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Position Vectors of Spacelike Biharmonic Horizontal Curves with Spacelike Binormal in Lorenzian Heisenberg Group Heis^3

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Abstract

In this paper, we show that every spacelike biharmonic curve in the Lorentzian Heisenberg group $Heis^3$ ***is a helix (both of whose curvature and torsion are constants). We give some characterizations for spacelike biharmonic curve by using the positions vectors of the curve.***

MSC: *31B30, 58E20*
Key words : *biharmonic curve, Heisenberg group, position vector.*

1 Introduction

Harmonic maps $f : (M, g) \longrightarrow (N, h)$ between Riemannian manifolds are the critical points of the energy

$$E(f) = \frac{1}{2}\int_M |df|^2 v_g, \tag{1.1}$$

and they are therefore the solutions of the corresponding Euler–Lagrange equation. This equation is given by the vanishing of the tension field

$$\tau(f) = \text{trace}\nabla df. \tag{1.2}$$

As suggested by Eells and Sampson in [4], we can define the bienergy of a map f by

$$E_2(f) = \frac{1}{2}\int_M |\tau(f)|^2 v_g, \tag{1.3}$$

and say that is biharmonic if it is a critical point of the bienergy.

Jiang derived the first and the second variation formula for the bienergy in [8], showing that the Euler–Lagrange equation associated to E_2 is

$$\begin{aligned}\tau_2(f) &= -\mathcal{J}^f(\tau(f)) = -\Delta\tau(f) - \mathrm{trace}R^N(df, \tau(f))\,df \\ &= 0,\end{aligned} \tag{1.4}$$

where $\mathcal{J}^f$ is the Jacobi operator of f . The equation $\tau_2(f) = 0$ is called the biharmonic equation. Since $\mathcal{J}^f$ is linear, any harmonic map is biharmonic. Therefore, we are interested in proper biharmonic maps, that is non-harmonic biharmonic maps.

In the last decade there have been a growing interest in the theory of biharmonic functions which can be divided into two main research directions. On the one side, the differential geometric aspect has driven attention to the construction of examples and classification results. The other side is the analytic aspect from the point of view of PDE: biharmonic functions are solutions of a fourth order strongly elliptic semilinear PDE.

In [1] the authors completely classified the biharmonic submanifolds of the three-dimensional sphere, while in [2] there were given new methods to construct biharmonic submanifolds of codimension greater than one in the n-dimensional sphere. The biharmonic submanifolds into a space of nonconstant sectional curvature were also investigated. The proper biharmonic curves on Riemannian surfaces were studied in [3]. Inoguchi classified the biharmonic Legendre curves and the Hopf cylinders in three-dimensional Sasakian space forms [6]. Then, Sasahara gave in [17] the explicit representation of the proper biharmonic Legendre surfaces in five-dimensional Sasakian space forms.

The second variation formula for biharmonic maps in spheres was deduced [14] and the stability of certain classes of biharmonic maps in spheres was discussed in [11]. Also, in [18] there were given some sufficient conditions for the instability of Legendre proper biharmonic submanifolds in Sasakian space forms and the author proved the instability of Legendre curves and surfaces in Sasakian space forms.

In this paper, we show that every spacelike biharmonic curve in the Lorentzian Heisenberg group $Heis^3$ is a helix (both of whose curvature and torsion are constants). We give some characterizations for spacelike biharmonic curve by using the positions vectors of the curve.

2 The Lorentzian Heisenberg Group $Heis^3$

The Heisenberg group $Heis^3$ is a Lie group which is diffeomorphic to $\mathbb{R}^3$ and the group operation is defined as

$$(x, y, z) * (\overline{x}, \overline{y}, \overline{z}) = (x + \overline{x}, y + \overline{y}, z + \overline{z} - \overline{x}y + x\overline{y}).$$

The identity of the group is $(0,0,0)$ and the inverse of (x,y,z) is given by $(-x,-y,-z)$. The left-invariant Lorentz metric on $Heis^3$ is

$$g = -dx^2 + dy^2 + (xdy + dz)^2.$$

The following set of left-invariant vector fields forms an orthonormal basis for the corresponding Lie algebra:

$$\left\{ \mathbf{e}_1 = \frac{\partial}{\partial z},\ \mathbf{e}_2 = \frac{\partial}{\partial y} - x\frac{\partial}{\partial z},\ \mathbf{e}_3 = \frac{\partial}{\partial x} \right\}. \tag{2.1}$$

The characterising properties of this algebra are the following commutation relations:

$$[\mathbf{e}_2, \mathbf{e}_3] = 2\mathbf{e}_1, \quad [\mathbf{e}_3, \mathbf{e}_1] = 0, \quad [\mathbf{e}_2, \mathbf{e}_1] = 0,$$

with

$$g(\mathbf{e}_1, \mathbf{e}_1) = g(\mathbf{e}_2, \mathbf{e}_2) = 1, \quad g(\mathbf{e}_3, \mathbf{e}_3) = -1. \tag{2.2}$$

Lemma 2.1 *For the covariant derivatives of the Levi-Civita connection of the left-invariant metric g, defined above the following is true:*

$$\nabla = \begin{pmatrix} 0 & \mathbf{e}_3 & \mathbf{e}_2 \\ \mathbf{e}_3 & 0 & \mathbf{e}_1 \\ \mathbf{e}_2 & -\mathbf{e}_1 & 0 \end{pmatrix}, \tag{2.3}$$

where the (i,j)-element in the table above equals $\nabla_{e_i} e_j$ for our basis

$$\{\mathbf{e}_k, k = 1,2,3\} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}.$$

We adopt the following notation and sign convention for Riemannian curvature operator:

$$R(X,Y)Z = -\nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z + \nabla_{[X,Y]} Z.$$

The Riemannian curvature tensor is given by

$$R(X,Y,Z,W) = g(R(X,Y)Z, W).$$

Moreover we put

$$R_{ijk} = R(\mathbf{e}_i, \mathbf{e}_j)\mathbf{e}_k, \quad R_{ijkl} = R(\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k, \mathbf{e}_l),$$

where the indices i, j, k and l take the values $1, 2$ and 3.

$$R_{121} = -\mathbf{e}_2, \quad R_{131} = -\mathbf{e}_3, \quad R_{232} = \mathbf{3e}_3$$

and

$$R_{1212} = -1, \quad R_{1313} = 1, \quad R_{2323} = -3. \tag{2.4}$$

3 Spacelike Biharmonic Curves In The Lorentzian Heisenberg Group $Heis^3$

Let $\gamma : I \longrightarrow Heis^3$ be a non geodesic spacelike curve on the Lorentzian Heisenberg group $Heis^3$ parametrized by arc length. Let $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ be the Frenet frame fields tangent to the Lorentzian Heisenberg group $Heis^3$ along γ defined as follows:

$\mathbf{t}$ is the unit vector field γ' tangent to γ, $\mathbf{n}$ is the unit vector field in the direction of $\nabla_{\mathbf{t}}\mathbf{t}$ (normal to γ), and $\mathbf{b}$ is chosen so that $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ is a positively oriented orthonormal basis. Then, we have the following Frenet formulas:

$$\begin{aligned} \nabla_{\mathbf{t}}\mathbf{t} &= \kappa_1 \mathbf{n} \\ \nabla_{\mathbf{t}}\mathbf{n} &= \kappa_1 \mathbf{t} + \kappa_2 \mathbf{b} \\ \nabla_{\mathbf{t}}\mathbf{b} &= \kappa_2 \mathbf{n}, \end{aligned} \tag{3.1}$$

where $\kappa_1 = |\tau(\gamma)| = |\nabla_{\mathbf{t}}\mathbf{t}|$ is the curvature of γ and κ_2 is its torsion and

$$\begin{aligned} g(\mathbf{t},\mathbf{t}) &= 1,\ g(\mathbf{n},\mathbf{n}) = -1,\ g(\mathbf{b},\mathbf{b}) = 1, \\ g(\mathbf{t},\mathbf{n}) &= g(\mathbf{t},\mathbf{b}) = g(\mathbf{n},\mathbf{b}) = 0. \end{aligned}$$

With respect to the orthonormal basis $\{e_1, e_2, e_3\}$ we can write

$$\begin{aligned} \mathbf{t} &= t_1\mathbf{e}_1 + t_2\mathbf{e}_2 + t_3\mathbf{e}_3, \\ \mathbf{n} &= n_1\mathbf{e}_1 + n_2\mathbf{e}_2 + n_3\mathbf{e}_3, \\ \mathbf{b} &= \mathbf{t} \times \mathbf{n} = b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3. \end{aligned}$$

Theorem 3.1 *$\gamma : I \longrightarrow Heis^3$ is a spacelike biharmonic curve if and only if*

$$\begin{aligned} \kappa_1 &= \text{constant} \neq 0, \\ \kappa_1^2 + \kappa_2^2 &= 1 - 4b_1^2, \\ \kappa_2' &= 2n_1 b_1. \end{aligned} \tag{3.2}$$

Proof. Using Eq. (2.4) and Eq. (3.1), we have Eq. (3.2).

Theorem 3.2 *Let $\gamma : I \longrightarrow Heis^3$ be a spacelike curve with constant curvature. If $\kappa_2' \neq 0$, then γ is not biharmonic.*

Proof. We can use Eq. (2.3) to compute the covariant derivatives of the vector fields $\mathbf{t}, \mathbf{n}$ and $\mathbf{b}$ as:

$$\begin{aligned} \nabla_{\mathbf{t}}\mathbf{t} &= t_1'\mathbf{e}_1 + (t_2' + 2t_1t_3)\mathbf{e}_2 + (t_3' + 2t_1t_2)\mathbf{e}_3, \\ \nabla_{\mathbf{t}}\mathbf{n} &= (n_1' + t_2n_3 - t_3n_2)\mathbf{e}_1 + (n_2' + t_1n_3 - t_3n_1)\mathbf{e}_2 \\ &\quad +(n_3' + t_2n_1 - t_1n_2)\mathbf{e}_3, \\ \nabla_{\mathbf{t}}\mathbf{b} &= (b_1' + t_2b_3 - t_3b_2)\mathbf{e}_1 + (b_2' + t_1b_3 - t_3b_1)\mathbf{e}_2 \\ &\quad +(b_3' + t_2b_1 - t_1b_2)\mathbf{e}_3. \end{aligned} \tag{3.3}$$

On the other hand, using Frenet formulas Eq. (3.1) and Eq. (3.6), we have

$$\begin{aligned} t_1' &= \kappa_1 n_1, \\ n_1' + t_2n_3 - t_3n_2 &= \kappa_1 t_1 + \kappa_2 b_1, \\ b_1' + t_2b_3 - t_3b_2 &= \kappa_2 n_1. \end{aligned} \tag{3.4}$$

Assume now that $\kappa_2' = 2n_1b_1 \neq 0$. Then using γ is biharmonic and Eq. (3.2), we obtained

$$-2\kappa_2'\kappa_2 = 8b_1b_1',$$

and

$$\kappa_2 n_1 b_1 = -2b_1b_1' \ .$$

Then

$$\kappa_2 = \frac{-2b_1'}{n_1}. \tag{3.5}$$

Using Eq. (3.4) and Eq. (3.5), we get

$$\kappa_2 = -\frac{2}{3} = \text{constant}.$$

Therefore also κ_2 is constant and we have a contradiction that is $\kappa_2' = n_1b_1 \neq 0$.

Corollary 3.3 *$\gamma : I \longrightarrow Heis^3$ is a spacelike biharmonic if and only if*

$$\begin{aligned} \kappa_1 &= \text{constant} \neq 0, \\ \kappa_2 &= \text{constant}, \\ n_1b_1 &= 0, \\ \kappa_1^2 + \kappa_2^2 &= 1 - 4b_1^2. \end{aligned} \tag{3.6}$$

Corollary 3.4 *If $\gamma : I \longrightarrow Heis^3$ is a spacelike biharmonic, then*

$$n_1 = 0.$$

Corollary 3.5 *If* $n_1 = 0$, *then*

$$\mathbf{t}(s) = \cosh \mu_0 \mathbf{e}_1 + \sinh \mu_0 \cosh \Omega(s)\mathbf{e}_2 + \sinh \mu_0 \sinh \Omega(s)\mathbf{e}_3, \tag{3.7}$$

where $\mu_0 \in \mathbb{R}$.

4 Position Vectors of a Spacelike Biharmonic Horizontal Curve in $Heis^3$

Consider a nonintegrable 2-dimensional distribution $(x,y) \longrightarrow \mathcal{H}_{(x,y)}$ in $\mathbb{R}^3 = \mathbb{R}^2_{(x,y)} \times \mathbb{R}_z$ defined as $\mathcal{H} = \ker \omega$, where ω is a 1-form on $\mathbb{R}^3$. The distribution $\mathcal{H}$ is called the horizontal distribution.

A curve $\gamma : I \longrightarrow Heis^3$ is called horizontal curve if $\gamma'(s) \in \mathcal{H}_{\gamma(s)}$, for every s.

Lemma 4.1 *Let* $\gamma : I \longrightarrow Heis^3$ *is a spacelike horizontal curve. Then*

$$z'(s) + x(s)y'(s) = 0. \tag{4.1}$$

Proof. Using the orthonormal left-invariant frame (2.1) we have

$$\begin{aligned} \gamma'(s) &= x'(s)\partial_x + y'(s)\partial_y + z'(s)\partial_z \\ &= x'(s)\mathbf{e}_3 + y'(s)\mathbf{e}_2 + \omega(\gamma'(s))\mathbf{e}_1. \end{aligned}$$

Then, $\gamma(s)$ is a spacelike horizontal curve iff

$$\gamma'(s) = x'(s)e_3 + y'(s)e_2, \quad \omega(\gamma'(s)) = z'(s) + x(s)y'(s). \tag{4.2}$$

We obtain Eq. (4.1) and lemma is proved.

Lemma 4.2 *If* $\gamma : I \longrightarrow Heis^3$ *is a spacelike horizontal curve, then*

$$x'(s)\mathbf{e}_3 + y'(s)\mathbf{e}_2 = x'(s)\frac{\partial}{\partial x} + y'(s)\frac{\partial}{\partial y} - x(s)y'(s)\frac{\partial}{\partial z}. \tag{4.3}$$

Theorem 4.3 *Let $\gamma : I \longrightarrow Heis^3$ is a spacelike horizontal biharmonic curve. Then, the position vector of γ*

$$\begin{aligned}\gamma(s) &= (\frac{1}{\kappa_1}\cosh(\kappa_1 s+\sigma)+c_1, \frac{1}{\kappa_1}\sinh(\kappa_1 s+\sigma)+c_2,\\ &\quad -\frac{s}{2\kappa_1}-\frac{1}{4\kappa_1}\cosh 2(\zeta s+\sigma)+\frac{c_1}{\kappa_1}\sinh(\kappa_1 s+\sigma)+c_3),\end{aligned}$$

where c_1, c_2, c_3 are constants of integration.

Proof. If $\gamma : I \longrightarrow Heis^3$ is a spacelike horizontal biharmonic curve, then we can write its position vector as follows:

$$\gamma(s) = x(s)\partial_x + y(s)\partial_y + z(s)\partial_z. \tag{4.4}$$

Differentiating Eq. (4.1) with respect to s and by using the corresponding orthonormal left-invariant frame (2.1), we find

$$\gamma'(s) = x'(s)\mathbf{e}_3 + y'(s)\mathbf{e}_2 + \omega(\gamma'(s))\mathbf{e}_1.$$

Since $|\nabla_t t| = \kappa_1$, we obtain

$$\Omega(s) = (\frac{\kappa_1 - \sinh 2\mu_0}{\sin \mu_0})s + \sigma, \tag{4.5}$$

where $\sigma \in \mathbb{R}$.

Using Eq. (3.7) and Lemma 4.1, we get

$$\cosh \mu_0 = 0 \text{ and } \sinh \mu_0 = -1. \tag{4.6}$$

Then from Eq. (4.4) and Eq. (4.5) , we get

$$\begin{aligned} x'(s) &= \sinh(\kappa_1 s+\sigma),\\ y'(s) &= \cosh(\kappa_1 s+\sigma),\\ z'(s)+x(s)y'(s) &= \cosh \mu_0 = 0.\end{aligned} \tag{4.7}$$

If the system Eq. (4.7) is integrated, we obtain

$$\begin{aligned} x(s) &= \frac{1}{\kappa_1}\cosh(\kappa_1 s+\sigma)+c_1,\\ y(s) &= \frac{1}{\kappa_1}\sinh(\kappa_1 s+\sigma)+c_2,\\ z(s) &= -\frac{s}{2\kappa_1}-\frac{1}{4\kappa_1}\sinh 2(\zeta s+\sigma)+\frac{c_1}{\kappa_1}\sinh(\kappa_1 s+\sigma)+c_3,\end{aligned} \tag{4.8}$$

where c_1, c_2, c_3 are constants of integration.

The picture of $\gamma(s)$ at $c_1 = c_2 = c_3 = c_4 = \kappa_1 = -\sigma = 1$ as following:

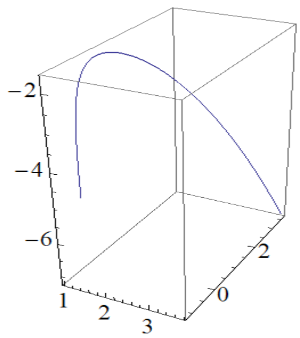

Figure 1:

5 Open Problems

In recent years, by the coming of the theory of relativity, researchers treated some of classical differential geometry topics to extend analogous problems to Lorentzian manifolds. In a similar way, we study a classical topic in Lorentzian Heisenberg group $Heis^3$. In this work, we have investigated spacelike horizontal biharmonic curves in Minkowski space. We have given some explicit characterizations of these curves in terms of Frenet's equations. Additionally, problems such as; investigation of null horizontal biharmonic curves or extending such kind curves to Lorentzian Heisenberg group $Heis^3$.

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