Sx, t), M(Sy, Sy, t)\} \\ &= (\alpha + \beta) M(Sx, Sy, t) \end{aligned}$$

a contradiction, since $(\alpha + \beta) > 1$. Therefore $Sx = Sy$. Therefore $Ax = Ay$ and Ax is unique. From Lemma 2.14, A and S have a unique fixed point.

Question 1: Are the above mentioned theorems true in a generalized fuzzy metric space?

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