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Operators of Minimal Norm Via Modified Green's Function in Two-dimensional Elastic Waves

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Abstract

The problem of non-uniqueness arising in the integral formulation of an exterior boundary value problem in two-dimensional elastic waves can be faced using the modified Green's function technique. In this work a criterion based on the minimization of the norm of the modified integral operator is established. As an example of the proposed procedure the case of the circle and perturbations of circle are examined.

Keywords: *Multipole coefficients, Green's function, integral equations of Fredholm type, elasticity.*
MSC (2000): *45B05, 34B27, 34B30.*

1 Introduction

As is well known, the problem of non-uniqueness arising in the integral formulation of an exterior boundary value problem has been treated with the addition of a series of outgoing waves to the free-space fundamental solution, that is, with the modified Green's function technique. This method was introduced by Jones [9] and Ursell [19] to treat the exterior Dirichlet and Neumann problem for the Helmholtz equation. The appropriate choice of the multipole coefficients of the added series to the free-space Green's function guarantees the unique solvability of the boundary integral equation which describes the problem. In [12] Kleinmann and Roach have shown that in addition to guaranteeing unique solvability of the integral equation, the multipole coefficients of the modification could be chosen so that the modified Green's function is the best approximation to the actual Green's function for the problem in the least squares sense. In [11] the same authors, motivated by a desire not only to ensure unique solvability but also to provide a constructive method of solving the integral equation, have chosen as a criterion the minimization of the norm of the modified integral operator. In [10] Kleinmann and Kress presented another criterion choosing the coefficients of the modification, that of the minimization of the condition number of the integral equation. All the above mentioned work referred to the acoustical case.

Applying the modified Green's function technique for the elastic case the problem of the irregular frequencies arising in the integral equation of Fredholm type can be removed as well. Although the main ideas in both the acoustic and elastic cases are the same, nevertheless in order to derive the corresponding results for the elasticity much more complicated procedures are required compared to the acoustical case. This is due to the complexity of the problem in elasticity. The first work which adopts the modified Green's function technique in elasticity is due to Jones [8], who examined the cavity in $\Re^3$. In [6] consideration to elastic problems in $\Re^2$ is also given. In [4] The exterior Dirichlet problem in $\Re^3$ is investigated by Argyropoulos, Kiriaki and Roach, and the non-uniqueness of the boundary integral equation is overcome with a suitable choice of mutipole coefficients in the modification. In [16, 17] we have presented another criterion choosing the coefficients of the modification, that of the minimization of the norm of the modified Green's function in the case of two-dimensional elastic waves.

In this work the criterion of operators of minimal norm via modified Green's function for two-dimensional elastic waves is investigated. If the norm of the modified integral operator can be made small enough then the modified integral equation can be solved by iteration. So, if the multipole coefficients of the modification are chosen so as to satisfy this criterion of the minimal norm, then the unique solvability of the integral equation is ensured. More precisely, in section 2 the modified Green's function technique is presented. The free space Green's function and the regular part are expressed via Hankel vector functions . In section 3 a criterion of optimal modification, the criterion of minimization of the norm of the modified integral operator is adopted and via this the optimal multipole coefficients for the modification are chosen. In section 4 the case of the circle is considered as an example of the proposed procedure. In section 5 boundaries which can be derived as perturbation of the circle are investigated. The same results for this work are given by Argyropoulos and Kiriaki [3] in the case of three-dimensional elastic waves.

2 The modified Green's function technique

In order to treat an exterior boundary value problem, we can reformulate it as an integral equation, using the direct or indirect method. An exterior Dirichlet boundary value problem in two-dimensional elastic waves can be described through a boundary integral equation of the form [18, 15, 7] :

$$\left(\frac{1}{2}I+\overline{K_0^*}\right)\varphi(p)=g(p)\quad p\in\partial D \qquad (2.1)$$

where g is a Holder continuous, density and the integral operator K_0 is defined as :

$$(K_0\varphi)(p)=\frac{1}{2\pi}\int_{\partial D}T_pG_0(p,q).\varphi(q).ds_q\quad p\in\partial D \qquad (2.2)$$

where $(*)$ denotes the L_2 adjoint operator and $(-)$, the complex conjugate. G_0 is the fundamental solution and T is the surface stress operator. The superscript (p) on T indicates the action of the operator on the point p.

In order to remove the lack of uniqueness which appears when the boundary value problem is formulated as a boundary integral equation we follow the modified Green's function technique. Introducing a regular solution $H(P,Q)$ [7] the modified Green's function is written as the superposition of the fundamental solution and the regular part as :

$$G_1(P,Q)=G_0(P,Q)+H(P,Q) \qquad (2.3)$$

We replace the kernel of (2.2) with a modified one defined through G_1, so the operator K_0 is modified to K_1. The boundary integral equation which we obtain following a layer theoretic approach is given by :

$$\left(\frac{1}{2}I+\overline{K_1^*}\right)\varphi(p)=g(p)\qquad p\in\partial D \qquad (2.4)$$

We note that operators K_0, K_1 are not compact, because their kernels are singular but the singular integral equation (2.1) and the corresponding modified equation (2.4) admit a regularization procedure as is described in [13].

In what follows we will take the eigenvector expansion for the Green's function introduced in [7]. So, for the free-space fundamental solution we have the following expansion :

$$G_0(P,Q)=\frac{i}{4\mu K^2}\sum_{m=0}^{\infty}\sum_{\sigma=1}^{2}\sum_{\ell=1}^{2}\left(F_m^{\sigma l}(P_>)\otimes F_m^{\sigma l}(P_<)\right) \qquad (2.5)$$

$F_m^{\sigma l}$ are the vector Hankel functions [7], where :

$$P_>=\begin{cases}P, & R_P>R_Q\\ Q, & R_P<R_Q\end{cases},\ \text{and}\ P_<=\begin{cases}P, & R_P<R_Q\\ Q, & R_P>R_Q\end{cases} \qquad (2.6)$$

The $\widehat{F}_m^{\sigma l}$ are obtained by changing the function of Hankel H_m^1 of the vector Hankel functions into the function of Bessel J_m^1 [1]. For the regular part of the modification, apart of the usage of dyads similar to those appeared in (2.5), we introduce cross terms as well, as in [7] :

$$H(P,Q) = \frac{i}{4\mu K^2} \sum_{m=0}^{\infty} \sum_{\sigma=1}^{2} \sum_{\ell=1}^{2} [a_m^{\sigma l} F_m^{\sigma l}(P) \otimes F_m^{\sigma l}(Q) \qquad (2.7)$$

$$+(-1)^{\sigma+l}\ b_m F_m^{\sigma l}(P) \otimes F_m^{(3-\sigma)(3-l)}(Q)]$$

where

$$F_m^{\sigma 1}(P) = grad(H_m^1(k\ R_P).E_m^{\sigma}(\theta_P)) \qquad (2.8)$$

$$F_m^{\sigma 2}(P) = rot\left(H_m^1(K\ R_P).E_m^{\sigma}(\theta_P).\widehat{e}_3\right)$$

and $a_m^{\sigma l}$, b_m are respectively, the simple and cross multipole coefficients. (R_P, θ_P) are the polar coordinates of the point P,

and $E_m^{\sigma}(\theta_P) = \sqrt{\varepsilon_m}.\{ \begin{matrix} \cos(m\theta_p) & \sigma = 1 \\ \sin(m\theta_p) & \sigma = 2 \end{matrix}$, with $\varepsilon_m = \{ \begin{matrix} 1, & m = 0 \\ 2, & m > 0 \end{matrix}$

In what follows, we will assume that the series in (2.7) converges uniformly. This is an assumption which usually is taken under consideration. As it has been proved in [7] the set $\left\{F_m^{\sigma l}\right\}_{m=0:\infty}^{\sigma l=1:2}$ is linearly independent and complete set in $L_2(\partial D)$. The elements of this set are not orthogonal so, in order to proceed, we need to define the following set $\left\{F_m^{\sigma l\ \perp}\right\}_{m=0:\infty}^{\sigma l=1:2}$ whose elements have the property :

$$< F_m^{\sigma l}, F_m^{\nu k\ \perp} >= \delta_{mn}\ \delta_{\sigma\nu} \qquad (2.9)$$

In fact, we can represent every element of the new set as a linear combination of Hankel vectors, so :

$$F_m^{\nu k\ \perp}(P) = \sum_{m=0}^{\infty} \sum_{\sigma=1}^{2} \sum_{\ell=1}^{2} C_{mn}^{\sigma\nu\ lk} F_m^{\sigma l}(P) \qquad (2.10)$$

Taking the inner products of (2.10) with $F_m^{\sigma l}(P)$ in the L_2 sense we conclude to a linear system with unknowns $C_{mn}^{\sigma\nu\ lk}$ having non-vanishing determinant, fact that is established by the linear independence of $\left\{F_m^{\sigma l}\right\}_{m=0:\infty}^{\sigma l=1:2}$. Solving this system we can uniquely calculate the coefficients $C_{mn}^{\sigma\nu\ lk}$ of (2.10). So $F_m^{\nu k\ \perp}$ can be computed through (2.10) explicitly. As it is obvious from their definition, $\left\{F_m^{\sigma l\ \perp}\right\}_{m=0:\infty}^{\sigma l=1:2}$ are linearly independent.

3 A criterion of optimal modification

In the sequel we will consider a different criterion for choosing the multipole coefficients in the modification (2.3), from the criterion presented in [15]. A similar criterion is considered for the acoustical case by Kleinmann and Roach [15]. As in their work it is mentioned this criterion does not only assure the unique solvability of the boundary integral equation but also leads to a constructive method of solving the equation. We will prove that the same holds for the two-dimensional elastic waves. This argument is established by the following theorem.

3.1 Theorem

The norm $\|K_1\|$ of the modified integral operator K_1 is minimized if we choose the multipole coefficients of the modification (2.3) through the relations

$$a_m^{\sigl} \cdot \frac{i}{4\mu K^2} = \frac{\overline{\beta_m^{\sigma l}}\ g_m^{\sigma l} - \alpha_m^{(3-\sigma)(3-l)}\ f_m^{\sigma l}}{\triangle_m^{\sigma l\ \prime}} \qquad (3.1)$$

$$\text{and} \qquad (-1)^{\sigma+l}\ b_m \cdot \frac{i}{4\mu K^2} = \frac{\beta_m^{\sigma l}\ f_m^{\sigma l} - \alpha_m^{\sigma l}\ g_m^{\sigma l}}{\triangle_m^{\sigma l\ \prime}}$$

where

$$\triangle_m^{\sigma l\ \prime} = \alpha_m^{\sigma l}\alpha_m^{(3-\sigma)(3-l)} - \left|\beta_m^{\sigma l}\right|^2 \qquad (3.2)$$

$$\alpha_m^{\sigma l} = \left\|TF_m^{\sigma l}\right\|^2 \qquad (3.3)$$

$$\beta_m^{\sigma l} = < TF_m^{\sigma l}, TF_m^{(3-\sigma)(3-l)} > \qquad (3.4)$$

$$f_m^{\sigma l} = < \overline{K_0^*}T\overline{F_m^{\sigma l}}, F_m^{\sigma l\ \perp} > \qquad (3.5)$$

$$g_m^{\sigma l} = < \overline{K_0^*}T\overline{F_m^{(3-\sigma)(3-l)}}, F_m^{\sigma l\ \perp} > \qquad (3.6)$$

Proof :

The operator norm will be minimized if the multipole coefficients in the modification minimize $\|K_1 w\|^2$ for each function $w \in L_2(\partial D)$. So we will calculate the norm of $\|K_1 w\|^2$, using the expansion for the kernel given by (2.5) and (2.7), we have :

$$\|K_1 w\|^2 = \|K_0 w\|^2 \qquad (3.7)$$

$$+\frac{i}{4\mu K^2}\sum_{m=0}^{\infty}\sum_{\sigma=1}^{2}\sum_{\ell=1}^{2}\Big(\overline{a_m^{\sigma l}} \ < K_0 w, TF_m^{\sigma l} >$$

$$+(-1)^{\sigma+l} \ \overline{b_m} \ < K_0 w, TF_m^{(3-\sigma)(3-l)} >\Big) < \overline{F_m^{\sigma l}}, w >$$

$$+\begin{pmatrix} a_m^{\sigma l} < TF_m^{\sigma l}, K_0 w > \\ +(-1)^{\sigma+l} \ \overline{b_m} \ < TF_m^{(3-\sigma)(3-l)}, K_0 w > \end{pmatrix} < F_m^{\sigma l}, \overline{w} >$$

$$+\left(\frac{i}{4\mu K^2}\right)^2\sum_{m=0}^{\infty}\sum_{\sigma=1}^{2}\sum_{\ell=1}^{2}\sum_{n=0}^{\infty}\sum_{\nu=1}^{2}\sum_{k=1}^{2}\Big(a_m^{\sigma l} \ \overline{a_n^{\nu k}} \ < TF_m^{\sigma l}, TF_n^{\nu k} >$$

$$+(-1)^{\nu+k} \ a_m^{\sigma l} \ \overline{b_n} \ < TF_m^{\sigma l}, TF_n^{(3-\nu)(3-k)} >$$

$$+(-1)^{\sigma+l} \ \overline{a_n^{\nu k}} \ \overline{b_m} \ < TF_m^{(3-\sigma)(3-l)}, TF_n^{\nu k} >$$

$$+(-1)^{\sigma+l+\nu+k} \ b_m \ \overline{b_n} \ < TF_m^{(3-\sigma)(3-l)}, TF_n^{(3-\nu)(3-k)} >\Big) < F_m^{\sigma l}, \overline{w} >< \overline{F_n^{\nu k}}, w >$$

In (3.7) the inner products and norms are in L_2 sense.

Necessary conditions for the minimum of (3.7) are the vanishing of the gradient, with respect to the coefficients. So, first, differentiating with respect to $a_n^{\nu k}$ and b_n we obtain the relations :

$$\begin{pmatrix} \frac{\partial\|K_1 w\|^2}{\partial a_n^{\nu k}} \\ \frac{\partial\|K_1 w\|^2}{\partial b_n} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \forall w \in L_2\left(\partial D\right) \qquad (3.8)$$

from this we conclude that :

$$K_0^* TF_n^{\nu k} - \frac{i}{4\mu K^2}\sum_{m=0}^{\infty}\sum_{\sigma=1}^{2}\sum_{\ell=1}^{2}\Big[\ \overline{a_n^{\nu k}} \ < T\overline{F_m^{\sigma l}}, T\overline{F_n^{\nu k}} > \qquad (3.9)$$

$$+(-1)^{\sigma+l} \ \ \overline{b_m} \ < T\overline{F_m^{(3-\sigma)(3-l)}}, T\overline{F_n^{\nu k}} >\Big] \overline{F_n^{\nu k}} = 0$$

$$K_0^* TF_n^{(3-\nu)(3-k)} - \frac{i}{4\mu K^2} \sum_{m=0}^{\infty} \sum_{\sigma=1}^{2} \sum_{\ell=1}^{2} \Big[\overline{a_m^{\sigma l}} < T\overline{F_m^{\sigma l}}, T\overline{F_n^{(3-\nu)(3-k)}} > \tag{3.10}$$

$$+ (-1)^{\sigma+l} \overline{b_m} < T\overline{F_m^{(3-\sigma)(3-l)}}, T\overline{F_n^{(3-\nu)(3-k)}} > \Big] \overline{F_n^{\nu k}} = 0$$

Taking the inner products of (3.9) and (3.10) with $F_n^{\nu k \perp}$, we conclude to a linear system 2×2, with non-vanishing determinant $\triangle_m^{\sigma l \prime}$, fact that is established by the linear independence of $\left\{F_n^{\nu k}\right\}_{n=0:\infty}^{\nu k=1:2}$ in $L_2(\partial D)$ [14] and the Schwartz inequality. The unique solution of this system gives us $a_m^{\sigma l}$ and b_m as they are expressed via (3.1).

It remains to prove that this choice of multipole coefficients provides a minimum, that is if we denote by K_1^0 the modified operator with the optimal multipole coefficients as specified by (3.1) and by K_1 the modified operator with any other choice of multipole coefficients, we have to verify that :

$$\left\| K_1^0 w \right\| \leq \| K_1 w \| \quad \forall w \in L_2(\partial D) \tag{3.11}$$

Let the multipole coefficients in the arbitrary modification be denoted by :

$$a_m^{\sigma l} = a_m^{\sigma l}(0) + \varepsilon_m^{\sigma l} \quad \textit{and} \quad b_m = b_m(0) + \delta_m \tag{3.12}$$

where $a_m^{\sigma l}(0)$ and $b_m(0)$ are defined by (3.1). Then we can calculate $\|K_1 w\|$ taking into account (3.12) :

$$\|K_1 w\|^2 = \left\| K_1^0 w \right\|^2 + < K_1^0 w, \sum_{m=0}^{\infty} \sum_{\sigma=1}^{2} \sum_{\ell=1}^{2} h_m^{\sigma l} < w, \overline{F_m^{\sigma l}} >> \tag{3.13}$$

$$+ < \sum_{m=0}^{\infty} \sum_{\sigma=1}^{2} \sum_{\ell=1}^{2} h_m^{\sigma l} < w, \overline{F_m^{\sigma l}} >, K_1^0 w >$$

$$\sum_{m=0}^{\infty} \sum_{\sigma=1}^{2} \sum_{\ell=1}^{2} \sum_{n=0}^{\infty} \sum_{\nu=1}^{2} \sum_{k=1}^{2} < h_m^{\sigma l}, h_n^{\nu k} >< w, \overline{F_m^{\sigma l}} >< \overline{w}, F_n^{\nu k} >$$

where

$$h_m^{\sigma l}(p) = \varepsilon_m^{\sigma l} \, TF_m^{\sigma l}(P) + (-1)^{\sigma+l} \delta_m TF_m^{(3-\sigma)(3-l)}(P) \tag{3.14}$$

From (3.9) and (3.10) we have the vanishing of all terms in the first two inner products of the sum in (3.13). Then (3.13) becomes :

$$\|K_1 w\|^2 = \left\|K_1^0 w\right\|^2 + \sum_{m=0}^{\infty}\sum_{\sigma=1}^{2}\sum_{\ell=1}^{2}\sum_{n=0}^{\infty}\sum_{\nu=1}^{2}\sum_{k=1}^{2} Z_m^{\sigma l}\ \overline{Z_n^{\nu k}} < h_m^{\sigma l}, h_n^{\nu k} > \qquad (3.15)$$

where :

$$Z_m^{\sigma l} =< w, \overline{F_m^{\sigma l}} > \qquad (3.16)$$

So to establish our argument we need to prove that the quantity :

$$\sum_{m=0}^{\infty}\sum_{\sigma=1}^{2}\sum_{\ell=1}^{2}\sum_{n=0}^{\infty}\sum_{\nu=1}^{2}\sum_{k=1}^{2} Z_m^{\sigma l}\ \overline{Z_n^{\nu k}} < h_m^{\sigma l}, h_n^{\nu k} > \qquad (3.17)$$

is positive semidefined.

But, by constructing an orthonormal set $\left\{U_m^{\sigma l}\right\}_{m=0:\infty}^{\sigma l=1:2}$ from $\left\{h_m^{\sigma l}\right\}_{m=0:\infty}^{\sigma l=1:2}$, e.g. by using a Gram-Schmidt procedure [14] and the linear independence of $\left\{h_m^{\sigma l}\right\}_{m=0:\infty}^{\sigma l=1:2}$, there exists a set of coefficients $\left\{d_{mp}^{\sigma\mu ls}\right\}$ such that :

$$h_m^{\sigma l} = \sum_{p=0}^{\infty}\sum_{\mu=1}^{2}\sum_{s=1}^{2} d_{mp}^{\sigma\mu ls} U_p^{\mu s} \qquad (3.18)$$

Then :

$$< h_m^{\sigma l}, h_n^{\nu k} >= \sum_{p=0}^{\infty}\sum_{\mu=1}^{2}\sum_{s=1}^{2}\sum_{q=0}^{\infty}\sum_{\lambda=1}^{2}\sum_{r=1}^{2} d_{mp}^{\sigma\mu ls}\ \overline{d_{nq}^{\nu\lambda kr}} < U_p^{\mu s}, U_q^{\lambda r} > \qquad (3.19)$$

$$= \sum_{p=0}^{\infty}\sum_{\mu=1}^{2}\sum_{s=1}^{2} d_{mp}^{\sigma\mu ls}\ \overline{d_{np}^{\nu\mu ks}} = D\ D^*$$

Where D is the matrix with elements $d_{mp}^{\sigma\mu ls}$ and D^* is the Hermitian conjugate. However, $D\ D^*$ is positive semidefined [14], which completes the proof.

4 Optimization for the case of the circle

4.1 Lemma

When ∂D is a circle of radius a then the expansion for $\left\{F_m^{\sigma l\ \perp}\right\}_{m=0:\infty}^{\sigma l=1:2}$ are given by the following equations :

$$F_m^{11\ \perp} = \frac{(a_m^2 F_m^{11} - c_m F_m^{22})}{\triangle_m} \quad , \quad F_m^{22\ \perp} = \frac{(a_m^1 F_m^{22} - \overline{c_m} F_m^{11})}{\triangle_m} \qquad (4.1)$$

$$F_m^{12\perp} = \frac{(a_m^1 F_m^{12} + \overline{c_m} F_m^{21})}{\triangle_m} \quad , \quad F_m^{21\perp} = \frac{(a_m^2 F_m^{21} + c_m F_m^{12})}{\triangle_m} \qquad (4.2)$$

where

$$a_m^1 = 2\pi a k^2 \left[\left| H_m^{'}(ka) \right|^2 + \frac{m^2}{(ka)^2} \left| H_m(ka) \right|^2 \right] \qquad (4.3)$$

$$a_m^2 = 2\pi a K^2 \left[\left| H_m^{'}(Ka) \right|^2 + \frac{m^2}{(Ka)^2} \left| H_m(Ka) \right|^2 \right] \qquad (4.4)$$

$$c_m = 2\pi a k K \left[\frac{m}{Ka} H_m^{'}(ka) \overline{H_m(Ka)} + \frac{m}{ka} H_m(ka) \overline{H_m^{'}(Ka)} \right] \qquad (4.5)$$

$$\triangle_m = a_m^1 a_m^2 - |c_m|^2 \qquad (4.6)$$

Proof :

Taking the inner product of (2.10) with $F_m^{\sigma l}$ and using (2.9), we obtain :

$$\sum_{p=0}^{\infty} \sum_{\mu=1}^{2} \sum_{s=1}^{2} C_{mn}^{\sigma\nu lk} \; I_{mn}^{\sigma\nu lk} = \delta_{mn}\delta_{\sigma\nu}\delta_{lk} \qquad (4.7)$$

with

$$I_{mn}^{\sigma\nu lk} = < F_m^{\sigma l}, F_n^{\nu k} > \qquad (4.8)$$

where the inner product in (4.8) is defined on the circle.

The values of (4.8) for each $\sigma\nu l, k = 1:2$ are given by [18] :

$$I_{mn}^{\sigma\nu 11} = a_m^1 \; \delta_{mn}\delta_{\sigma\nu} \qquad (4.9)$$

$$I_{mn}^{\sigma\nu 22} = a_m^2 \; \delta_{mn}\delta_{\sigma\nu} \qquad (4.10)$$

$$I_{mn}^{\sigma\nu 12} = -(-1)^{\sigma} c_m \; \delta_{mn}(1 - \delta_{\sigma\nu}) \qquad (4.11)$$

$$I_{mn}^{\sigma\nu 21} = -(-1)^{\nu} \; \overline{c_m} \; \delta_{mn}(1 - \delta_{\sigma\nu}) \qquad (4.12)$$

using (4.9)-(4.12), we conclude to a linear systems :

$$\left\{ \begin{array}{l} a_m^1 C_{mn}^{\sigma 1 l 1} + \overline{c_m} C_{mn}^{\sigma 2 l 2} = \delta_{mn}\delta_{\sigma 1}\delta_{l1} \\ c_m C_{mn}^{\sigma 1 l 1} + a_m^2 C_{mn}^{\sigma 2 l 2} = \delta_{mn}\delta_{\sigma 2}\delta_{l2} \end{array} \right. \qquad (4.13)$$

$$\{ \begin{array}{l} a_m^1 C_{mn}^{\sigma 2l1} + \overline{c_m} C_{mn}^{\sigma 1l2} = \delta_{mn}\delta_{\sigma 2}\delta_{l1} \\ -c_m C_{mn}^{\sigma 2l1} + a_m^2 C_{mn}^{\sigma 1l2} = \delta_{mn}\delta_{\sigma 1}\delta_{l2} \end{array} \quad (4.14)$$

with the same non-vanishing determinant given by (4.6), fact that is established by the linear independence of $\left\{F_m^{\sigma l}\right\}_{m=0:\infty}^{\sigma l=1:2}$ in $L_2(\partial D)$ and the Schwartz inequality. The unique solution of the system (4.13) gives us $F_m^{11\ \perp}$ and $F_m^{22\ \perp}$ as they are expressed via (4.1), and the unique solution of the system (4.14) gives us $F_m^{12\ \perp}$ and $F_m^{21\ \perp}$ as they are expressed via (4.2).

4.2 Theorem

When ∂D is a circle of radius a then the optimal multipole coefficients of the modification (2.3), which minimize the norm of the modified integral operator, given by (3.1), take the form :

$$a_m^{11} = a_m^{21} = -\frac{1}{2}\left[\frac{\left(\widehat{\alpha}_m^1 \alpha_m^2 - \overline{\beta_m}\widehat{\beta}_m\right)}{\triangle_m'} + \frac{\left(\widehat{a}_m^1 a_m^2 - \overline{c_m}\widehat{c}_m\right)}{\triangle_m}\right] \quad (4.15)$$

$$a_m^{12} = a_m^{22} = -\frac{1}{2}\left[\frac{\left(\alpha_m^1 \widehat{\alpha}_m^2 - \beta_m\widehat{\psi}_m\right)}{\triangle_m'} + \frac{\left(a_m^1 \widehat{a}_m^2 - c_m\widehat{d}_m\right)}{\triangle_m}\right] \quad (4.16)$$

$$b_m = \frac{1}{2}\left[\frac{\left(\widehat{\alpha}_m^1 \beta_m - \alpha_m^1\widehat{\beta}_m\right)}{\triangle_m'} + \frac{\left(\widehat{a}_m^2 \overline{c_m} - a_m^2\widehat{d}_m\right)}{\triangle_m}\right] \quad (4.17)$$

$$b_m = \frac{1}{2}\left[\frac{\left(\overline{\beta_m} \widehat{\alpha}_m^2 - \alpha_m^2\widehat{\psi}_m\right)}{\triangle_m'} + \frac{\left(c_m \widehat{a}_m^1 - \widehat{c}_m a_m^1\right)}{\triangle_m}\right] \quad (4.18)$$

where

$$\widehat{a}_m^1 = 2\pi a k^2 \left[J_m'(ka)\,\overline{H_m'(ka)} + \frac{m^2}{(ka)^2} J_m(ka)\,\overline{H_m(ka)}\right] \quad (4.19)$$

$$\widehat{a}_m^2 = 2\pi a K^2 \left[J_m'(Ka)\,\overline{H_m'(Ka)} + \frac{m^2}{(Ka)^2} J_m(Ka)\,\overline{H_m(Ka)}\right] \quad (4.20)$$

$$\widehat{c}_m = 2\pi akK\left[\frac{m}{Ka}J_m^{'}(ka)\,\overline{H_m(Ka)} + \frac{m}{ka}J_m(ka)\,\overline{H_m^{'}(Ka)}\right] \qquad (4.21)$$

$$\widehat{d}_m = 2\pi akK\left[\frac{m}{ka}J_m^{'}(Ka)\,\overline{H_m(ka)} + \frac{m}{Ka}J_m(Ka)\,\overline{H_m^{'}(Ka)}\right] \qquad (4.22)$$

$$\widehat{\alpha}_m^1 = 2\pi a\left[\begin{array}{c} k^4\left(2\mu J_m^{''}(ka) - \lambda J_m(ka)\right)\left(2\mu\overline{H_m^{''}(ka)} - \lambda\overline{H_m(ka)}\right) \\ +\left(\frac{2\mu m}{a}\right)^2\left(k\overline{H_m^{'}(ka)} - \frac{\overline{H_m(ka)}}{a}\right)\left(kJ_m^{'}(ka) - \frac{J_m(ka)}{a}\right)\end{array}\right] \qquad (4.23)$$

$$\widehat{\alpha}_m^2 = 2\pi a\left[\begin{array}{c} (\mu K^2)^2\left(2J_m^{''}(Ka) + J_m(Ka)\right)\left(2\overline{H_m^{''}(Ka)} + \overline{H_m(Ka)}\right) \\ +\left(\frac{2\mu m}{a}\right)^2\left(KJ_m^{'}(Ka) - \frac{J_m(Ka)}{a}\right)\left(K\overline{H_m^{'}(Ka)} - \frac{\overline{H_m(Ka)}}{a}\right)\end{array}\right] \qquad (4.24)$$

$$\widehat{\beta}_m = 4\pi\mu m\left[\begin{array}{c} k^2\left(2\mu J_m^{''}(ka) - \lambda J_m(ka)\right)\left(K\overline{H_m^{'}(Ka)} - \frac{\overline{H_m(Ka)}}{a}\right) \\ +\mu K^2\left(kJ_m^{'}(ka) - \frac{J_m(ka)}{a}\right)\left(2\overline{H_m^{''}(Ka)} + \overline{H_m(Ka)}\right)\end{array}\right] \qquad (4.25)$$

$$\widehat{\psi}_m = 4\pi\mu m\left[\begin{array}{c} k^2\left(2\mu\overline{H_m^{''}(ka)} - \lambda\overline{H_m(ka)}\right)\left(KJ_m^{'}(Ka) - \frac{J_m(Ka)}{a}\right) \\ +\mu K^2\left(k\overline{H_m^{'}(ka)} - \frac{\overline{H_m(ka)}}{a}\right)\left(2\overline{J_m^{''}(Ka)} + J_m(Ka)\right)\end{array}\right] \qquad (4.26)$$

$$\triangle_m^{'} = (2\pi\mu)^2\left|\begin{array}{c} \mu k^2K^2\left(2\mu H_m^{''}(ka) - \lambda H_m(ka)\right)\left(2H_m^{''}(Ka) + H_m(Ka)\right) \\ -\left(\frac{2\mu m}{a}\right)^2\left(kH_m^{'}(ka) - \frac{H_m(ka)}{a}\right)\left(KH_m^{'}(Ka) - \frac{H_m(Ka)}{a}\right)\end{array}\right| \qquad (4.27)$$

Proof :

To calculate the multipole coefficients $a_m^{\sigma l}$ and b_m, we must calculate the values of $f_m^{\sigma l}$ and $g_m^{\sigma l}$ given by (3.5) and (3.6). We have [18] :

$$\overline{K_0^*}T\overline{F_m^{\sigma l}}(p) = \int_{\partial D} T\overline{F_m^{\sigma l}}(q)\,T_qG_0(q,p)\,ds_q \qquad (4.28)$$

using the following expansion for the fundamental solution [18] :

$$G_0\left(q,p\right)=\frac{i}{4\mu K^2}\frac{1}{2}\sum_{n=0}^{\infty}\sum_{\nu=1}^{2}\sum_{k=1}^{2}F_n^{\nu k}\left(q\right)\otimes\widehat{F}_n^{\nu k}\left(P\right)+\widehat{F}_n^{\nu k}\left(q\right)\otimes F_m^{\sigma l}\left(p\right) \tag{4.29}$$

we obtain :

$$f_m^{\sigma l}=\frac{i}{8\mu K^2}\left[\begin{array}{c} < T\widehat{F}_m^{\sigma l}, TF_m^{\sigma l} > \\ \sum_{n=0}^{\infty}\sum_{\nu=1}^{2}\sum_{k=1}^{2} < TF_n^{\nu k}, TF_m^{\sigma l} >< \widehat{F}_n^{\nu k}, F_m^{\sigma l\ \perp} > \end{array}\right] \tag{4.30}$$

In the same way, we can obtain :

$$g_m^{\sigma l}=\frac{i}{8\mu K^2}\left[\begin{array}{c} < T\widehat{F}_m^{\sigma l}, TF_m^{(3-\sigma)(3-l)} > \\ \sum_{n=0}^{\infty}\sum_{\nu=1}^{2}\sum_{k=1}^{2} < TF_n^{\nu k}, TF_m^{(3-\sigma)(3-l)} >< \widehat{F}_n^{\nu k}, F_m^{\sigma l\ \perp} > \end{array}\right] \tag{4.31}$$

Using (4.19)-(4.27) and (4.3)-(4.6), we obtain the following relations :

$$f_m^{11}=f_m^{21}=\frac{i}{8\mu K^2}\left[\widehat{\alpha}_m^1+\widehat{\alpha}_m^1\frac{\left(\widehat{a}_m^1a_m^2-\overline{c_m}\widehat{c}_m\right)}{\triangle_m}+\overline{\beta_m}\frac{\left(\widehat{d}_ma_m^2-\overline{c_m}\widehat{a}_m^2\right)}{\triangle_m}\right] \tag{4.32}$$

$$f_m^{12}=f_m^{22}=\frac{i}{8\mu K^2}\left[\widehat{\alpha}_m^2+\widehat{\alpha}_m^2\frac{\left(a_m^1\widehat{a}_m^2-c_m\widehat{d}_m\right)}{\triangle_m}+\beta_m\frac{\left(\widehat{c}_ma_m^1-c_m\widehat{a}_m^1\right)}{\triangle_m}\right] \tag{4.33}$$

$$g_m^{11}=-g_m^{21}=\frac{i}{8\mu K^2}\left[\widehat{\beta}_m+\beta_m\frac{\left(\widehat{a}_m^1a_m^2-\overline{c_m}\widehat{c}_m\right)}{\triangle_m}+\alpha_m^2\frac{\left(\widehat{d}_ma_m^2-\overline{c_m}\widehat{a}_m^2\right)}{\triangle_m}\right] \tag{4.34}$$

$$g_m^{22}=-g_m^{12}=\frac{i}{8\mu K^2}\left[\widehat{\gamma}_m+\overline{\beta_m}\frac{\left(a_m^1\widehat{a}_m^2-c_m\widehat{d}_m\right)}{\triangle_m}+\alpha_m^1\frac{\left(\widehat{c}_ma_m^1-c_m\widehat{a}_m^1\right)}{\triangle_m}\right] \tag{4.35}$$

using (4.32)-(4.35), we obtain the expressions of the multipole coefficients $a_m^{\sigma l}$ and b_m as they are expressed via (4.15)-(4.18).

4.3 Theorem

When ∂D is a circle of radius a, and when *the optimal multipole coefficients of the modification (2.3), is given by (4.15)-(4.18), then we have :*

$$\|K_1\|=0$$

Proof :

In view of lemma 4.1 and theorem 4.2, the modified Green's function admits the following development :

$$G_1(p,q) = \frac{i}{4\mu K^2}\sum_{m=0}^{\infty}\sum_{\sigma=1}^{2}\sum_{\ell=1}^{2} F_m^{\sigma l}(P_<) \otimes \left[\begin{array}{c} \widehat{F}_m^{\sigma l}(P_<) + a_m^{\sigma l} F_m^{\sigma l}(P_<) \\ +(-1)^{\sigma+l} b_m F_m^{(3-\sigma)(3-l)}(P_<) \end{array}\right]$$

$$= \frac{1}{2}\left[G^D(p,q) + G^N(p,q)\right] \qquad (4.37)$$

where G^D is the Green's function for the exterior Dirichlet problem while G^N is the Green's function for the exterior Neumann problem for the circle. So :

$$G^D(p,q) = 0 \quad and \quad T_q G^N(p,q) = 0 \quad for \quad R_p > a, \quad R_q = a \qquad (4.38)$$

After calculations we conclude :

$$T_q G^D(p,q) = -T_q G^N(p,q) \quad for \quad R_p = R_q = a \qquad (4.39)$$

so on the circle holds :

$$T_q G_1(p,q) = 0 \qquad (4.40)$$

but this last result implies that :

$$K_1 w = 0 \qquad \forall w \in L_2(\partial D) \qquad (4.41)$$

Hence $\|K_1 w\| = 0$ and the integral equation is uniquely solvable.

5 Modification for a perturbation of a circle

As in [11] we will consider a family of non-circular boundaries given parametrically by the relation :

$$R_\varepsilon = a + \varepsilon\varphi(\theta_p) \qquad 0 \le \theta_p \le 2\pi \qquad (5.1)$$

where φ and $\frac{\partial\varphi}{\partial\theta}$ are all bounded. Using the estimates for the multipole vectors which are established in [18] :

$$F_m^{\sigma l}(P_\varepsilon) = F_m^{\sigma l}(P_a) + O(\varepsilon) \qquad (5.2)$$

$$TF_m^{\sigma l}(P_\varepsilon) = TF_m^{\sigma l}(P_a) + O(\varepsilon) \qquad (5.3)$$

$$< F_m^{\sigma l}, F_n^{\nu k} >_\varepsilon = < F_m^{\sigma l}, F_n^{\nu k} >_a + O(\varepsilon) \qquad (5.4)$$

$$< TF_m^{\sigma l}, TF_n^{\nu k} >_\varepsilon = < TF_m^{\sigma l}, TF_n^{\nu k} >_a + O(\varepsilon) \qquad (5.5)$$

$$F_m^{\sigma l\ \perp}(P_\varepsilon) = F_m^{\sigma l\ \perp}(P_a) + O(\varepsilon) \qquad (5.6)$$

where P_ε is a point in the perturbed circle while P_a describes points on the circle of radius a, and $< \ , \ >_\varepsilon$ is the inner product on the perturbed circle, and $< \ , \ >_a$ is the inner product on the circle.

5.1 Theorem

When ∂D is defined by (5.1), then the optimal multipole coefficients of the modification (2.3), which minimize the norm of the modified integral operator, given by (3.1), take the form :

$$a_m^{11} = a_m^{11}(a) + O(\varepsilon) \qquad (5.7)$$

$$a_m^{12} = a_m^{12}(a) + O(\varepsilon) \qquad (5.8)$$

$$a_m^{21} = a_m^{21}(a) + O(\varepsilon) \qquad (5.9)$$

$$a_m^{22} = a_m^{22}(a) + O(\varepsilon) \qquad (5.10)$$

$$b_m = b_m(a) + O(\varepsilon) \qquad (5.11)$$

where $a_m^{\sigma l}(a)$ and $b_m(a)$ are the optimal multipole coefficients for the circle of radius a.

Proof :

From (5.5), we conclude :

$$\alpha_m^{\sigma l}(\varepsilon) = \alpha_m^{\sigma l}(a) + O(\varepsilon) \qquad (5.12)$$

$$\beta_m^{\sigma l}(\varepsilon) = \beta_m^{\sigma l}(a) + O(\varepsilon) \qquad (5.13)$$

$$\triangle_m^{\sigma l\ \prime}(\varepsilon) = \triangle_m^{\sigma l\ \prime}(a) + O(\varepsilon) \qquad (5.14)$$

using (5.2), (5.6), (5.12)-(5.14), we obtain :

$$f_m^{\sigma l}(\varepsilon) = f_m^{\sigma l}(a) + O(\varepsilon) \qquad (5.15)$$

$$g_m^{\sigma l}(\varepsilon) = g_m^{\sigma l}(a) + O(\varepsilon) \qquad (5.16)$$

which leads to (5.7)-(5.11).

5.2 Theorem

When ∂D is defined by (5.1), and when *the optimal multipole coefficients of the modification (2.3), is given by (5.7)-(5.11), then we have :*

$$\|K_1\| = O(\varepsilon)$$

Proof :

In view of theorem 5.1, we have :

$$T_{p_\varepsilon} G_0(p_\varepsilon, q_\varepsilon) = T_{p_a} G_0(p_a, q_a) + O(\varepsilon) \qquad (5.18)$$

$$T_{q_\varepsilon} G_0(p_\varepsilon, q_\varepsilon) = T_{q_a} G_0(p_a, q_a) + O(\varepsilon) \qquad (5.19)$$

$$T_{p_\varepsilon} G_1(p_\varepsilon, q_\varepsilon) = T_{p_a} G_1(p_a, q_a) + O(\varepsilon) \qquad (5.20)$$

$$T_{p_\varepsilon} G_1(p_\varepsilon, q_\varepsilon) = T_{p_a} G_1(p_a, q_a) + O(\varepsilon) \qquad (5.21)$$

$$(K_1^\varepsilon w)(p_\varepsilon) = (K_1^a w)(p_a) + O(\varepsilon) \qquad (5.22)$$

using (4.41), (5.22) becomes :

$$(K_1^\varepsilon w)(p_\varepsilon) = O(\varepsilon) \qquad (5.23)$$

which leads to $\|K_1\| = O(\varepsilon)$.

6 Open problems

1- Investigate an other special cases.
2- Investigate an other criterion of optimization choosing the multipole coefficients of the modification, that of the minimization of the condition number of the integral equation (in the case of three dimensions, see [10] for acoustical case and [2] for elastical case).
3- Establish the numerical results for this work (for numerical results see [7] and [13]).

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