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Numbers Which Factor as Their Digital Sum Times a Prime

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Abstract

A natural number n is Niven when n is divisible by its digital sum, denoted by $S(n)$. We study Niven numbers which come in the form $n = S(n) \times p$, for some prime number p. Two constructions of such numbers are given, with suggestions for further studies and conjectures to solve.

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1 Introduction

The function $S(n)$, over the domain of the natural numbers, denote the sum of the digits in n. For example, $S(7902) = 7 + 9 + 2 = 18$. The name of this function $S(n)$ is the *digital sum* of n. The number n is called *Niven* when n is a multiple of $S(n)$. For example, the number 7902 is a Niven number because $7902 = 18 \times 439$, where $18 = S(7902)$.

Niven numbers were named after Ivan Niven, the mathematician who made popular the study of such numbers [4]. There is obviously an abundance of Niven numbers, for example the sequence 110, 1100, 11000, ... are all Niven numbers of digital sum 2. Even so, if $N(x)$ denotes the number of Niven numbers up to x, then it has been shown [5] that

$$\lim_{x\to\infty} \frac{N(x)}{x} = 0.$$

Later, De Koninck and Doyon [1] showed that

$$x^{1-\varepsilon} \ll N(x) \ll \frac{x \log\log x}{\log x},$$

where $\varepsilon > 0$ is any small number. Finally, in 2003, De Koninck, Doyon and Kátai [2] obtained the long awaited asymptotic formula for $N(x)$ by showing that

$$N(x) = (c_0 + o(1))\frac{x}{\log x} \quad (x \to \infty),$$

where $c_0 = \dfrac{10}{27}\log 10$.

There exist many ways in which the concept of Niven numbers can be extended. For instance, it is common to consider Niven numbers for integers in base b other than $b = 10$.

Our purpose in this article is to introduce a special subfamily of Niven numbers given by those n for which $n/S(n)$ is prime. An example is again given by the number 7902, since $7902 = 18 \times 439$, where $18 = S(7902)$ and 439 is a prime number. We give several numerical facts, methods of constructions, and conjectures based on our observations.

2 Basic Results

Throughout this article, N^* stands for the subset of Niven numbers defined by

$$N^* = \{n \in N \mid n = S(n) \times p, \text{ for some prime } p\},$$

where N is the set of Niven numbers. The are only seven elements of N^* up to 100, i.e.,

$$\begin{aligned} 18 &= 9 \times 2 \\ 21 &= 3 \times 7 \\ 27 &= 9 \times 3 \\ 42 &= 6 \times 7 \\ 45 &= 9 \times 5 \\ 63 &= 9 \times 7 \\ 84 &= 12 \times 7 \end{aligned}$$

We shall now introduce two particular ways in which one may construct elements of N^*. The first one involves the use of repunits, or string of ones, which is usually denoted $R_n = (10^n - 1)/9 = 111\dots111$. The following theorem was first given by McDaniel [6].

Theorem 2.1. *Let $R_n = (10^n - 1)/9$. If $m < 9R_n$ then $S(9mR_n) = 9n$.*

Clearly then, with the choice of $m = n$ in the above theorem we get $S(9nR_n) = 9n$. Hence $9nR_n \in N^*$, provided that R_n is prime:

Theorem 2.2. *Suppose that $R_n = (10^n - 1)/9$ is a prime number. Then $9nR_n \in N^*$.*

Although it is generally conjectured that infinitely many repunits R_n are primes, up to date only five of them are known to be primes—corresponding to $n = 2$, 19, 23, 317, and 1031.

The second construction relies on the seemingly abundant prime numbers whose digital sums equal 4.

Theorem 2.3. *Let p be a prime with $S(p) = 4$. Then $12p \in N^*$.*

Proof. Such primes p must have digits which consist of just 0, 1, 2, or 3. In any case, the digits in $2p$ are each 6 or less, hence adding $10p$ to $2p$ digit by digit will never exceed 9. Thus we have $S(12p) = S(10p) + S(2p) = S(p) + 2S(p) = 3S(p) = 12$ as desired. □

To provide examples based on Theorem 2.3, we searched for primes of digital sum 4, up to 1000 digits long, using a computer program. In this interval, we found forty six suitable primes p for which $12p \in N^*$ by considering only six different forms of p. The results are displayed in Table 1.

Table 1: Examples of primes p for which $12p \in N^*$

p	k
$3 \times 10^k + 1$	1, 3, 7, 10, 28, 36, 67, 81, 147, 483, 643
$12 \times 10^k + 1$	2, 38, 80
$21 \times 10^k + 1$	1, 3, 5, 21, 43, 45, 49, 369, 645
$102 \times 10^k + 1$	1, 3, 4, 148, 189, 511
$111 \times 10^k + 1$	8, 16, 20, 62, 64, 76, 100, 142, 230, 670
$120 \times 10^k + 1$	1, 7, 33, 67, 138, 414, 450

Note that the list given in Table 1 can be extended indefinitely both horizontally and vertically, but there is no proof that such primes cannot be exhausted.

3 Open Problem

A very reasonable conjecture is that the set N^* is infinite. This would clearly follow if we could solve the following problem.

Problem 1. *Show that there are infinitely many prime numbers of digital sums equal four.*

This problem is a particular case of an open problem discussed at length in the recent book of De Koninck [3, page 75], which is certainly very hard to prove despite the fact that it is supported by numerical evidence.

Instead of proving Problem 1, it is very likely that one may discover other constructions of elements of N^* by trying out other multipliers, other than $m = 12$, and other digital sum $S(p)$, preferably quite small, so as to come with the right equality $S(mp) = m$. We leave this as a challenge to the reader.

Problem 2. *Find another pair of integers* (k, m)*, other than* $(4, 12)$*, such that if a prime number* p *has digital sum* $S(p) = k$*, then the number* $mp \in N^*$*. Furthermore, prove that there exist infinitely many such pairs.*

Another way to attack the problem of proving that the set N^* is infinite, is to consider the counting function for N^*, as follows.

Let $N^*(x)$ stand for the number of elements in N^* less than or equal to x. We then provide in Table 2 a hint of the growth of $N^*(x)$ for every $x = 10^k$, up to $k = 7$.

Table 2: The growth of $N^*(x)$ up to $x = 10^7$

x	$N^*(x)$	$\lfloor x/\log^2 x \rfloor$	$(x/\log^2 x)/N^*(x)$
100	7	4	0.674
1 000	62	20	0.338
10 000	336	117	0.350
100 000	1 915	754	0.394
1 000 000	12 259	5 239	0.427
10 000 000	86 356	38 492	0.446

In this initial range, we note that $\dfrac{x/\log^2 x}{N^*(x)}$ seems to approach a constant. We conjecture, perhaps prematurely, that $N^*(x) \sim \dfrac{cx}{\log^2 x}$ for some constant $c > 0$.

Lastly, one may consider extending this study on N^* by attempting to answer the following two questions.

Problem 3. *For every prime* p*, is there a number* m *such that* $mp \in N^*$*? If not, can we identify which primes* p *will yield a positive answer?*

Problem 4. *For every number* m*, can we find* $n \in N^*$ *such that* $S(n) = m$*? If so, can we give descriptions to the smallest of such element* n *with respect to* m*?*

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