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Various Proofs for the Decrease Monotonicity of the Schatten's Power Norm, Various Families of $\mathbb{R}^n-$Norms and Some Open Problems

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Abstract

Let $1 \leq p \leq \infty$ be a (extended) real number, $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$ and the map $p \longrightarrow \| x \|_p := (\sum_{i=1}^n |x_i|^p)^{1/p}$. In the present paper, some different proofs for the decrease monotonicity of $p \longrightarrow \| x \|_p$ are given. Afterwards, we construct an iterative algorithm that converges point-wisely to a parameterized norm Θ_p. At the end, the interconnection between $\|.\|_p$ and Θ_p, together with extensions for $2-$norms, are putted as open problems.

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1 Introduction

Let $\mathbb{R}^n$ be the real finite dimensional space. In the literature, three special norms of $\mathbb{R}^n$ are well known: for $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$,

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|, \quad \|x\|_2 = \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2}, \quad \|x\|_\infty = \max_{1\leq i\leq n} |x_i|,$$

which satisfy the following inequalities

$$\|x\|_\infty \leq \|x\|_2 \leq \|x\|_1 \leq n\|x\|_\infty.$$

The second norm $\|.\|_2$, so-called euclidian norm, derives from the standard inner product defined by

$$x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n,\ y = (y_1, y_2, ..., y_n) \in \mathbb{R}^n, \quad \langle x, y\rangle = \sum_{i=1}^{n} x_i y_i.$$

There are many other norms of $\mathbb{R}^n$, so-called Schatten power norm, extending the above three ones: for $1 \le p \le \infty$ (extended) real number, we set

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}.$$

Such parameterized norm immediately gives $\|.\|_1$ and $\|.\|_2$ for $p = 1$ and $p = 2$ respectively, and extends $\|.\|_\infty$ in the following sense

$$\forall x \in \mathbb{R}^n \qquad \|x\|_\infty = \lim_{p\to\infty} \|x\|_p.$$

If we denote by p^* the conjugate of p defined by $1/p+1/p^* = 1$ i.e $p^* = p/(p-1)$ with the convention $p = 1, p^* = \infty$ and $p = \infty, p^* = 1$, the following inequality, so-called the Hölder inequality in $\mathbb{R}^n$, is well known: for all $x, y \in \mathbb{R}^n$ there holds

$$|\langle x, y\rangle| \le \|x\|_p \|y\|_{p^*}.$$

For $p = p^* = 2$, the above Hölder inequality is exactly the classical Cauchy-Schwartz one. Further, it is not difficult to prove the following inequality:

$$\forall x \in \mathbb{R}^n \qquad \|x\|_p \le \frac{1}{p}\|x\|_1 + \frac{1}{p^*}\|x\|_\infty. \tag{1.1}$$

Indeed, writing

$$\sum_{i=1}^{n} |x_i|^p = \sum_{i=1}^{n} |x_i|^{p-1}|x_i| \le \left(\max_{1\le i\le n} |x_i|\right)^{p-1} \sum_{i=1}^{n} |x_i|,$$

we deduce that

$$\left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} \le \left(\max_{1\le i\le n} |x_i|\right)^{(p-1)/p} \left(\sum_{i=1}^{n} |x_i|\right)^{1/p},$$

which, with the Young inequality, yields the desired result (1.1).
Finally, we may state the following theorem that is also known in the literature.

Theorem 1.1. *Let $1 \le p \le \infty$ be a (extended) real number. For fixed vector $x := (x_1, x_2, ..., x_n) \in \mathbb{R}^n$, the mapping*

$$p \longrightarrow \| x \|_p := \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p},$$

is monotone decreasing, that is, $p > q \ge 1 \implies \| x \|_p \le \| x \|_q$.

In the following section, we will present some different proofs of the above theorem. For this, we state the next lemma which will be needed in the sequel.

Lemma 1.2. *Let* $c_1, c_2, ..., c_N$ *be* N *positive real numbers and* $0 < m < 1$. *Then there holds*

$$\left(\sum_{i=1}^{N} c_i\right)^m \leq \sum_{i=1}^{N} c_i^m.$$

Proof. It is a simple exercise for the reader. □

2 Various Proofs of Theorem 1.1

In what follows, we will present seven different proofs of Theorem 1.1.

Proof 1. The short proof, based on Lemma 1.2, is first as follows. If $p > q \geq 1$ then $0 < q/p < 1$ and Lemma 1.2 with $m = q/p,\ c_i = |x_i|^p$ gives

$$\left(\sum_{i=1}^{n} |x_i|^p\right)^{q/p} \leq \sum_{i=1}^{n} |x_i|^q.$$

The desired result follows.

Proof 2. The second proof, simple and elementary, is based on the homogeneity principle of a norm. We can assume without loss the generality that $\| x \|_q = 1$. Then, $|x_i| \leq 1$ and so $|x_i|^p \leq |x_i|^q$, for all $i = 1, 2, ..., n$. It follows that

$$\sum_{i=1}^{n} |x_i|^p \leq \sum_{i=1}^{n} |x_i|^q = 1,$$

and so

$$\left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} \leq 1 = \|x\|_q,$$

which is the desired result.

Proof 3. The third proof is an explicit and manual statement of the above one. Setting $|x_i| = a_i \geq 0$, we can assume that $\sum_{i=1}^n a_i > 0$. We write successively

$$\frac{(\sum_{i=1}^n a_i^p)^{1/p}}{(\sum_{i=1}^n a_i^q)^{1/q}} = \left(\frac{\sum_{i=1}^n a_i^p}{(\sum_{i=1}^n a_i^q)^{p/q}}\right)^{1/p}$$

$$= \left(\sum_{i=1}^{n} \frac{(a_i^q)^{p/q}}{(\sum_{i=1}^n a_i^q)^{p/q}}\right)^{1/p} = \left(\sum_{i=1}^{n} \left(\frac{a_i^q}{\sum_{i=1}^n a_i^q}\right)^{p/q}\right)^{1/p}.$$

Since

$$0 \leq \frac{a_i^q}{\sum_{i=1}^n a_i^q} \leq 1 \text{ and } p/q \geq 1,$$

we deduce that

$$\left(\frac{a_i^q}{\sum_{i=1}^n a_i^q}\right)^{p/q} \leq \frac{a_i^q}{\sum_{i=1}^n a_i^q}.$$

Summing over i from 1 to n we infer that

$$\sum_{i=1}^n \left(\frac{a_i^q}{\sum_{i=1}^n a_i^q}\right)^{p/q} \leq \sum_{i=1}^n \frac{a_i^q}{\sum_{i=1}^n a_i^q} = 1,$$

from which we deduce

$$\left(\sum_{i=1}^n \left(\frac{a_i^q}{\sum_{i=1}^n a_i^q}\right)^{p/q}\right)^{1/p} \leq 1.$$

The desired result follows.

Proof 4. The fourth proof is also simple and elementary, with explicit computations. We can assume that $a_i := |x_i| > 0$ for all $i = 1, 2, ..., n$. Let us set

$$\Psi(p) = exp\left(\frac{1}{p} ln\left(\sum_{i=1}^n a_i^p\right)\right),$$

for $p \in]1, +\infty[$. We wish to establish that the map $p \longmapsto \Psi(p)$ is monotone decreasing. A simple computation of $\Psi'(p)$, after a reduction, yields

$$\Psi'(p) = \frac{1}{p} \frac{\sum_{i=1}^n a_i^p ln\ a_i - \frac{1}{p}(ln \sum_{i=1}^n a_i^p)(\sum_{i=1}^n a_i^p)}{\sum_{i=1}^n a_i^p} \Psi(p).$$

The fact that

$$a_i \leq \left(\sum_{i=1}^n a_i^p\right)^{1/p},$$

for all $i = 1; 2, ..., n$ implies that

$$ln\ a_i \leq \frac{1}{p} ln \sum_{i=1}^n a_i^p,$$

for all $i = 1, 2, ..., n$, and thus

$$\sum_{i=1}^n a_i^p ln\ a_i \leq \frac{1}{p}\left(ln \sum_{i=1}^n a_i^p\right) \sum_{i=1}^n a_i^p.$$

Combining the previous, we then have proved

$$\forall p \in]1,\infty[\qquad \Psi^{'}(p) \leq 0,$$

which yields the desired result.

Proof 5. Here, we use a mathematical induction on n. For $n=1$ the desired result is trivial. Assume that, for $p>q$,

$$\left(\sum_{i=1}^{k} a_i^p\right)^{1/p} \leq \left(\sum_{i=1}^{k} a_i^q\right)^{1/q}.$$

We can write

$$\left(\sum_{i=1}^{k+1} a_i^p\right)^{1/p} = \left(\sum_{i=1}^{k} a_i^p + a_{k+1}^p\right)^{1/p},$$

which, according to the induction hypothesis, becomes

$$\left(\sum_{i=1}^{k+1} a_i^p\right)^{1/p} \leq \left(\left(\sum_{i=1}^{k} a_i^q\right)^{p/q} + a_{k+1}^p\right)^{1/p},$$

or again

$$\left(\sum_{i=1}^{k+1} a_i^p\right)^{1/p} \leq \left(\left(\left(\sum_{i=1}^{k} a_i^q\right)^{p/q} + a_{k+1}^p\right)^{q/p}\right)^{1/q}.$$

Now, Lemma 1.2 with

$$N=2,\ m=q/p \in]0,1[,\ c_1 = \left(\sum_{i=1}^{k} a_i^q\right)^{p/q},\ c_2 = a_{k+1}^p$$

yields the following

$$\left(\sum_{i=1}^{k+1} a_i^p\right)^{1/p} \leq \left(\sum_{i=1}^{k} a_i^q + a_{n+1}^q\right)^{1/q}.$$

Summarizing what previous, the desired result follows.

Proof 6. This proof is based on a duality principle using the Hölder inequality. It is easy to deduce that

$$\|x\|_p = max\,\{|\langle x,y\rangle|,\ \ \|y\|_{p^*} \leq 1\} = max\{|\langle x,y\rangle|,\ \sum_{i=1}^{n} |y_i|^{p^*} \leq 1\}.$$

If $p \geq q$ then $p^* \leq q^*$ and so there holds

$$\{y \in \mathbb{R}^n,\ \sum_{i=1}^{n} |y_i|^{p^*} \leq 1\} \subset \{y \in \mathbb{R}^n,\ \sum_{i=1}^{n} |y_i|^{q^*} \leq 1\}.$$

Then we can write

$$\|x\|_p = max\{|\langle x, y\rangle|,\ \sum_{i=1}^{n} |y_i|^{p^*} \leq 1\} \leq max\{|\langle x, y\rangle|,\ \sum_{i=1}^{n} |y_i|^{q^*} \leq 1\} = \|x\|_q,$$

which is the desired result.

Proof 7. Here we only present the key idea of this proof and we omit the details to the reader. First, we easily show the theorem when p and q are both integers. Secondly, if p and q are rational the proof of the desired result can be reduced, with a simple manipulation, to the above case. Finally, for p and q real numbers, it is sufficient to remark that the map $p \longmapsto \|x\|_p$, for fixed $x \in \mathbb{R}^n$, is continuous and the desired result follows by density of $\mathbb{Q}$ in $\mathbb{R}$.

3 Another Family of $\mathbb{R}^n-$Norms

As already pointed, the aim of this section turns out of to construct another family of norms in $\mathbb{R}^n$ indexed by $p \in [0, +\infty]$. We need additional basic notions which we will state below. Let α be a norm in $\mathbb{R}^n$. The dual of α is the norm α^* defined by

$$\forall x \in \mathbb{R}^n \qquad \alpha^*(x) = max\{|\langle x, y\rangle|,\ \alpha(y) \leq 1\} = max\{\frac{|\langle x, y\rangle|}{\alpha(y)},\ y \neq 0\}.$$

It is easy to see that $\alpha^{**} := (\alpha^*)^* = \alpha$ and, if α and β are two norms of $\mathbb{R}^n$ such that $\alpha \geq \beta$ (i.e $\alpha(x) \geq \beta(x)$ for every $x \in \mathbb{R}^n$) then $\alpha^* \leq \beta^*$. The norm α will be called a normalized norm if $\alpha(e_i) = 1$ for all $i = 1, 2, ..., n$, where $(e_1, e_2, ..., e_n)$ refers to the canonical basis of $\mathbb{R}^n$. It is clear that the set of all normalized norms of $\mathbb{R}^n$ is not stable for the operation addition but it is a convex set. It is easy to see that if α is a normalized norm then so is α^*. The above norm $\|.\|_p$ is a normalized norm for all $1 \leq p \leq \infty$, with the relationship $(\|.\|_p)^* = \|.\|_{p^*}$. In particular, $\|.\|_1^* = \|.\|_\infty,\ \|.\|_\infty^* = \|.\|_1,\ \|.\|_2^* = \|.\|_2$, and it is not hard to verify that $\|.\|_2$ is the unique self-dual norm of $\mathbb{R}^n$.
A sequence $(\alpha^k)_k$ of norms in $\mathbb{R}^n$ will be called (point-wise) converging if the two following conditions are both satisfied
(i) For all $x \in \mathbb{R}^n$, $\alpha^k(x)$ converges in $\mathbb{R}$,
(ii) Setting $\alpha(x) := lim_{k\to\infty}\alpha^k(x)$, then α defines a norm in $\mathbb{R}^n$.
It is clear that (i) doesn't ensure (ii). In fact, if (α^k) satisfies (i) with $lim_{k\to\infty}\alpha^k(x) =$

$\alpha(x)$ then α is only a semi-norm (i.e the separation axiom of a norm is not generally satisfied by α). For example, $\alpha^k(x) = \|x\|/(k+1)$, for a given norm $\|.\|$ of $\mathbb{R}^n$, satisfies (i) but not (ii). However, the following result holds.

Lemma 3.1. *Let $(\alpha^k)_k$ be a sequence of normalized norms of $\mathbb{R}^n$ with $lim_{k\to\infty}\alpha^k(x) = \alpha(x)$ for every $x \in \mathbb{R}^n$. Then, α is a normalized norm of $\mathbb{R}^n$. If moreover $(\alpha^k)_k$ is monotone decreasing then $\big((\alpha^k)^*\big)_k$ converges to α^*.*

Proof. It is a simple exercise for the reader. □

The following lemma will be needed in the sequel.

Lemma 3.2. *Let α and β be two norms of $\mathbb{R}^n$. Then, for all $1 \leq p \leq \infty$, there holds*

$$\left(\frac{1}{p}\alpha + \frac{1}{p^*}\beta\right)^* \leq \frac{1}{p}\alpha^* + \frac{1}{p^*}\beta^*. \tag{3.1}$$

If $\alpha \geq \beta$ then the map $p \longmapsto \frac{1}{p}\alpha + \frac{1}{p^*}\beta$ is (point-wise) monotone decreasing.

Proof. By virtu of the definition of the dual norm, it is sufficient to prove that

$$\left(\frac{1}{p}a + \frac{1}{p^*}b\right)^{-1} \leq \frac{1}{p}a^{-1} + \frac{1}{p^*}b^{-1},$$

for all real numbers $a > 0, b > 0$. Since the real mapping $x \longmapsto 1/x$ is convex on $]0, +\infty[$, the desired inequality follows. The second part of the lemma is immediate so completes the proof. □

Now, for $1 \leq p \leq \infty$, we define a sequence $(\Theta_p^k)_k$ by the following iterate process

$$\Theta_p^{k+1} = \frac{1}{p}\Theta_p^k + \frac{1}{p^*}\left(\Theta_{p^*}^k\right)^*, \ k \geq 0; \quad \Theta_p^0 = \|.\|_1 \tag{3.2}$$

It is easy to see that Θ_p^k is a normalized norm for all p and $k \geq 0$, with

$$\Theta_p^1 = \frac{1}{p}\|.\|_1 + \frac{1}{p^*}\|.\|_\infty.$$

It is clear that $\Theta_1^k = \|.\|_1$ and $\Theta_\infty^k = \|.\|_\infty$ for all $k \geq 0$. Below, we can then assume that $1 < p < \infty$.

Proposition 3.3. *For all $1 < p < \infty$, the following assertions are met*
(i) For every $k \geq 0$, $\left(\Theta_{p^*}^k\right)^* \leq \Theta_p^k$
(ii) The sequence $(\Theta_p^k)_k$ is monotone decreasing.

Proof. (i) For $k = 0$ it is trivial. By (3.2) with Lemma 3.2, we obtain

$$\left(\Theta_{p^*}^{k+1}\right)^* \leq \frac{1}{p^*}\left(\Theta_{p^*}^{k}\right)^* + \frac{1}{p}\Theta_p^k = \Theta_p^{k+1},$$

which gives the desired inequality.
(ii) Substituting the above inequality in (3.2), we immediately deduce the decrease monotonicity of the sequence $(\Theta_p^k)_k$. This completes the proof. □

Theorem 3.4. *The norm sequence $(\Theta_p^k)_k$ converges (point-wisely) to a normalized norm Θ_p of $\mathbb{R}^n$, with the following estimation*

$$\forall k \geq 0 \quad 0 \leq \Theta_p^k - \Theta_p \leq \frac{1}{p^k}\left(\|.\|_1 - \|.\|_\infty\right). \tag{3.3}$$

Proof. By Proposition 3.3, for all $k \geq 0$ we have

$$\|.\|_\infty \leq ... \leq \left(\Theta_{p^*}^{k-1}\right)^* \leq \left(\Theta_{p^*}^{k}\right)^* \leq \Theta_p^k \leq \Theta_p^{k-1} \leq ... \leq \|.\|_1 \tag{3.4}$$

It follows that, for all $x \in \mathbb{R}^n$, the real sequences $\left(\Theta_p^k(x)\right)_k$ is monotone decreasing and lower bounded and so converges in $\mathbb{R}$. Since Θ_p^k is a normalized norm, the first part of Lemma 3.1 tells us that $\left(\Theta_p^k\right)_k$ converges (point-wisely) to a normalized norm denoted by Θ_p. To prove the estimation (3.3), we first remark that

$$\left(\Theta_{p^*}^{k}\right)^* \leq \Theta_p \leq \Theta_p^k,$$

which with the fact that

$$\Theta_p^{k+1} - \Theta_p = \frac{1}{p}\left(\Theta_p^k - \Theta_p\right) + \frac{1}{p^*}\Big(\left(\Theta_{p^*}^{k}\right)^* - \Theta_p\Big),$$

yields the following double inequality

$$0 \leq \Theta_p^{k+1} - \Theta_p \leq \frac{1}{p}\left(\Theta_p^k - \Theta_p\right),$$

for all $k \geq 0$. The desired result follows by a mathematical induction with a simple manipulation. The proof of the theorem is complete. □

Corollary 3.5. *For all $1 < p < \infty$, the following properties hold true*

(i) $\Theta_p \leq \frac{1}{p}\|.\|_1 + \frac{1}{p^*}\|.\|_\infty$

(ii) $(\Theta_p)^* = \Theta_{p^*}$. *In particular, one has* $\Theta_2 = \|.\|_2$.

Proof. (i) By the decrease monotonicity of $\left(\Theta_p^k\right)_k$ we have

$$\Theta_p^k \leq \Theta_p^1 = \frac{1}{p}\|.\|_1 + \frac{1}{p^*}\|.\|_\infty.$$

Letting $k \to \infty$ in this later inequality we obtain the desired result.
(ii) Since $\left(\Theta_p^k\right)_k$ converges to Θ_p for every $1 < p < \infty$ then $\left(\Theta_{p^*}^k\right)_k$ converges to Θ_{p^*}. Algorithm (3.2), with the second part of Lemma 3.1, yields when $k \to \infty$ the first relation of (ii). In particular, if $p = 2$ then $p^* = 2$ and so $(\Theta_2)^* = \Theta_2$. Since $\|.\|_2$ is the unique self-dual norm of $\mathbb{R}^n$, the second relation of (ii) is so obtained, thus concludes the proof. □

Proposition 3.6. *For $1 < p < \infty$, the map $p \longmapsto \Theta_p$ is monotone decreasing.*

Proof. By a mathematical induction on $k \geq 1$, we wish to establish that $\Theta_p^k \leq \Theta_q^k$ for $p \geq q$. For $k = 1$, it is immediate from the second part of Lemma 3.2, since $\|.\|_\infty \leq \|.\|_1$. Now, assume that for p and q such that $p \geq q$ we have $\Theta_p^k \leq \Theta_q^k$. One has

$$\Theta_p^{k+1} = \frac{1}{p}\Theta_p^k + \frac{1}{p^*}\left(\Theta_{p^*}^k\right)^* \leq \frac{1}{p}\Theta_q^k + \frac{1}{p^*}\left(\Theta_{q^*}^k\right)^*. \tag{3.5}$$

According to Proposition 3.3,(i), we have $\left(\Theta_{q^*}^k\right)^* \leq \Theta_q^k$ which, with the second part of Lemma 3.2 and (3.5) yields

$$\Theta_p^{k+1} \leq \frac{1}{q}\Theta_q^k + \frac{1}{q^*}\left(\Theta_{q^*}^k\right)^* = \Theta_q^{k+1}.$$

Summarizing, for all $k \geq 0$ and $p \geq q$ we have $\Theta_p^k \leq \Theta_q^k$. Letting $k \to \infty$ in this latter inequality we deduce the desired result. □

A Hölder type inequality satisfied by the norm Θ_p is recited in the following.

Proposition 3.7. *Let $1 < p < \infty$ and $x, y \in \mathbb{R}^n$ then one has*

$$|\langle x, y\rangle| \leq \Theta_p(x)\Theta_{p^*}(y)$$

Proof. Algorithm (3.2), with the definition of the dual of a norm, gives

$$\Theta_p^{k+1}(x) \geq \frac{1}{p}\Theta_p^k(x) + \frac{1}{p^*}\left(\Theta_{p^*}^k\right)^*(x) \geq \frac{1}{p}\Theta_p^k(x) + \frac{1}{p^*}\frac{|\langle x, y\rangle|}{\Theta_{p^*}^k(y)},$$

for all $x, y \in \mathbb{R}^n$ with $y \neq 0$. By Young inequality we deduce that

$$\Theta_p^{k+1}(x) \geq \left(\Theta_p^k(x)\right)^{1/p} \frac{|\langle x, y\rangle|^{1/p^*}}{\left(\Theta_{p^*}^k(y)\right)^{1/p^*}}.$$

This, when $k \to \infty$ and after a simple reduction, yields the desired inequality thus completes the proof. □

4 Some Open Problems

As we have seen throughout the previous study, the norms $\|.\|_p$ and Θ_p have similar properties. In summary, we have obtained the following:

1. $\|.\|_p$ and Θ_p are both normalized norms of $\mathbb{R}^n$, for all $1 \le p \le \infty$,

2. $p \longmapsto \|.\|_p$ and $p \longmapsto \Theta_p$ are both monotone decreasing,

3. $(\|.\|_p)^* = \|.\|_{p^*}$ and $(\Theta_p)^* = \Theta_{p^*}$ for all $1 \le p \le \infty$,

4. $\|.\|_p \le \dfrac{1}{p}\|.\|_1 + \dfrac{1}{p^*}\|.\|_\infty$ and $\Theta_p \le \dfrac{1}{p}\|.\|_1 + \dfrac{1}{p^*}\|.\|_\infty$, for all $1 \le p \le \infty$,

5. $|\langle x, y\rangle| \le \|x\|_p\|y\|_{p^*}$ and $|\langle x, y\rangle| \le \Theta_p(x)\Theta_{p^*}(y)$, for all $1 \le p \le \infty$ and $x, y \in \mathbb{R}^n$,

6. $\|.\|_1 = \Theta_1,\ \|.\|_2 = \Theta_2$ and $\|.\|_\infty = \Theta_\infty$.

This allows us to put the following.

Open Problem 1. What is the explicit form of Θ_p? What is the interconnection between $\|.\|_p$ and Θ_p? Is it possible to compare (algebraically) these two norms?

Now, in order to state a conjecture we need some additional notions about tensorial product theory. For further details, the reader can consult [1, 2] for example. Let $n \ge 2$ be an integer and a chosen decomposition $n = rs$ with r, s integers. Then, $\mathbb{R}^n = \mathbb{R}^r \otimes \mathbb{R}^s$ and every $x \in \mathbb{R}^n$ can be written as follows

$$x = \sum_{i=1}^{N} y_i \otimes z_i,\ y_i \in \mathbb{R}^r,\ z_i \in \mathbb{R}^s,\ N \ge 1.$$

Let $1 < p < \infty$ and $\|.\|_{\mathbb{R}^r}$ and $\|.\|_{\mathbb{R}^s}$ be two fixed norms of $\mathbb{R}^r$ and $\mathbb{R}^s$ respectively. For all $x \in \mathbb{R}^n$ we define

$$d_p^{r,s}(x) = \underset{x=\sum_{i=1}^N y_i\otimes z_i}{inf} \left\{ \left(\sum_{i=1}^{N} \|y_i\|_{\mathbb{R}^r}^p\right)^{1/p} \left(sup \left\{ \sum_{i=1}^{N} |\langle z_i, t\rangle|^{p^*},\ \|t\|_{\mathbb{R}^s} \le 1 \right\}\right)^{1/p^*} \right\},$$

where the "inf" is taken over all finite decomposition $\sum_i y_i \otimes z_i$ of $x \in \mathbb{R}^n = \mathbb{R}^r \otimes \mathbb{R}^s$.

For fixed integers r, s such that $n = rs$, the above expression of $d_p^{r,s}$ defines a norm on $\mathbb{R}^n$, called cross norm on $\mathbb{R}^r \otimes \mathbb{R}^s$. After these notions, our conjecture is now recited in the following.

Conjecture. Let $n \geq 2$ be a integer and $1 < p < \infty$. Then there exist r, s integers with $n = rs$ and two norms $\|.\|_{\mathbb{R}^r}$, $\|.\|_{\mathbb{R}^s}$ such that $\Theta_p = d_p^{r,s}$.

Finally, we briefly recall the notions of $2-$norms in order to state our second open problem. For further details concerning the introduction of $2-$norms, the reader can consult [3] for example. Let E be a linear vector space on a field $\mathbb{K} = \mathbb{R}, \mathbb{C}$. A binary map $\|., .\|$ defined from $E \times E$ into $[0, +\infty[$ is called $2-$norm if the four following conditions are simultaneously satisfied:

(i) $\|u, v\| = 0$ if and only if u and v are linearly dependent,
(ii) $\|u, v\| = \|v, u\|$ for all $u, v \in E$,
(iii) $\|\lambda.u, v\| = |\lambda|.\|u, v\|$ for all $u, v \in E$ and $\lambda \in \mathbb{K}$,
(iv) $\|u + v, w\| \leq \|u, w\| + \|v, w\|$ for all $u, v, w \in E$.
Now, our question arising from the above can be recited as follows:

Open Problem 2. What should be the reasonable analogues of the above family of norms for $2-$norms?

In this direction, we invite the reader to consult [4].

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