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Discrete Operator and Functional Means Can Be Reduced To The Continuous Arithmetic Mean

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Abstract

In this paper, we introduce the functional arithmetic mean of a continuous family for which we associate its conjugate dual in the sense of convex analysis. Afterwards, we prove that the discrete operator and functional means, already discussed in the literature, can be obtained as special cases from our continuous arithmetic mean. Such functional approach gives, in a fast way, that of positive operators and allows us to introduce a chaotic geometric operator mean whose the extension for functional variable will be putted as an open problem.

Keywords: *Convex analysis; operator mean; functional mean; continuous arithmetic functional mean; chaotic geometric operator mean; probability measure.*

MSC (2000): 46A20, 52A41, 49-XX, 52-XX.

1 Introduction

In recent few years, an enormous amount of effort by some authors has been devoted to introduce various operator means, see [3, 5, 6, 11, 19] for instance and the related references cited therein. Recently, these operator means have been extended from the case that the variables are positive operators to the case that the variables are convex functionals, see [1, 4, 7, 12, 15, 18]. The key idea of such extension turns out of that the conjugate operation in convex analysis can be interpreted as an inverse in a certain sense. All previous works, as we know, concern the finite discrete families for defining operator and functional means. In the present paper, we will introduce a generalized concept, namely the arithmetic functional mean of a continuous family, to which we associate its dual for the conjugate operation in convex analysis. Afterwards, we show throughout various examples, that the discrete operator and functional means, already stated in the literature, can be obtained as particular cases of our present functional mean. Further, others operator and functional notions, as the functional logarithm, the relative entropy and the operator sign, can be also considered here as special examples. Our functional approach gives immediately that of operator and allows us to introduce a chaotic operator mean of a continuous family of positive operators whose analogue for functional variables will be putted as an open problem.

2 Basic Notions

This section is devoted to state some basic notions about convex analysis that are needed in the sequel. For further details, we refer the reader to [2, 10, 23] for example. Let E be a real or complex locally convex space, E^* its topological dual, and $< .,. >$ the duality bracket between E and E^*. The notation $\overline{\mathbb{R}}^E$ refers to the space of all functions defined from E into $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ equipped with the point-wise partial ordering,

$$\forall\ f,\ g \in \overline{\mathbb{R}}^E, \quad f \leq g \iff \forall u \in E \quad f(u) \leq g(u),$$

where we extend the structure of $\mathbb{R}$ on $\overline{\mathbb{R}}$ by setting

$$\forall\ \ x \in \overline{\mathbb{R}}, \quad -\infty \leq x \leq +\infty, \qquad (+\infty) + x = +\infty, \quad 0.(+\infty) = +\infty.$$

Given a functional $f : E \to \widetilde{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$, the notation f^* stands for the conjugate of f defined by

$$\forall u^* \in E^* \qquad f^*(u^*) = \sup_{u \in E}\{Re < u, u^* > - f(u)\},$$

where $Re < u^*, u >$ denotes the real part of the complex number $< u^*, u >$. It is clear that, if $f \leq g$ then $g^* \leq f^*$.
Let $f \in \widetilde{\mathbb{R}}$ and $\lambda > 0$ be a real, we define the functionals $\lambda.f$ and $f.\lambda$ by

$$\forall u \in E, \quad (\lambda.f)(u) = \lambda.f(u) \ \text{ and } \ (f.\lambda)(u) = \lambda.f(\frac{u}{\lambda}).$$

With this, it is not hard to see that

$$(\lambda.f)^* = f^*.\lambda \ \text{ and } \ (f.\lambda)^* = \lambda.f^*.$$

The bi-conjugate of f is the functional $f^{**} : E \longrightarrow \widetilde{\mathbb{R}}$ defined as follows

$$\forall u \in E \quad f^{**}(u) := (f^*)^*(u) = \sup_{u^* \in E^*}\{Re < u^*, u > - f^*(u^*)\}.$$

If we denote by $\Gamma_0(E)$ the cone of lower semi-continuous (l.s.c) convex functionals from E into $\mathbb{R}\cup\{+\infty\}$ not identically equal to $+\infty$, it is well-known that, $f^{**}\leq f$ and, $f^{**}=f$ if and only if $f\in\Gamma_0(E)$.
An important and typical example of $\widetilde{\mathbb{R}}^E-$functional is f_A defined by

$$\forall u\in E \qquad f_A(u)=(1/2)Re<Au,u>,$$

where $A:E\longrightarrow E^*$ is a bounded linear operator. We say that f_A is quadratic in the sense that $f(t.u)=|t|^2f(u)$ for all $u\in E$ and all number complex t. It is easy to see that the conjugate operation preserves the quadratic character, and it is well-known that $f_A\in\Gamma_0(E)$ if and only if A is a self-adjoint positive operator defined from E into E^*. In this latter case, if moreover A is invertible then, f_A^* takes the explicit form,

$$\forall u^*\in E^* \qquad f_A^*(u^*)=(1/2)<A^{-1}u^*,u^*>.$$

In another way, $f_A^*=f_{A^{-1}}$ and so, as already pointed out, the conjugate operation can be interpreted as a reasonable extension from the case that the variable is a linear positive operator to the case that the variable is a convex functional. This latter point is more explained by a set of works published in the same sense, see [1, 7, 12, 13, 14, 15, 16, 18].

3 Functional Arithmetic Mean of a Continuous Family

Let T be a nonempty set, μ a probability measure on T and E a locally convex linear space. For fixed $t\in T$, let $f(t,.):H\longrightarrow\widetilde{\mathbb{R}}$ be a functional family defined from E into $\overline{\mathbb{R}}=\mathbb{R}\cup\{-\infty,+\infty\}$. To simplify the writing, we set $f(t,.)=f_t$ and $F=(f_t)_{t\in T}$, $F^\star=(f_t^*)_{t\in T}$, where $f_t^*:E^*\longrightarrow\overline{\mathbb{R}}$ is the conjugate of f_t for fixed $t\in T$. We say that $F=(f_t)_{t\in T}$ is convex (resp.quadratic, $\Gamma_0(E)$) if f_t is convex (resp.quadratic, $\Gamma_0(E)$) for every $t\in T$. For two given families $F=(f_t)_{t\in T}$ and $G=(g_t)_{t\in T}$, we write $F\leq G$ if and only if $f_t(u)\leq g_t(u)$ for all $t\in T$ and $u\in E$. It is clear that, if $F\leq G$ then $G^*\leq F^*$. Also, if T is finite with $card\,T=k$ we write $F=(f_1,f_2,...,f_k)$.
Our first central definition is recited in the following.

Definition 3.1. With the above, the arithmetic functional mean of the continuous family $F=(f_t)_{t\in T}$, with respect to the probability measure μ, is the map $\mathcal{A}(F):E\longrightarrow\overline{\mathbb{R}}$ defined by

$$\forall u\in E \qquad \mathcal{A}(F)(u)=\int_T f_t(u)d\mu(t). \tag{3.1}$$

In order to simplify the presentation for the reader, we briefly write

$$\mathcal{A}(F) = \int_T f_t d\mu(t),$$

instead of (3.1). The following result is immediate from the definition.

Proposition 3.1. *With the above, the following statements hold*
1. For all constant family $F = (f)$ one has $\mathcal{A}(f) = f$.
2. The map $F \longmapsto \mathcal{A}(F)$ is linear and applies $Conv(E)$ into $Conv(E)$.
3. If $F = (f_t)_{t\in T}$ and $G = (g_t)_{t\in T}$ are such that $F \leq G$ then $\mathcal{A}(F) \leq \mathcal{A}(G)$.

Proposition 3.2. *If the family $F = (f_t)_{t\in T}$ is quadratic so is $\mathcal{A}(F)$. Moreover, if $f_t = f_{A_t}$, where $A := (A_t)_{t\in T}$ is a family of bounded linear positive operators from E into E^*, then $\mathcal{A}(F) = f_{\mathcal{A}(A)}$ with*

$$\mathcal{A}(A) = \int_T A_t d\mu(t).$$

Proof. It is a simple exercise for the reader. □

By analogy with Definition 3.1, the map $A \longmapsto \mathcal{A}(A)$ will be called the arithmetic operator mean of the continuous family $A = (A_t)_{t\in T}$, with respect to μ. Now, we wish to state our second central definition.

Definition 3.2. Let $\mathcal{A}(F)$ be a functional arithmetic mean as in the above. The dual of the map $F \longmapsto \mathcal{A}(F)$ is defined by

$$\mathcal{A}^\star(F) := (\mathcal{A}(F^\star))^* = \left(\int_T f_t^* d\mu(t)\right)^*. \tag{3.2}$$

From this definition, the following elementary properties are immediate.

Proposition 3.3. *The dual arithmetic mean enjoys the following properties:*
1. $F \longmapsto \mathcal{A}^\star(F)$ is an application from $\widetilde{\mathbb{R}}^E$ into $\Gamma_0(E)$.
2. For every constant family $F = (f)$ with $f \in \Gamma_0(E)$ we have $\mathcal{A}^\star(f) = f$.
3. For all $F = (f_t)_{t\in T}$ and a real $\lambda > 0$ one has

$$\mathcal{A}^\star(\lambda.F) = \lambda.\mathcal{A}^\star(F) \quad \textit{and} \quad \mathcal{A}^\star(F.\lambda) = \mathcal{A}^\star(F).\lambda$$

4. If $F \leq G$ then $\mathcal{A}^\star(F) \leq \mathcal{A}^\star(G)$.

Proposition 3.4. *The dual arithmetic mean conserves the quadratic character. Moreover, if $f_t = f_{A_t}$, where $A := (A_t)_{t\in T}$ is a family of bounded positive linear operators from E into E^*, then $\mathcal{A}^\star(F) = f_{\mathcal{A}^\star(A)}$ with*

$$\mathcal{A}^\star(A) = \left(\int_T A_t^{-1} d\mu(t)\right)^{-1}.$$

With the above, and as for the functional case, the map $A \longmapsto \mathcal{A}^\star(A)$ is called the dual operator arithmetic mean of $A \longmapsto \mathcal{A}(A)$.
As out of pointed and for all operator mean, if a positive operator A is not invertible we replace A^{-1} by

$$A^+ = \lim_{\epsilon \to 0^+} (A + \epsilon.I)^{-1},$$

for the sake of convenience. Now, we state an interesting inequality which, as we will see later, yields immediately various inequalities differently discussed in the literature.

Proposition 3.5. *The functional arithmetic mean $\mathcal{A}(F)$ and its dual satisfy the following inequality*

$$\mathcal{A}^\star(F) \ \le \ \mathcal{A}(F).$$

Proof. By definition, we can write for all $u \in E$

$$\mathcal{A}^\star(F)(u) = \left(\mathcal{A}(F^\star)\right)^*(u) = \sup_{u^* \in E^*} \{Re < u^*, u > -\mathcal{A}(F^\star)(u^*)\}.$$

According to Definition 3.1, we obtain by substituting the expression of $\mathcal{A}(F^\star)$

$$\mathcal{A}^\star(F)(u) = \sup_{u^* \in E^*} \int_T \left(Re < u^*, u > -f_t^*(u^*)\right) d\mu(t),$$

or again

$$\mathcal{A}^\star(F)(u) \le \int_T \sup_{u^* \in E^*} \{Re < u^*, u > -f_t^*(u^*)\} d\mu(t).$$

It follows that

$$\mathcal{A}^\star(F)(u) \le \int_T f_t^{**}(u) d\mu(t) \le \int_T f_t(u) d\mu(t) = \mathcal{A}(F)(u),$$

which completes the proof. □

4 Discrete Means and Continuous Arithmetic Mean

In this section, we wish to show throughout various examples that the known discrete operator and functional means can be obtained from the above continuous arithmetic mean by choosing a convenient family $(f_t)_{t \in T}$ and a probability measure μ. Moreover, many other functions involving operators or convex functionals, discussed in the literature, are also obtained here as particular cases. To simplify the presentation, we precise that all functionals considered later, denoted by $f, g...$, belong to $\widetilde{\mathbb{R}}^E$ or, if it is necessary, to $\Gamma_0(E)$, and all operators, denoted by $A, B...$, are bounded linear (eventually positive invertible) defined from E into E^*.

Example 4.1. Operator and Functional Arithmetic and Harmonic Means.
Let us consider $T = \{1, 2, ..., k\}$, $F = (f_1, f_2, ..., f_k)$ and the discrete probability measure $\mu(t) = 1/k$ for every $t \in T$.
By Definition 3.1, the arithmetic mean of the finite family $F = (f_1, f_2, ..., f_k)$ is so given by

$$\mathcal{A}(f_1, f_2, ..., f_k) = \frac{\sum_{i=1}^{k} f_i}{k},$$

which is the functional arithmetic mean of $f_1, f_2, ..., f_k$, see [1, 4, 15].
By Definition 3.2, the dual arithmetic mean is here given by

$$\mathcal{A}^\star(f_1, f_2, ..., f_k) = \left(\frac{\sum_{i=1}^{k} f_i^*}{k}\right)^*,$$

which is the functional harmonic mean of $f_1, f_2, ..., f_k$, [15](introduced at the first time in [1] for $k = 2$).
If the functional variables $f_1, f_2, ..., f_k$ are quadratic associated with operators $A_1, A_2, ..., A_k$ respectively then $\mathcal{A}(f_1, f_2, ..., f_k)$ and $\mathcal{A}^\star(f_1, f_2, ..., f_k)$ are also quadratic associated, respectively, to the arithmetic and harmonic operator means.
Otherwise, Proposition 3.5 gives here the arithmetic-harmonic functional (resp.operator, scalar) inequality.

Example 4.2. Operator and Functional Geometric Means.
Let $T = (0, +\infty[$ and the family $F = (f_t)_{t\in T}$ where

$$\forall t \in T \qquad f_t = \left(\frac{1}{1+t} f^* + \frac{t}{1+t} g^*\right)^*.$$

With the probability measure

$$d\mu(t) = \frac{dt}{\pi\sqrt{t}(1+t)},$$

$\mathcal{A}(F)$ is nothing other than the geometric functional mean of f and g introduced in [1, 15]. See also [7] for another view-point of this functional mean.
Analogously, considering the quadratic case $f = q_A$ and $g = q_B$, with A, B are two positive operators, we obtain $\mathcal{A}(F) = q_{\mathcal{A}(A)}$, where $\mathcal{A}(A)$ coincides with the integral representation in Kubo-Ando theory [9] for the construction of the geometric operator mean.

Example 4.3. Power Operator and Functional Means.
This example concerns an extension of the two above ones. Here, p is a real such that $0 < p < 1$, $T = (0, +\infty[$, μ is the probability measure

$$\forall t > 0 \qquad d\mu(t) = \frac{sin\ p\pi}{\pi} \frac{t^{p-1}}{1+t} dt.$$

The family $F = (f_t)_{t\in T}$ is respectively defined in the following cases:
• For $t \in (0, +\infty[$ we set

$$f_t = \frac{1}{1+t} f + \frac{t}{1+t} g.$$

By Definition 3.1, with a computation (see [15]), we obtain

$$\mathcal{A}(F) = (1-p).f + p.g,$$

which is the power arithmetic functional mean of f and g.
• The power harmonic functional mean of f and g,

$$\mathcal{H}(F) = \left((1-p).f^* + p.g^*\right)^*,$$

can be deduced, as in the previous examples, by duality from the above. The details are here left to the reader.
• Now, let $F = (f_t)_t$ defined as follows

$$\forall t \geq 0 \qquad f_t = \left(\frac{1}{1+t}.f^* + \frac{t}{1+t}.g^*\right)^*.$$

Definition 3.1 gives here the power geometric functional mean introduced in [15] (and called by the authors, the $(1-p,p)$-functional mean of f and g). For $p = 1/2$, we find the (arithmetic, harmonic and geometric) functional means discussed in the above examples. The quadratical case yields immediately, in a fast way, the power arithmetic, harmonic and geometric operator means.

Example 4.4. Operator and Functional Logarithmic Means.
Let $T = [0,1]$, μ the Lebesgue measure and $F = (f_t)_{t\in T}$ with

$$\forall t \in [0,1] \qquad f_t = t.f + (1-t).g.$$

By Definition 3.2, the dual arithmetic mean of the family $F = (f_t)_{t\in T}$ is

$$\mathcal{A}^\star(F) = \left(\int_0^1 (t.f + (1-t).g)^* dt\right)^* := \mathcal{L}(f,g),$$

which is the logarithmic functional mean of f and g introduced by the author in [18]. In the quadratical case, we find immediately

$$\mathcal{L}(A,B) = \left(\int_0^1 (t.A + (1-t).B)^{-1} dt\right)^{-1},$$

which for the scalar case yields the classical logarithmic mean,

$$\mathcal{L}(a,b) = \frac{a-b}{ln\ a - ln\ b} \text{ if } a \neq b, \ \ \mathcal{L}(a,a) = a.$$

Proposition 3.5 yields here the arithmetic-logarithmic functional (resp.operator, scalar) inequality.

Example 4.5. Operator and Functional Means of several variables.
The arithmetic and harmonic functional (and operator) means of several variables was introduced in Example 4.1. In this example, we will be interested by the

geometric and logarithmic functional means of several variables.
Let $m \geq 2$ be an integer and $T = \Delta_{m-1}$ given by

$$\Delta_{m-1} = \left\{ t = (t_1, t_2, ..., t_{m-1}) \in I\!R^{m-1}, \sum_{i=1}^{m-1} t_i \leq 1, t_i \geq 0, 1 \leq i \leq m-1 \right\},$$

and we set $t_m = 1 - \sum_{i=1}^{m-1} t_i$. With this, we investigate the two following situations.
• Consider $d\mu(t) = (m-1)! dt_1 dt_2 ... dt_{m-1}$ the $(m-1)$-product probability on Δ_{m-1} and $F = (f_t)$ where

$$\forall t \in \Delta_{m-1} \qquad f_t = \sum_{i=1}^{m} t_i f_i,$$

with $f_1, f_2, ..., f_m \in \Gamma_0(E)$ such that $\bigcap_{i=1}^m dom\ f_i \neq \emptyset$. By Definition 3.2, the dual of $\mathcal{A}(F)$ is given by

$$\mathcal{A}^\star(F) = \left((m-1)! \int_{\Delta_{m-1}} \left(\sum_{i=1}^{m} t_i f_i \right)^* dt_1 dt_2 ... dt_{m-1} \right)^*,$$

which is the logarithmic functional mean of $f_1, f_2, ..., f_m$ introduced by the author in [18]. The quadratic case is immediately obtained by taking $f_i = f_{A_i}, 1 \leq i \leq m$, where $(A_i)_{i=1}^m$ is a family of positive operators, see [18] for further details.
• Now, let $d\mu(t)$ be defined by

$$d\mu(t) = \frac{1}{(\Gamma(1/m))^m} \prod_{i=1}^{m-1} t_i^{1/m-1} dt_1 dt_2 ... dt_{m-1},$$

and the family $F = (f_t)$ such that

$$f_t = \left(\sum_{i=1}^{m} t_i . f_i^* \right)^*,$$

with $f_1, f_2, ..., f_m \in \Gamma_0(E)$ such that $\bigcap_{i=1}^m dom\ f_i \neq \emptyset$. Definition 3.1 gives

$$\mathcal{A}(F) = \frac{1}{(\Gamma(1/m))^m} \int_{\Delta_{m-1}} \left(\sum_{i=1}^{m} t_i . f_i^* \right)^* \left(\prod_{i=1}^{m-1} t_i^{1/m-1} \right) dt_1 dt_2 ... dt_{m-1},$$

which we propose here as the geometric functional mean involving m variables. If $f_1, f_2, ..., f_m$ are quadratic, we obtain the Kosaki geometric operator mean from which we have inspired our functional case.
For $m = 2$ we obtain respectively the geometric and logarithmic means of two convex functionals discussed in the above.
We left the reader to say what the type functional (resp.operator, scalar) inequality can be derived from Proposition 3.5 for this example.

Example 4.6. Operator and Functional Logarithm.
First, the reader will do well to distinguish the concept discussed here with that of Example 4.4.
Let E be a Hilbert space, $T = [0,1]$, μ the Lebesgue measure and

$$f_t = \frac{\sigma - ((1-t).\sigma + t.f)^*}{t},$$

where $\sigma = (1/2)\|.\|^2$ and $f \in \Gamma_0(E)$. By Definition 3.1, the arithmetic mean of $F = (f_t)_{t\in T}$ is

$$\mathcal{A}(F) = \int_0^1 \frac{\sigma - ((1-t).\sigma + t.f)^*}{t} dt := \mathcal{L}(f),$$

which is the functional logarithm in convex analysis introduced in [13]. If $f = f_A$, where $A : E \longrightarrow E$ is a positive invertible operator, then we obtain the classical operator logarithm. As proved in [13], for all $f \in \Gamma_0(E)$ the functional relation $\mathcal{L}(f^*) = -\mathcal{L}(f)$ extends that of operator $Log\ A^{-1} = -Log\ A$, and the map $f \longmapsto \mathcal{L}(f)$ is concave (with respect to the point-wise ordering). With this, it is not hard to deduce that

$$\mathcal{L}(\mathcal{A}^\star(F)) \ \le\ \mathcal{A}(\mathcal{L}(F)) \ \le\ \mathcal{L}(\mathcal{A}(F)),$$

for every family $F = (f_t)_{t\in T}$ with $\mathcal{L}(F) = (\mathcal{L}(f_t))_{t\in T}$.

Example 4.7. Relative Operator and Functional Entropies.
Let E be a (normed) space, $T = (0,1]$, μ the Lebesgue measure and $F = (f_t)_{t\in T}$ with

$$f_t = \frac{((1-t).f^* + t.g^*)^* - f}{t}.$$

In this case, $\mathcal{A}(F)$ is given by

$$\mathcal{A}(F) = \int_0^1 \frac{((1-t).f^* + t.g^*)^* - f}{t} dt,$$

which is exactly the relative functional entropy in convex analysis introduced by the author in [20]. For the quadratic case $A = (A_t)_{t\in T}$ we obtain

$$\mathcal{A}(A) = \int_0^1 \frac{((1-t).A^{-1} + t.B^{-1})^{-1} - A}{t} dt,$$

whose the explicit expression, in the hilbertian case, is given by

$$S(A/B) = A^{1/2} Log\left(A^{-1/2} B A^{-1/2}\right) A^{1/2}.$$

We notice that, as pointed in [19], the functional logarithm presented in the above Example 4.6, is a particular case of the relative functional entropy.

Example 4.8. Function Operator Sign.
Let H be a Hilbert space and $P : H \longrightarrow H$ a linear operator having no pure imaginary eigenvalues. Let us consider

$$T = [0,\pi/2],\ \ A_t = P\left((sin^2 t)I + (cos^2 t)P^2\right)^{-1},\ \ d\mu(t) = (2/\pi)dt.$$

By Definition 3.1, the arithmetic operator mean of the family $A = (A_t)_{t\in T}$ is given by

$$\mathcal{A}(A) = \frac{2}{\pi}P\int_0^{\pi/2}\big((sin^2 t)I + (cos^2 t)P^2\big)^{-1}dt,$$

or equivalently, by the variable change $s = tan\ t$,

$$\mathcal{A}(A) = \frac{2}{\pi}P\int_0^{+\infty}\big(t^2.I + P^2\big)^{-1}dt,$$

which is the known function sign of P known in the literature (see [8] and the reference therein). The explicit formula of $sign(P)$ is given by

$$sgn(P) = P(P^2)^{-1/2}.$$

To determine how to obtain the function sign involving functional variable and extending that of operator is not obvious and appears to be interesting (see [17] for further open problems). The main difficulty of such extension comes from the fact that the above function sign was defined for an operator non necessary positive. We left the reader to say why the next expression

$$\frac{2}{\pi}\int_0^{\pi/2}\big((sin^2 t).f^* + (cos^2 t).f\big)^* dt$$

is not a good extension of the function sign from operator to functional.

5 Chaotic Operator Means

In [11], the authors have introduced a chaotic geometric mean of several positive invertible operators. In general, this operator mean is different from the classical one. Precisely, for the case of two positive operators, the chaotic geometric operator mean,

$$G(A,B) = exp((1/2)Log\ A + (1/2)Log\ B),$$

is generally different from the following one

$$\mathcal{G}(A,B) = A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}$$

differently discussed in the literature. To justify this latter point it is sufficient to say that, for the partial ordering $\leq$:

$$A \leq B \iff A, B \text{ are symmetric and } B - A \text{ is positive},$$

the arithmetic geometric mean inequality,

$$\mathcal{G}(A,B) \leq (1/2)A + (1/2)B,$$

is well known while the following one

$$G(A,B) \leq (1/2)A + (1/2)B,$$

doesn't in general hold. However, for the chaotic order $\preceq$:

$$A \preceq B \iff Log\ A \leq Log\ B,$$

the above inequality remains clearly true, that is

$$G(A,B) \preceq (1/2)A + (1/2)B,$$

since the maps $A \longmapsto Log\ A$ and $A \longmapsto exp\ A$ are mutually reverse and $A \longmapsto Log\ A$ is concave with respect to the $\leq$ order.
In this section we wish to introduce, following our above approach, a chaotic mean of a continuous family of positive operators $A = (A_t)_{t\in T}$ with respect to the probability measure μ. Precisely, we state the following.

Definition 5.1. With the above, the chaotic mean of the operator family $A = (A_t)_{t\in T}$ is defined by

$$\mathcal{C}(A) = exp \int_T Log\ A_t d\mu(t).$$

From this definition the elementary properties of $A \longmapsto \mathcal{C}(A)$ can be stated in a simple manner. In particular, setting $A^{-1} = (A_t^{-1})_{t\in T}$, it is easy to see the following result.

Proposition 5.1. *For all family $A = (A_t)_{t\in T}$ of positive definite operators, the following assertions hold true*
1. The map $A \longmapsto \mathcal{C}(A)$ is self-dual in the sense

$$\left(\mathcal{C}(A)\right)^{-1} = \mathcal{C}(A^{-1}).$$

2. The following chaotic mean inequality holds

$$\mathcal{C}(A) \preceq \mathcal{A}(A).$$

We end this section by stating two examples whose the first justifies that our approach extends that of the discrete chaotic geometric mean introduced in [11] while the second one gives the chaotic identric operator mean.

Example 5.1. Chaotic Operator Geometric Mean.
Let $A = (A_t)_{t\in T}$ be the discrete family with $T = \{1,2,...,m\}$ and the discrete probability measure $\mu(i) = \alpha_i$ for all $i = 1,2,...,m$, with $\alpha = (\alpha_1, \alpha_2, ..., \alpha_m)$ is a probability vector, i.e, $\alpha_i \geq 0$ for every $i = 1,2,...,m$ and $\sum_{i=1}^m \alpha_i = 1$. In this

case, $\mathcal{C}(A)$ defined above is nothing other than the discrete chaotic geometric mean of $A_1, A_2, ..., A_m$ given by

$$G_\alpha(A_1, A_2, ..., A_m) = exp\left(\sum_{i=1}^{m} \alpha_i Log\ A_i\right).$$

In particular, for $m = 2$ and $\alpha = (1/2, 1/2)$ we find the previous mentioned operator $G(A_1, A_2)$.

Example 5.2. Chaotic Operator Logarithmic Mean.
Let A and B be two positive definite operators and set $A_t = (1-t)A + tB$ for $t \in [0, 1]$. Then, the associate chaotic operator mean is given by

$$I(A, B) := exp \int_0^1 Log((1-t)A + tB)dt,$$

which we will call here the chaotic identric (or exponential) operator mean of A and B. For further details concerning this operator mean we can consult [22].

6 Open Problem

After defining a chaotic mean of a continuous family of operators, we end this paper by arising a important general question about chaotic functional mean: What should be the analogue of $\mathcal{C}(A)$ when the operator family variables $A = (A_t)$ are convex functionals? In particular, what is the reasonable analogue of $G_(A_1, A_2, ..., A_m)$ (resp.$I(A, B)$) from the case that the variables are positive operators to the case that the variables are convex functionals? Following Definition 5.1, such analogue need the extensions of the maps $A \longmapsto Log\ A$ and $A \longmapsto exp\ A$ from linear bounded positive operator variables to convex functional variables. Note that, the extension of the first map $A \longmapsto Log\ A$ has been introduced in [13] while that of $A \longmapsto exp\ A$ is not done yet.

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