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Maximal and Minimal Surfaces of Factorable Surfaces in $Heis_3$

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Abstract

In this paper, we discuss notions of Gaussian curvature and mean curvature for factorable surfaces in 3-dimensional Heisenberg group.

MSC: *49Q05, 53A10, 53B30, 53C50.*
Keywords: *Heisenberg group, Lorentz metric, maximal surface, minimal surface.*

1 Introduction

A maximal surface in Lorentz- Minkowski space $\mathbb{L}^3$ is a spacelike surface with zero mean curvature. Maximal surface share many interesting properties with their counterparts, minimal surfaces, in $\mathbb{R}^3$. For example, they are critical points of area variations and also admit Enneper- Weierstrass representations. Today, minimal surfaces appear in various guises in diverse areas of mathematics, physics, chemistry and computer graphics, but have also been used in differential geometry to study basic properties of immersed surfaces in contact manifolds. It is well known that a minimal surface in $\mathbb{R}^3$ can be extended along its boundary if the boundary lies in a plane meeting the minimal surface orthogonally. This fact also holds for maximal surface L^3 where the plane is assumed to be timelike since spacelike and lightlike planes can not meet a maximal surface orthgonally, expect at singular points, [1] .

In [4] it is stuied factorable surfaces in 3- dimensional Minkowski space and it is given that classification of such surfaces for mean curvature and Gaussian curvature. They searched differences between these surfaces.

In this paper, we characterize factorable surfaces in $(Heis_3, g_3)$. We study mean and Gaussian curvature and their properties. We find conditions for factorable surfaces when they are minimal or maximal.

2 Preliminaries

In 3- dimensional Minkowski space, surfaces different according to spacelike direction, timelike direction and lightlike direction, the facorable surface in $Heis_3$ can be considered as the following six types,

■ type 1: along spacelike direction and spacelike direction;

■ type 2: along spacelike direction and timelike direction;

■ type 3: along lightlike direction and lightlike direction;

■ type 4: along lightlike direction and spacelike direction;

■ type 5: along timelike direction and lightlike direction;

■ type 6: along timelike direction and timelike direction.

In [4] they stuied along lightlike direction and lightlike direction (type 3), along lightlike direction and spacelike direction (type 4).

The curvatures of a smooth surface are local measures of its shape. Given a surface$M$ smoothly immersed in $\boldsymbol{E}^3$, we understand its local shape by looking at the Gauss map $G : M \longrightarrow S^2$ given by the unit normal vector $G = \nu_p$ at each point $p \in M$. The derivative of the Gauss map at p is a linear map from T_pM to $T_{Vp}S^2$. Since these spaces are naturally identified, being parallel planes in $\boldsymbol{E}^3$, we can view the derivative as an endomorphism $Sp : T_pM \longrightarrow T_pM$. The map Sp is called the shape operator. The shape operator is the complete second-order invariant (or curvature) which determines the original surface M.

Let M be a 2-manifold and $\Omega : M \longrightarrow \left(\tilde{M}^3, \tilde{g}\right)$ an immersion into a Lorentzian 3-manifold. We denote by g the pull-backed tensor field of $\tilde{g}$ by Ω :

$$g = \tilde{g}\left(d\Omega, d\Omega\right).$$

Then,

i. (M,g) is said to be non-degenerate if g is non-degenerate, i.e., $det(g)\neq 0$ on M.

ii. (M,g) is said to be a spacelike surface if g is a Riemannian metric, i.e., $det(g)>0$.

iii. (M,g) is said to be a timelike surface if $det(g)<0$.

Let M be a spacelike surface or timelike surface in $\tilde{M}$. Then we can take a local unit normal vector field N such that

$$\tilde{g}\left(N,N\right)=\varepsilon,\ \varepsilon=\begin{cases} 1 & M \text{ is spacelike} \\ -1 & M \text{ is timelike} \end{cases}.$$

The constant ε is called the sign of M. A spacelike surface is said to be a *maximal surface* if $H=0$. A timelike surface is said to be a *extremal surface (or minimal surface)* if $H=0$.

Heisenberg group is the matrice group which is given by

$$Heis_3=\left\{\begin{bmatrix} 1 & x_1 & z_1 \\ 0 & 1 & y_1 \\ 0 & 0 & 1 \end{bmatrix},x_1,y_1,z_1\in\mathbb{R}\right\}.$$

In Heisenberg group the group multiplication is

$$(x_1,y_1,z_1)\cdot(x_2,y_2,z_2)=\left(x_1+x_2,y_1+y_2,z_1+z_2+\frac{1}{2}x_1y_2-\frac{1}{2}y_1x_2\right).$$

The Lie algebra of the $Heis_3$ is

$$\boldsymbol{heis}_3=\left\{\begin{bmatrix} 0 & u_1 & u_3 \\ 0 & 0 & u_2 \\ 0 & 0 & 0 \end{bmatrix},u_1,u_2,u_3\in\mathbb{R}\right\}.$$

The orthonormal basis of the $\boldsymbol{heis}_3$ are

$$E_1=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\quad E_2=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},\quad E_3=\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

of the tangent space at the identity.

In [5] and [6] it is shown that three dimensional Heisenberg group has the following left-invariant Lorentz metrics

$$\begin{aligned} g_1 &= -dx^2+dy^2+(xdy+dz)^2, \\ g_2 &= dx^2+dy^2-(xdy+dz)^2, \\ g_3 &= dx^2+(xdy+dz)^2-((1-x)\,dy-dz)^2, \end{aligned}$$

and some geometric properties of the Heisenberg group $Heis_3$ endowed with a Lorentz metric are studied.

Each left invariant Lorentz metric on the Heisenberg group $Heis_3$ is isometric to one of the metrics g_1, g_2, g_3, [6].

The left invariant Lorentz metric g_3 is flat, [5]. A surface M in $Heis_3$ is called factorable surface if M can be written as

$$\begin{aligned} r(u,v) &= (x, y, f(x)g(y)) \text{ or} \\ r(u,v) &= (x, f(x)g(z), z) \text{ or} \\ r(u,v) &= (f(y)g(z), y, z). \end{aligned}$$

3 Factorable Surfaces in $(Heis_3, g_3)$

Theorem 3.1 *Let M be a factorable surface which is parametrized as*

$$r(u,v) = (x, y, z(x,y)) = (x, y, f(x)g(y)).$$

(**i**) If Gaussian curvature K of M is zero, we have the following condition

$$z(x,y) = \lambda\mu.$$

(**ii**) If the mean curvature H of M is zero, it is one of the following surfaces or an open part of them

a) $z(x,y) = (\lambda x + \mu)(\zeta y + \eta)$.

b) $z(x,y) = \lambda(\zeta y + \eta)$.

c) $z(x,y) = f(x)\left(\dfrac{1-2x}{2f(x)}y\right)$.

Proof. The Gaussian curvature and mean curvature, respectively, of the surface M is

$$K = \frac{A}{(f'(x)g(y)(2x + 2f(x)g'(y) - 1))^2},$$

where $A = f''(x)f(x)g''(y)g(y) - f'(x)g'(y)(1 + f'(x)g'(y))$,

$$H = \frac{B}{(f'(x)g(y)(2x + 2f(x)g'(y) - 1))^2},$$

where $B = f''(x)g(y)[2x + 2f(x)g'(y) - 1] + f(x)g''(y)$.

(**i**) Let $p(x) = \dfrac{df}{dx}$, $q(x) = \dfrac{dg}{dy}$. We can see that

$$A = \frac{dp}{df}f(x)\frac{dq}{dg}g(y) - p(x)q(y)(1 + p(x)q(y)).$$

So

$$K=0 \iff \frac{dp}{df}f(x)\frac{dq}{dg}g(y)=p(x)q(y)(1+p(x)q(y)). \qquad (3.1)$$

From solution of the differential equatin (3.1), we can obtain that

$$f(x)=\lambda,\ g(x)=\mu.$$

(ii) Let $p(x)=\dfrac{df}{dx}$, $q(x)=\dfrac{dg}{dy}$. We can see that

$$B=\frac{dp}{df}g(y)\left[2x'2f(x)q(y)-1\right]+f(x)\frac{dq}{dg}. \qquad (3.2)$$

From solution of the differential equatin (3.2), we have following choices;

a) If $\dfrac{dq}{dg}=0$, we have $\dfrac{dp}{df}=0$. So $f(x)=(\lambda x+\mu)$ and $g(y)=(\zeta y+\eta)$.

• If $y>(Cx-B)\dfrac{1}{A}$, where $ac=A$, $A\neq 0$ and $ad-2bc-2x+1=B$, the surface is spacelike. So, $r(u,v)$ is a minimal surface.

• If $y>(Cx-B)\dfrac{1}{A}$, the surface is timelike. From here we can deduce that $r(u,v)$ is a maximal surface.

b) From $p=0$ and $\dfrac{dq}{dg}=0$, we have $f(x)=\lambda$, $g(y)=\zeta y+\eta$.

• If $x>-B$, where $B=f(x)a-\dfrac{1}{4}$, the surface is spacelike. So, $r(u,v)$ is a minimal surface.

• If $x<-B$, where $B=f(x)a-\dfrac{1}{4}$, the surface is timelike. So, $r(u,v)$ is a maximal surface.

c) From $\dfrac{dp}{df}=0$, we have $f(x)g'(y)=\dfrac{1-2x}{2}\Longrightarrow g(y)=\dfrac{1-2x}{2f(x)}y$. So we can deduce that $z(x,y)=f(x)\left(\dfrac{1-2x}{2f(x)}y\right)$.

• If $g(y)\left(1-\dfrac{y}{2}\right)>(x^2-x)e^{-x}$, the surface is spacelike. So, $r(u,v)$ is a minimal surface.

• If $g(y)\left(1-\dfrac{y}{2}\right)<(x^2-x)e^{-x}$, the surface is timelike. So, $r(u,v)$ is a maximal surface.

Theorem 3.2 *Let M be a factorable surface which is parametrized as*

$$r(u,v) = (x(y,z), y, z) = (f(y)g(z), y, z).$$

(**i**) If Gaussian curvature K of M is zero, we have following conditions;If $y > (Cx - B)\frac{1}{A}$, where $ac = A$, $A \neq 0$ and $ad - 2bc - 2x + 1 = B$, the surface is spacelike. So, $r(u,v)$ is a minimal surface.

a) $x(y,z) = \lambda g(z)$.

b) $x(y,z) = \lambda \eta$.

c) $x(y,z) = (\lambda y + \mu)\eta$.

(**ii**) If the mean curvature H of M is zero, it is one of the following surface or an open part of it

$$x(y,z) = \zeta = \frac{1}{2}.$$

Proof. Gaussian curvature and mean curvature, respectively, of the surface M is

$$K = \frac{C}{\left(f'(y)^2 g(z)^2 + 2f(y)g(z) - 1\right)^2 \left(f'(y)f(y)g'(z)g(z) + 1\right)^2},$$

where

$$\begin{aligned} C &= -\left(f''(y)g(z) + f'(y)g(z)f(y)g'(z)\right)\left(f'(y)^2 g(z)^2 + 2f(y)g(z) - 1\right) \\ &\quad - \left(f'(y)g'(z) - f(y)^2 g'(z)^2\right) f'(y)g'(z), \end{aligned}$$

$$H = \frac{D}{\left(f'(y)^2 g(z)^2 + 2f(y)g(z) - 1\right)^2 \left(f'(y)f(y)g'(z)g(z) + 1\right)^2},$$

where

$$\begin{aligned} D &= \left(f'(y)^2 g(z)^2 + 2f(y)g(z) - 1\right)^2 \\ &\quad - \left(\left(f''(y)g(z) + f'(y)g(z)f(y)g'(z)\right)\left(f'(y)f(y)g'(z)g(z) + 1\right)\right). \end{aligned}$$

(**i**) Let $p(y) = \frac{df}{dy}$, $q(z) = \frac{dg}{dz}$. We can see that

$$\begin{aligned} C &= -\left(p(y)\frac{dp}{df}g(z) + p(y)g(z)f(y)q(z)\right)\left(p(y)^2 g(z)^2 + 2f(y)g(z) - 1\right) \\ &\quad - \left(p(y)q(z) - f(y)^2 q(z)^2\right) p(y)q(z). \end{aligned} \tag{3.3}$$

From (3.3), we have the following solutions:

a) If $p(y) = 0$, we have that $f(y,z) = \lambda g(z)$.

b) If $p(y) = 0$ and $q(z) = 0$, we have that $f(y,z) = \lambda\eta$.

c) If $\dfrac{dp}{df} = 0$ and $q(z) = 0$, we have $f(y,z) = (\lambda y + \mu)\gamma$.

(**ii**) Let $p(y) = \dfrac{df}{dy}$, $q(z) = \dfrac{dg}{dz}$. We can see that

$$\begin{aligned} D &= \left(p(y)^2 g(z)^2 + 2f(y)g(z) - 1\right)^2 \\ &\quad - \left(p(y)\frac{dp}{df}g(z) + p(y)g(z)f(y)q(z)\right)\left(p(y)f(y)q(z)g(z) + 1\right). \end{aligned} \tag{3.4}$$

From (3.4), we obtain

$$H = 0 \iff p(y) = 0 \;\text{ and }\; 2f(y)g(z) - 1 = 0.$$

So if we say $f(y) = \lambda$, we have that $g(z) = \dfrac{1}{2\lambda}$. From these computations, we can deduce that

$$f(y)g(z) = \frac{1}{2}.$$

Since $\det g = 0$, we can deduce that the surface is degenerate.

Theorem 3.3 *Let M be a factorable surface which is parametrized as*

$$r(u,v) = (x, f(x)g(z), z).$$

(**i**) If Gaussian curvature K of M is zero, we have following conditions;

a) $y(x,z) = \lambda g(z)$.
b) $y(x,z) = f(x)\eta$.

(**ii**) If the mean curvature H of M is zero, it is one of the following surface or an open part of it

$$y(x,z) = \lambda(\xi z + \eta)$$

Proof. (i) Gaussian curvature and mean curvature, respectively, of the surface M is

$$K = \frac{E}{\left(1 + f'(x)^2 g(z)^2 (2x-1)\right)^2 \left(f(x)g'(z)(2xf(x)g'(z) + 2)\right)^2},$$

where

$$\begin{aligned} E &= f''(x) f(x) g''(z) g(z) (-f(x) g'(z) + x f(x) g'(z) + 1) \\ &\quad - x f'(x) f(x) g'(z) g(z) - f'(x) g'(z)^2 (2x(1-x) f(x) g'(z) + 1) \\ &\quad \left(-f(x)^2 g'(z) + x(1-x) f(x) f'(x) g'(z) + x f'(x)\right). \end{aligned}$$

Let $p(x) = \dfrac{df}{dx}$, $q(z) = \dfrac{dg}{dz}$. We can see that

$$\begin{aligned} E &= \frac{dp}{df} f(x) \frac{dq}{dz} g(z) (-f(x) q(z) + x f(x) q(z) + 1) \\ &\quad - x p(x) f(x) q(z) g(z) - x p(x) f(x) q(z) g(z) \\ &\quad - p(x) q(z)^2 (2x(1-x) f(x) q(z) + 1) \\ &\quad \left(-f(x)^2 q(z) + x(1-x) f(x) p(x) q(z) + x p(x)\right). \end{aligned}$$

a) If $p(x) = 0$, we have that $f(x,z) = \lambda g(z)$.

b) If $q(y) = 0$, we have that $f(x,z) = f(x)\eta$.

(ii)

$$H = \frac{F}{\left(1 + f'(x)^2 g(z)^2 (2x-1)\right)^2 (f(x) g'(z) (2x f(x) g'(z) + 2))^2},$$

where

$$\begin{aligned} F &= \left(f''(x) g(z) + f''(x) g(z)^3 f'(x)^2 (2x-1)\right) [-f(x) g'(z) + x f(x) g'(z) + 1] \\ &\quad - x f'(x) f(x) g'(z) g(z) - x f'(x)^3 g(z)^3 f(x) g'(z) (2x-1) \\ &\quad + f(x) g''(z) f(x) g'(z) (2x f(x) g'(z) + 2). \end{aligned}$$

Let $p(x) = \dfrac{df}{dx}$, $q(z) = \dfrac{dg}{dz}$. We can see that

$$\begin{aligned} F &= \left(\frac{dp}{df} g(z) + \frac{dp}{df} g(z)^3 p(x)^2 (2x-1)\right) [-f(x) q(z) + x f(x) q(z) + 1] - \\ &\quad x p(x) f(x) q(z) g(z) - x p(x)^3 g(z)^3 f(x) q(z) (2x-1) + \\ &\quad f(x) \frac{dq}{dg}(z) f(x) q(z) (2x f(x) q(z) + 2). \end{aligned}$$

If we want to obtain the condition for the mean curvature is zero, we can find that

$$p(x) = 0 \ \text{ and } \ \frac{dq}{dz} = 0.$$

So, we can deduce that

$$y(x,z) = \lambda(\xi z + \eta).$$

If

$$f(x) > \frac{2}{2x\xi - \xi},$$

the factorable surface is a maximal surface.

If

$$f(x) < \frac{2}{2x\xi - \xi},$$

the factorable surface is a minimal surface.

4 Graphic Examples in Matlab

Let consider Theorem 3.1. If we take

$$z(x,y) = (\lambda x + \mu)(\eta y + \zeta),$$

for

$$\lambda = 2, \ \mu = 3, \ \eta = 5, \ \zeta = 4$$

Table 1 and Table 2 are as following.

In Table 1, we obtain Contour values for factorable surfaces by the program Matlab. In Table 2, it is obtained that contour curves and the function .

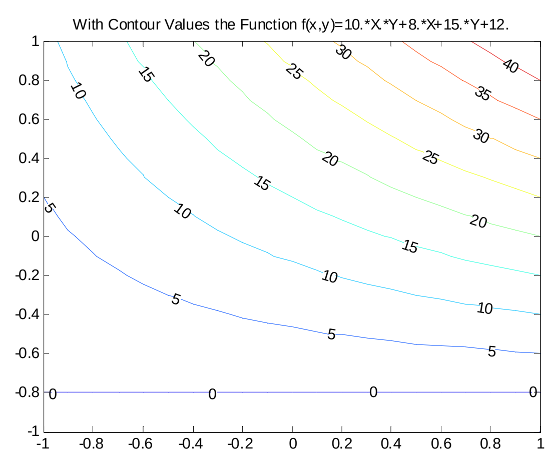

Table 1

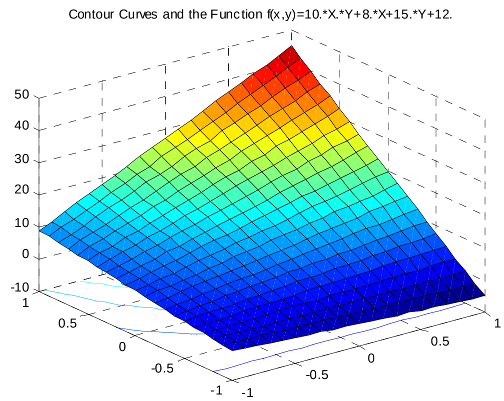

Table 2

For Theorem 3.2, we can take

$$x(y,z) = (\lambda y + \mu)(\eta z + \zeta).$$

For

$$\lambda = 2,\ \mu = 3,\ \eta = 5,\ \zeta = 4,$$

Table 3 and Table 4 are as following:

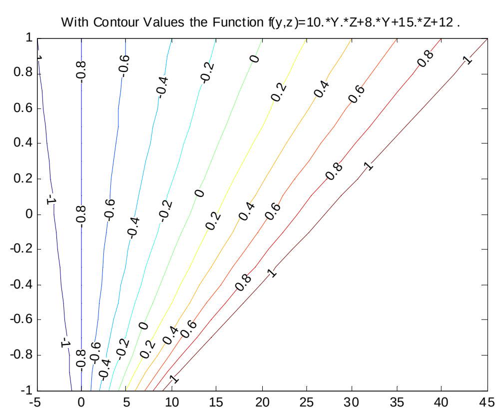

Table 3

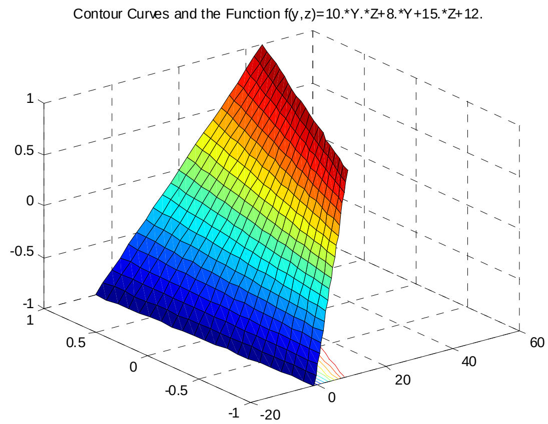

Table 4

For Theorem 3.3, we take

$$y\left(x,z\right)=\left(\lambda x+\mu\right)\left(\eta z+\zeta\right).$$

For

$$\lambda=2,\ \mu=3,\ \eta=5,\ \zeta=4$$

Table 5 and Table 6 are as following:

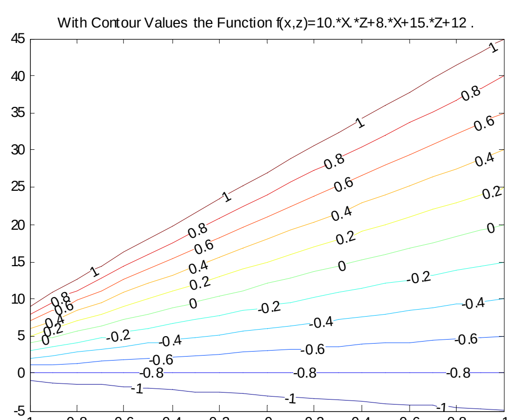

Table 5

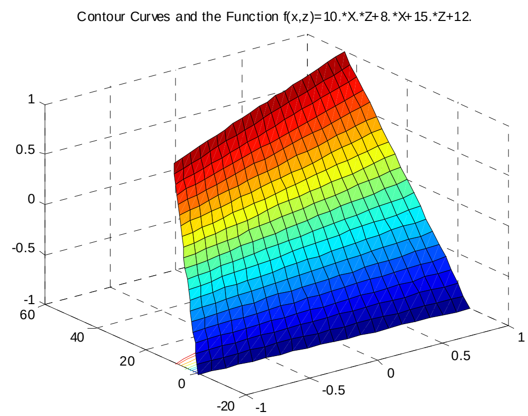

Table 6

5 Open Problem

In this paper, we discuss notions of Gauss curvature and mean curvature for factorable surfaces in 3-dimensional Heisenberg group. Properties of Gauss curvature and mean curvature can be examine in Galilean Spaces.

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