Int. J. Open Problems Compt. Math., Vol. 3, No. 3, September 2010
ISSN 1998-6262; Copyright ©ICSRS Publication, 2010
www.i-csrs.org

Left Multipliers and Lie Ideals in Rings With Involution

Oukhtite Lahcen

Université Moulay Ismaïl, Faculté des Sciences et Techniques
Département de Mathématiques, Groupe d'Algèbre et Applications
B. P. 509 Boutalamine, Errachidia; Maroc
e-mail: oukhtitel@hotmail.com

Abstract

The purpose of this paper is to explore commutativity of rings with involution satisfying certain identities involving left multiplier acting on Lie ideals. Some well known results concerning generalized derivations of prime rings have also been extended to Lie ideals.

2000 Mathematics Subject Classification: *16W10, 16W25, 16U80.*
Keywords: *rings with involution, $*$-prime rings, generalized derivations, left multipliers.*

1 Introduction

In the starting of this paper we should give some of the terminology to be used in the sequel. Throughout the present paper, R will denote an associative ring with center $Z(R)$, not necessarily with an identity element. For any $x, y \in R$, as usual $[x, y] = xy - yx$ and $x \circ y = xy + yx$ will denote the well-known Lie and Jordan products, respectively. We shall make extensive use of basic commutator identities: $[xy, z] = x[y, z] + [x, z]y$, $[x, yz] = y[x, z] + [x, y]z$. Recall that R is said to be 2-torsion free if whenever $2x = 0$, with $x \in R$, then $x = 0$. R is prime if $aRb = 0$ implies $a = 0$ or $b = 0$. If R is equipped with an involution $*$, then R is $*$-prime if $aRb = aRb^* = 0$ yields $a = 0$ or $b = 0$. Note that every prime ring having an involution $*$ is $*$-prime but the converse is in general not true. For example, if R^o denotes the opposite ring of a prime ring R, then $R \times R^o$ equipped with the exchange involution $*_{ex}$,

defined by $*_{ex}(x,y) = (y,x)$, is $*_{ex}$-prime but not prime. This example shows that every prime ring can be injected in a $*$-prime ring and from this point of view $*$-prime rings constitute a more general class of prime rings.
An additive subgroup U of R is said to be a Lie ideal of R if $[u,r] \in U$ for all $u \in U$ and $r \in R$. A Lie ideal U which satisfies $U^* = U$ is called a $*$-Lie ideal. An additive mapping $H : R \to R$ is called a left (resp. right) multiplier if $H(xy) = H(x)y$ (resp. $H(xy) = xH(y)$), holds for all $x, y \in R$. A multiplier is an additive mapping which is both right as well as left multiplier. Considerable work has been done on left (right) multipliers in prime and semiprime rings during the last couple of decades (see [11-13] for a partial bibliography). An additive mapping $d : R \to R$ is said to be a derivation if $d(xy) = d(x)y + xd(y)$ holds for all x, y in R. An additive mapping $F : R \to R$ is called a generalized derivation if there exists a derivation $d : R \to R$ such that $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in R$. Obviously, generalized derivation with $d = 0$ covers the concept of left multipliers.
There has been an ongoing interest concerning the relationship between the commutativity of a prime ring R and the behavior of a generalized derivation of R, with associated *nonzero* derivation. Many of obtained results extend other ones proven previously just for the action of the generalized derivation on the whole ring. In this direction, it seems natural to ask what we can say about the commutativity of R if the generalized derivation is replaced by a left multiplier. In the present paper, we have investigated this problem for certain situations involving left multipliers acting on Lie ideals.
Throughout, $(R, *)$ will be a *non-prime* ring with involution and $Sa_*(R) := \{r \in R / r^* = \pm r\}$ the set of symmetric and skew symmetric elements of R.

2 The conditions $F([x,y]) = \pm [x,y]$

We begin with the following lemmas which are essential for developing the proof of our results.

Lemma 2.1 ([6], Lemma 4) *If $U \not\subseteq Z(R)$ is a $*$-Lie ideal of a 2-torsion free $*$-prime ring R and $a, b \in R$ such that $aUb = a^*Ub = 0$, then $a = 0$ or $b = 0$.*

Lemma 2.2 ([7], Lemma 2.3) *Let $0 \neq U$ be a $*$-Lie ideal of a 2-torsion free $*$-prime ring R. If $[U,U] = 0$, then $U \subseteq Z(R)$.*

The principal result of this section is the following theorem.

Theorem 2.3 *Let U be a nonzero $*$-Lie ideal of a 2-torsion free ring R and let F be a left multiplier such that $F([x,y]) = [x,y]$ for all $x, y \in U$. If R is $*$-prime, then F is trivial or $U \subseteq Z(R)$.*

Proof. Assume that $U \not\subseteq Z(R)$. From $F([x, y]) = [x, y]$ it follows that

$$F([[u, v], w]) = [[u, v], w] \quad \text{for all} \;\; u, v, w \in U. \tag{1}$$

On the other hand, the fact that F is a left multiplier yields

$$F([[u, v], w]) = [u, v]w - F(w)[u, v] \quad \text{for all} \;\; u, v, w \in U. \tag{2}$$

Comparing (1) and (2) we conclude that

$$[[u, v], w] = [u, v]w - F(w)[u, v] \;\; \text{for all} \;\; u, v, w \in U, \tag{3}$$

whence it follows that

$$(F(w) - w)[u, v] = 0 \;\; \text{for all} \;\; u, v, w \in U. \tag{4}$$

Replacing u by $[u, zr]$ in (4), where $z \in U$ and $r \in R$, we obtain

$$(F(w) - w)[z[u, r], v] + (F(w) - w)[[u, z]r, v] = 0. \tag{5}$$

Applying (4), equation (5) becomes

$$(F(w) - w)z[[u, r], v] = 0 \;\; \text{for all} \;\; u, v, w, z \in U, r \in R,$$

and therefore

$$(F(w) - w)U[[u, r], v] = 0 \;\; \text{for all} \;\; u, v, w \in U, r \in R. \tag{6}$$

Since U is a $*$-ideal, then (6) assures that

$$(F(w) - w)U[[u, r], v]^* = 0 \;\; \text{for all} \;\; u, v, w \in U, r \in R. \tag{7}$$

(6) and (7) together with Lemma 2.1 assure that

$$F(w) - w = 0 \;\; \text{for all} \;\; w \in U \quad \text{or} \quad [[u, r], v] = 0 \;\; \text{for all} \;\; u, v \in U, \; r \in R.$$

Assume that

$$[[u, r], v] = 0 \;\; \text{for all} \;\; u, v \in U, \; r \in R. \tag{8}$$

Substituting rv for r in (8) and employing (8) we find that

$$[r, v][u, v] = 0 \;\; \text{for all} \;\; u, v \in U, \;\; r \in R. \tag{9}$$

Replacing r by rs in (9), where $s \in R$, we get $[r, v]s[u, v] = 0$ and thus

$$[r, v]R[u, v] = 0 \;\; \text{for all} \;\; u, v \in U, \;\; r \in R. \tag{10}$$

Let $v \in U \cap Sa_*(R)$; since $U^* = U$, from (10) it follows that

$$[r,v]R[u,v]^* = 0 \text{ for all } u \in U,\ r \in R. \tag{11}$$

Using the $*$-primeness of R, (10) together with (11) give $[r,v] = 0$ for all $r \in R$ or $[u,v] = 0$ for all $u \in U$. Accordingly, either $[U,v] = 0$ or $v \in Z(R)$, in which case $[U,v] = 0$. In conclusion we have

$$[U,v] = 0 \text{ for all } v \in U \cap Sa_*(R). \tag{12}$$

Let $w \in U$; as $w^* - w \in U \cap Sa_*(R)$, then (12) yields $[U, w^* - w] = 0$ so that

$$[y,w^*] = [y,w] \text{ for all } y \in U. \tag{13}$$

Replacing v by w^* in (10) and using (13) we get

$$[r,w^*]R[u,w] = 0 \text{ for all } u \in U,\ r \in R$$

which leads to

$$[r,w]^*R[u,w] = 0 \text{ for all } u \in U,\ r \in R. \tag{14}$$

Since $[r,w]R[u,w] = 0$ by (10), then (14) together with the $*$-primeness of R assure that $[u,w] = 0$ for all $u \in U$ or $[r,w] = 0$ for all $r \in R$. Hence, $[U,w] = 0$ or $w \in Z(R)$, in which case $[U,w] = 0$. Consequently,

$$[U,w] = 0 \text{ for all } w \in U,$$

proving that $[U,U] = 0$ which, by view of Lemma 2.2, contradicts $U \not\subseteq Z(R)$. Hence, necessarily

$$F(w) = w \quad \text{for all } w \in U. \tag{15}$$

Let $s \in R$, for all $x \in U$ the fact that $F([x,s]) = [x,s]$ implies that

$$(F(s) - s)x = 0 \quad \text{for all } x \in U. \tag{16}$$

Replacing s by rs in (16), where $r \in R$, we get $(F(r) - r)sx = 0$ and thus

$$(F(r) - r)Rx = 0 \quad \text{for all } x \in U,\ r \in R. \tag{17}$$

Since $U^* = U$, the relation (17) leads to

$$(F(r) - r)Rx^* = 0 \quad \text{for all } x \in U. \tag{18}$$

Applying the $*$-primeness of R, because of $0 \neq U$, from (17) and (18) it follows that $F(r) = r$ for all $r \in R$, proving that F is trivial. ∎

The following example proves necessity of $*$-primeness condition in Theorem

2.3.

Example 1. Let $S = I\!R[X] \times I\!R[X]$; if we define an addition on S by componentwise and multiplication by $(P_1, P_2)(Q_1, Q_2) = (P_1Q_2 - P_2Q_1, 0)$, then S is a ring such that $s^2 = 0$ for all $s \in S$. Moreover, S is noncommutative and $st = -ts$ for all $s, t \in S$. Let F be the additive mapping defined on the ring $R = \left\{ \begin{pmatrix} x & 0 \\ y & x \end{pmatrix} / x, y \in S \right\}$ by $F\left(\begin{pmatrix} x & 0 \\ y & x \end{pmatrix} \right) = \begin{pmatrix} x & 0 \\ y - x & x \end{pmatrix}$. Clearly, F is a nontrivial left multiplier of R. Since $st = -ts$ for all $s, t \in S$, it is easy to check that the map $* : R \longrightarrow R$ defined by $\begin{pmatrix} x & 0 \\ y & x \end{pmatrix}^* = \begin{pmatrix} -x & 0 \\ -y & -x \end{pmatrix}$ is an involution.
On the other hand, if we set $a = \begin{pmatrix} 0 & 0 \\ s & 0 \end{pmatrix} \in R$, where $s \neq 0$, then $aRa = 0$ and $aRa^* = 0$; proving that R is a non $*$-prime ring. Let $U = \left\{ \begin{pmatrix} 0 & 0 \\ y & 0 \end{pmatrix} / y \in S \right\}$; it is clear that U is a $*$-Lie ideal of R such that $F([u, v]) = [u, v]$ for all $u, v \in U$. Moreover, if $s, t \in S$ are such that $st \neq 0$, then $u = \begin{pmatrix} 0 & 0 \\ s & 0 \end{pmatrix} \in U$ and $r = \begin{pmatrix} t & 0 \\ s & t \end{pmatrix} \in R$ and $[u, r] \neq 0$, proving that $U \not\subseteq Z(R)$. Accordingly, in Theorem 2.3 the hypothesis of $*$-primeness is crucial.

Remark 1. In Theorem 2.3, if we replace the Lie ideal by an ideal of R, then 2-torsion freeness hypothesis can be omitted.

Using the fact that a $*$-prime ring having a nonzero central $*$-ideal must be commutative, Theorem 2.3 together with Remark 1 yield a commutativity criterium as follows:

Corollary 2.4 *Let $(R, *)$ be a ring with involution and let F be a nontrivial left multiplier such that $F([x, y]) = [x, y]$ for all x, y in a nonzero $*$-ideal I of R. If R is $*$-prime, then R is commutative.*

As a first application of Theorem 2.3 we get an improved version of ([7], Theorem 2.1) as follows:

Lemma 2.5 *Let $(R, *)$ be a ring with involution and let F be a generalized derivation of R such that $F([x, y]) = [x, y]$ for all x, y in a nonzero $*$-ideal I of R. If R is $*$-prime, then R is commutative.*

Proof. Assume that F is a generalized derivation associated with a derivation d. If F is trivial, then the condition $F([x, y]) = [x, y]$ for all $x, y \in I$, reduces to $[x, y] = [x, y]$ for all $x, y \in I$, is trivial in which case our theorem is without interest. Accordingly, one can assume that F is nontrivial. Now, when the associated derivation $d = 0$, then F is a nontrivial left multiplier and using Corollary 2.4 we get the required result. On the other hand if $d \neq 0$, then the proof follows from Theorem 2.1 of [8]. Indeed, neither d commutes with $*$ nor 2-torsion freeness is necessary in ([8], Theorem 2.1). ∎

In ([1], Theorem 2.1) it is proved that if a prime ring R admits a nonzero left multiplier H, with $H(x) \neq x$ for all x in a nonzero ideal I of R, such that $H([x, y]) = [x, y]$ for all $x, y \in I$, then R is commutative. However, one can easily see that the condition $H(x) \neq x$ for all $x \in I$ can be replaced by *there exists* $x \in I$ such that $H(x) \neq x$, that is H is nontrivial. Using this fact, the following Theorem extends Theorem 2.1 of [1] to Lie ideals.

Theorem 2.6 *Let R be a 2-torsion free prime ring and let U be a nonzero Lie ideal of R. If R admits a left multiplier F such that $F([x, y]) = [x, y]$ for all $x, y \in U$, then F is trivial or $U \subseteq Z(R)$.*

Proof. Let $\mathcal{F}$ be the additive mapping defined on $= R \times R^0$ by $(x, y) = (F(x), y)$. Clearly, $\mathcal{F}$ is a left multiplier of . Moreover, if we set $W = U \times U$, then W is a $*_{ex}$-Lie ideal of . As $F([x, y]) = [x, y]$ for all $x, y \in U$, it's easy to check that $([u, v]) = [u, v]$ for all $u, v \in W$. Since is a $*_{ex}$-prime ring, in view of Theorem 2.3 we deduce that $W \subseteq Z()$ or $\mathcal{F}$ is trivial. Accordingly, either $U \subseteq Z(R)$ or F is trivial. ∎

Now if F is a left multiplier such that $F \neq -Id$, then $-F$ is a nontrivial left multiplier. This fact together with Theorem 2.6 yield the following result:

Theorem 2.7 *Let R be a 2-torsion free prime ring and let U be a nonzero Lie ideal of R. If R admits a left multiplier F such that $F([x, y]) = -[x, y]$ for all $x, y \in U$, then $U \subseteq Z(R)$ or $F(r) = -r$ for all $r \in R$.*

The following theorem extends Theorem 2.5 of [1] to Lie ideals.

Theorem 2.8 *Let R be a 2-torsion free prime ring and let U be a nonzero Lie ideal of R. If R admits a left multiplier F such that neither F nor $-F$ is trivial, then the following conditions are equivalent:*
(1) *$F([x, y]) = [x, y]$ for all $x, y \in U$;*
(2) *$F([x, y]) = -[x, y]$ for all $x, y \in U$;*
(3) *for all $x, y \in U$, either $F([x, y]) = [x, y]$ or $F([x, y]) = -[x, y]$;*
(4) *$U \subseteq Z(R)$.*

Proof. Clearly, (4) $\Longrightarrow$ (1), (4) $\Longrightarrow$ (2) and (4) $\Longrightarrow$ (3). On the other hand, Theorem 2.6 yields (1) $\Longrightarrow$ (4) and from Theorem 2.7 it follows that (2) $\Longrightarrow$ (4). For (3) $\Longrightarrow$ (4), let us consider $U_1 = \{x \in U \,/\, F([x, y]) = [x, y]$ for all $y \in U\}$ and $U_2 = \{x \in U \,/\, F([x, y]) = -[x, y]$ for all $y \in U\}$. It is clear that U_1 and U_2 are additive subgroups of U such that $U = U_1 \cup U_2$. But a group cannot be a union of two of its proper subgroups and hence $U = U_1$ or $U = U_2$. Thus, we find that either $F([x, y]) = [x, y]$ for all $x, y \in U$ and $U \subseteq Z(R)$ by Theorem 2.6 or $F([x, y]) = -[x, y]$ for all $x, y \in U$ in which case $U \subseteq Z(R)$ by Theorem 2.7. ∎

Using Theorem 2.6, even without the characteristic assumption on the ring, we get an improved version of ([1], Theorem 2.1) as follows:

Corollary 2.9 *Let R be a prime ring and let I be a nonzero ideal of R. If R admits a nontrivial left multiplier F such that $F([x, y]) = [x, y]$ for all $x, y \in I$, then R is commutative.*

We conclude this section with the following theorem which improve the results proved in [1], [3], [4], [10] and [11].

Theorem 2.10 *Let R be a prime ring and let I be a nonzero ideal of R. If R admits a generalized derivation F associated with a derivation d such that $F([x, y]) = [x, y]$ (resp. $F([x, y]) = -[x, y]$) for all $x, y \in I$, then R is commutative.*

Proof. We can assume that neither F nor $-F$ is trivial. Indeed, if F (resp. $-F$) is trivial, then the condition $F([x, y]) = [x, y]$ (resp. $F([x, y]) = -[x, y]$) for all $x, y \in I$ reduces in both the cases to $[x, y] = [x, y]$, in which case our theorem is without interest.
When the associated derivation $d = 0$, then using Theorem 2.8, even without 2-torsion freeness condition, we get $I \subseteq Z(R)$ and therefore R is commutative. If $d \neq 0$, then the proof follows from ([10], Theorem 2.1). ∎

3 The conditions $F(x \circ y) = \pm x \circ y$

In this section, we replace the product $[x, y]$ by $x \circ y$ and obtain results similar to those of the first section. The following theorem proves that the conclusion of Theorems 1 still hold for the Jordan product.

Theorem 3.1 *Let U be a nonzero $*$-Lie ideal of a 2-torsion free ring R and let F be a left multiplier such that $F(x \circ y) = x \circ y$ for all $x, y \in U$. If R is $*$-prime, then F is trivial or $U \subseteq Z(R)$.*

Proof. From $F(x \circ y) = x \circ y$ for all $x, y \in U$ it follows that

$$(F(x) - x)y = (y - F(y))x \quad \text{for all } x, y \in U. \tag{19}$$

Replacing y by $[x, r]$ in (19), where $r \in R$, we get

$$(F(x) - x)xr = (F(r) - r)x^2 \quad \text{for all } x \in U,\ r \in R. \tag{20}$$

Since $(F(x)-x)x=0$, by (19), then (20) becomes

$$(F(r)-r)x^2=0 \quad \text{for all } x\in U,\ r\in R. \tag{21}$$

Replacing r by rs in (21), where $s\in R$, we obtain

$$(F(r)-r)sx^2=0$$

and therefore

$$(F(r)-r)Rx^2=0 \quad \text{for all } x\in U,\ r\in R. \tag{22}$$

Since $U^*=U$, then (22) implies that

$$(F(r)-r)R(x^2)^*=0 \quad \text{for all } x\in U,\ r\in R. \tag{23}$$

Using (22) together with (23), the $*$-primeness of R assures that

$$F(r)=r \quad \text{for all } r\in R \text{ or } x^2=0 \quad \text{for all } x\in U.$$

Hence either F is trivial or $x^2=0$ for all $x\in U$.
If $x^2=0$ for all $x\in U$, then $uv+vu=0$ for all $u,v\in U$ and thus

$$u[v,r]+[v,r]u=0 \quad \text{for all } u,v\in U,\ r\in R. \tag{24}$$

Replacing r by rs in (24), where $s\in R$, and employing (24) we obtain

$$[v,r][s,u]+[u,r][v,s]=0 \quad \text{for all } u,v\in U,\ r,s\in R. \tag{25}$$

Taking $s=u$ in (25) we find that

$$[u,r][v,u]=0 \quad \text{for all } u,v\in U,\ r\in R. \tag{26}$$

Since equation (26) is the same as equation (9), arguing as in the proof of Theorem 2.3, we arrive at $[U,U]=0$ and Lemma 2.2 assures that $U\subseteq Z(R)$. ∎

Example 2. In hypothesis of Theorem 3.1 the $*$-primeness condition is necessary. Indeed, in Example 1 it is clear that $F(x\circ y)=x\circ y$ for all $x,y\in U$, but neither $U\nsubseteq Z(R)$ nor F is trivial.

Remark 2. In Theorem 3.1, if we replace the Lie ideal by an ideal of R, then 2-torsion freeness hypothesis can be omitted.

Using Theorem 3.1 together with Remark 2, we obtain a commutativity criterium as follows:

Corollary 3.2 *Let $(R,*)$ be a ring with involution and let F be a nontrivial left multiplier such that $F(x\circ y)=x\circ y$ for all x,y in a nonzero $*$-ideal I of R. If R is $*$-prime, then R is commutative.*

From Theorem 3.1 we get an improved version of ([7], Theorem 2.2) as follows:

Theorem 3.3 *Let $(R, *)$ be a ring with involution and let F be a generalized derivation of R such that $F(x \circ y) = x \circ y$ for all x, y in a nonzero $*$-ideal I of R. If R is $*$-prime, then R is commutative.*

Proof. Assume that F is a generalized derivation associated with a derivation d. If F is trivial, then the condition $F(x \circ y) = x \circ y$ for all $x, y \in I$ reduces to $x \circ y = x \circ y$ for all $x, y \in I$, in which case our theorem is without interest. Accordingly, one can assume that F is nontrivial. Now, when the associated derivation $d = 0$, then F is a nontrivial left multiplier and using Corollary 3.2 we get the required result. If $d \neq 0$, then the proof follows from Theorem 2.2 of [8]. Indeed, neither d commutes with $*$ nor 2-torsion freeness is necessary in ([8], Theorem 2.2). ∎

In ([1], Theorem 2.3) it is proved that if a prime ring R admits a nonzero left multiplier H, with $H(x) \neq x$ for all x in a nonzero ideal I of R, such that $H(x \circ y) = x \circ y$ for all $x, y \in I$, then R is commutative. However, it's easy to verify that the condition $H(x) \neq x$ for all $x \in I$ can be replaced by *there exists* $x \in I$ such that $H(x) \neq x$, that is H is nontrivial. Using this fact, the following Theorem extends Theorem 2.3 of [1] to Lie ideals.

Theorem 3.4 *Let R be a 2-torsion free prime ring and let U be a nonzero Lie ideal of R. If R admits a left multiplier F such that $F(x \circ y) = x \circ y$ for all $x, y \in U$, then F is trivial or $U \subseteq Z(R)$.*

Proof. Extend F on a left multiplier $\mathcal{F}$ on the $*_{\text{ex}}$-prime ring $= R \times R^0$ by $(x, y) = (F(x), y)$. If we set $W = U \times U$, then W is a nonzero $*_{\text{ex}}$-Lie ideal of . Moreover, it's easy to check that $(x \circ y) = x \circ y$ for all $x, y \in W$. Applying Theorem 3.1, we conclude that $\mathcal{F}$ is trivial or $W \subseteq Z(\mathcal{R})$, whence it follows that either F is trivial or $U \subseteq Z(R)$. ∎

From Theorem 3.4, we obtain the following result which extends Theorem 2.4 of [1] to Lie ideals.

Theorem 3.5 *Let R be a 2-torsion free prime ring and let U be a nonzero Lie ideal of R. If R admits a left multiplier F such that $F(x \circ y) = -x \circ y$ for all $x, y \in U$, then $U \subseteq Z(R)$ or $F(r) = -r$ for all $r \in R$.*

Using Theorem 3.4, even without the characteristic assumption on the ring, we get an improved version of ([1], Theorem 2.3) as follows:

Corollary 3.6 *Let R be a prime ring and let I be a nonzero ideal of R. If R admits a nontrivial left multiplier F such that $F(x \circ y) = x \circ y$ for all $x, y \in I$, then R is commutative.*

Arguing as in the proof of Theorem 2.10, Theorem 3.4 and Theorem 3.5 yield the following result which improve Theorem 2.7 of [1] and Theorem 2.3 & 2.4 of [10].

Theorem 3.7 *Let R be a prime ring and let I be a nonzero ideal of R. If R admits a generalized derivation F associated with a derivation d such that $F(x \circ y) = x \circ y$ (resp. $F(x \circ y) = -x \circ y$) for all $x, y \in I$, then R is commutative.*

References

[1] M. Ashraf and A. Shakir, "On left multipliers and the commutativity of prime rings", *Demonstratio Math.*, Vol. 41, No.4 (2008), pp. 764-771.

[2] M. Ashraf, A. Ali and A. Shakir, "Some commutativity theorems for rings with generalized derivations", *Southeast Asian Bull. Math.* Vol. 31, No.2 (2007), pp. 415-421.

[3] M. Ashraf and N. Rehman, "On commutativity of rings with derivations", *Results Math.* Vol. 42 (2002), pp. 3-8.

[4] M. Hongan, "A note on semiprime rings with derivation", *Int. J. Math. Math. Sci.* Vol. 2 (1997), pp. 413-415.

[5] B. Hvala, "Generalized derivations in rings", *Comm. Algebra*, Vol. 26, No.4 (1998), pp. 1147-1166.

[6] L. Oukhtite, S. Salhi, "Centralizing automorphisms and Jordan left derivations on σ-prime rings", *Advances in Algebra*, Vol. 1, No.1 (2008), pp. 19-26.

[7] L. Oukhtite, S. Salhi, "Lie ideals and derivations of σ-prime rings", *Int. J. Algebra*, Vol. 1, No.1 (2007), pp. 25-30.

[8] L. Oukhtite, S. Salhi and L. Taoufiq, "On generalized derivations and commutativity in σ-prime rings", *Int. J. Algebra*, Vol. 1, No.5 (2007), pp. 227-230.

[9] E. C. Posner, "Derivations in prime rings", *Proc. Amer. Math. Soc.*, Vol. 8 (1957), pp. 1093-1100.

[10] M. A. Quadri, M. S. Khan and N. Rehman, "Generalized derivations and commutativity of prime rings", *Indian J. Pure Appl. Math.*, Vol. 34, No.9 (2003), pp. 1393-1396.

[11] N. Rehman, "On commutativity of rings with generalized derivations", *Math. J. Okayama Univ.*, Vol. 44 (2002), pp. 43-49.

[12] J. Vukman, "Centralizer on semiprime rings", *Comment. Math. Univ. Carolinae*, Vol. 42 (2001), pp. 237-245.

[13] J. Vukman, "An identity related to centralizer in semiprime rings", *Comment. Math. Univ. Carolinae*, Vol. 40 (1999), pp. 447-456.

[14] B. Zalar, "On Centralizer of semiprime rings", *Comment. Math. Univ. Carolinae*, Vol. 32 (1991), pp. 609-614.