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Ideal Congruences on a C-algebra

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Abstract

The notion of an ideal congruence corresponding to a given ideal of a C-algebra A is introduced and proved several results on these congruences. It is proved that the map $I \mapsto \phi_I$ is a homomorphism of $\Im(A)$, the lattice of ideals of a C-algebra A into the lattice Con(A) of all congruences on A. The map is an injection if and only if the C-algebra A is a Boolean algebra.

Keywords: *Boolean algebra, C-algebra, Congruence, Ideal*

1 Introduction

In [2] Guzman and Squier introduced the variety of C-algebras as the variety generated by the three element algebra $C = \{T, F, U\}$, which is the algebraic form of the three valued conditional logic. They proved that C and the two element Boolean algebra $B = \{T, F\}$ are the only subdirectly irreducible C-algebras and that the variety of C-algebras is a minimal cover of the variety of Boolean algebras. In [7] Swamy,U.M. et al., introduced the concept of the Centre $\mathbb{B}(A)$, proved that $\mathbb{B}(A)$ is a Boolean algebra and every factor congruence is of the form θ_a for some $a \in \mathbb{B}(A)$. Later Kalesha Vali, Sundarayya and Swamy,U.M.[3] introduced the notion of an ideal in a C-algebra and proved that the set of all ideals forms an algebraic distributive lattice under the set inclusion ordering. In this paper we give some equivalent conditions for a C-algebra to become a Boolean algebra in terms of fundamental congruences.

Later, the notion of the ideal congruence on a C-algebra A corresponding to an ideal I of A is introduced and some important properties of ϕ_I are derived. This notion is analogous to that in the theory of distributive lattices and it is proved that the map $I \mapsto \phi_I$ is a homomorphism and the map is one-one if and only if the C-algebra A is a Boolean algebra.

2 C-algebra

In this section we recall the definition of a C-algebra and some results from [2, 4, 5, 7]. Let us start with the definition of a C-algebra.

Definition 2.1 ([2]). *By a C-algebra we mean an algebra of type* $(2, 2, 1)$ *with binary operations* $\wedge$ *and* $\vee$ *and unary operation* $'$ *satisfying the following identities.*

(1) $x'' = x$ *(2)* $(x \wedge y)' = x' \vee y'$
(3) $(x \wedge y) \wedge z = x \wedge (y \wedge z)$ *(4)* $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$
(5) $(x \vee y) \wedge z = (x \wedge z) \vee (x' \wedge y \wedge z)$ *(6)* $x \vee (x \wedge y) = x$
(7) $(x \wedge y) \vee (y \wedge x) = (y \wedge x) \vee (x \wedge y)$.

Example 2.2 ([2]). *The three element algebra* $C = \{T, F, U\}$ *with the operations given by the following tables is a C-algebra.*

$\wedge$	T	F	U
T	T	F	U
F	F	F	F
U	U	U	U

$\vee$	T	F	U
T	T	T	T
F	T	F	U
U	U	U	U

x	x'
T	F
F	T
U	U

Note 2.3 ([2]). *The identities 2.1(1), 2.1(2) imply that the variety of C-algebras satisfies all the dual statements of 2.1(2) to 2.1(7). In this vein, if* (m, n) *is a law satisfied by C-algebras, then* $(m, n)'$ *will denote the dual of* (m, n) *obtained by interchanging* $\wedge$ *and* $\vee$*. Furthermore,* $\wedge$ *and* $\vee$ *are not commutative in* C*. The ordinary distributive law of* $\wedge$ *over* $\vee$ *fails in* C*. Every Boolean algebra is a C-algebra.*

Now we give some results on C-algebra collected from [2, 4, 5, 7].

Lemma 2.4. *Every C-algebra satisfies the following identities:*

(1) $x \wedge x = x$ *(2)* $x \wedge x' = x' \wedge x$
(3) $x \wedge y \wedge x = x \wedge y$ *(4)* $x \wedge x' \wedge y = x \wedge x'$
(5) $x \wedge y = (x' \vee y) \wedge x$ *(6)* $x \wedge y = x \wedge (y \vee x')$
(7) $x \wedge y = x \wedge (x' \vee y)$ *(8)* $x \wedge y \wedge x' = x \wedge y \wedge y'$
(9) $(x \vee y) \wedge x = x \vee (y \wedge x)$ *(10)* $x \wedge (x' \vee x) = (x' \vee x) \wedge x = (x \vee x') \wedge x = x$.

The dual statements of the above identities are also valid in a C-algebra.

Definition 2.5. *An element z of a C-algebra A is called a left zero for $\wedge$ if $z \wedge x = z$ for all $x \in A$.*

By Lemma 2.4, $x \wedge x'$ is a left zero for $\wedge$, for all $x \in A$. In fact, any left zero for $\wedge$ must be of the form $x \wedge x'$ for some $x \in A$.

3 Some properties of Congruences

In this section, we introduce a congruence ϕ_x in addition to that the existing congruence $\theta_x = \{(a, b) \in A \times A \mid x \wedge a = x \wedge b\}$ [2] on a C-algebra and discuss various properties of these congruences and establish a relation between θ_x and ϕ_x. We recall that $\phi_x \vee \phi_y = \phi_x \cup \phi_y \cup (\phi_x \circ \phi_y) \cup (\phi_x \circ \phi_y \circ \phi_x) \cup (\phi_x \circ \phi_y \circ \phi_x \circ \phi_y) \cup ...$ where $\phi_x \circ \phi_y = \{(p, q) \in A \times A \mid (p, r) \in \phi_y, (r, q) \in) \phi_x$ for some $r \in A\}$. We shall begin with the following.

Definition 3.1. *For any element x of a C-algebra A, we define*

$$\phi_x = \{(a, b) \in A \times A \mid x \vee a = x \vee b\}.$$

Lemma 3.2. *ϕ_x is a congruence on a C-algebra A, for any $x \in A$.*

Proof. Straight forward verification. □

Lemma 3.3. *The following hold for any elements x and y of a C-algebra.*
(1) $\phi_{x \vee y} = \phi_x \vee \phi_y = \phi_y \circ \phi_x \circ \phi_y = \phi_x \circ \phi_y \circ \phi_x$
(2) $\phi_x \cap \phi_y \subseteq \phi_{x \wedge y} \subseteq \phi_x$
(3) $\phi_{x \vee y} = \phi_{y \vee x}$.

In general, $\phi_{x \wedge y} \nsubseteq \phi_x \cap \phi_y$ for any elements x and y in the C-algebra $C = \{T, F, U\}$ in which $\phi_U = C \times C$, $\phi_F = \triangle_C$ and $\phi_{U \wedge F} = \phi_U = C \times C$. However, we have $\phi_{x \wedge y} \subseteq \phi_x$ in general. But $\phi_{x \wedge y} \nsubseteq \phi_y$ is not always true. In fact, we have the following.

Lemma 3.4. *The following are equivalent for any C-algebra $(A, \wedge, \vee, ')$.*
(1) $\phi_{x \wedge y} \subseteq \phi_y$ for all $x, y \in A$
(2) $(A, \wedge, \vee, ')$ is a Boolean algebra
(3) $\phi_{x \wedge y} = \phi_{y \wedge x}$ for all $x, y \in A$
(4) $\phi_{x \wedge y} = \phi_x \cap \phi_y$ for all $x, y \in A$.

Proof. (1) ⇒ (2): suppose that $\phi_{x \wedge y} \subseteq \phi_y$. By dual of (1) in Lemma 2.4 and by Definition 3.1 we have $((x \wedge y) \vee y, y) \in \phi_{x \wedge y}$. By hypothesis $((x \wedge y) \vee y, y) \in \phi_{x \wedge y} \subseteq \phi_y$. Thus $y \vee (x \wedge y) \vee y = y \vee y = y$, for any $x, y \in A$. Therefore $y \vee (x \wedge y) = y$ for all $x, y \in A$. It is well known from [7] that a C-algebra A is a Boolean algebra if and only if $a \vee (b \wedge a) = a$ for all $a, b \in A$. Therefore $(A, \wedge, \vee, ')$ is a Boolean algebra.
(2) ⇒ (3) is trivial, since $\wedge$ is commutative in a Boolean algebra.
(3) ⇒ (4): For any $x, y \in A$, $\phi_{x \wedge y} = \phi_{y \wedge x} \subseteq \phi_y$ and hence,
$\phi_{x \wedge y} \subseteq \phi_x \cap \phi_y \subseteq \phi_{x \wedge y}$ (by Lemma 3.3). (4) ⇒ (1) is trivial. □

In general, $\phi_{x\wedge y}$ and $\phi_{y\wedge x}$ may not coincide; for, consider the three element C-algebra $C = \{T, F, U\}$ in which $\phi_U = C \times C$ and $\phi_F = \triangle_C$ are congruences. Here $\phi_{F\wedge U} = \phi_F \neq \phi_U = \phi_{U\wedge F}$.

Next we recollect the fundamental congruence corresponding to an element in a C-algebra, defined in [2].

Definition 3.5 ([2]). *For any element x of a C-algebra A, $\theta_x = \{(a, b) \in A \times A \mid x \wedge a = x \wedge b\}$ is a congruence on A.*

Lemma 3.6. *For any $x \in A$, $\theta_x = \phi_{x'} = \{(a, b) \in A \times A \mid x' \wedge a = x' \wedge b\}$.*

Proof. Follows from (7) of Lemma 2.4 and its dual $(7)'$. □

4 Ideal Congruences

We introduce the notion of the ideal congruence on a C-algebra A corresponding to an ideal I of A. This notion is analogous to that in the theory of distributive lattices. Because of the commutativity of the operations in the lattice case, we get stronger properties of ideal congruences. However, many of these properties can be extended to the case of C-algebras also. We begin with the following which are collected from [1, 3].

A nonempty subset I of a C-algebra A is said to be an ideal of A if it satisfies (i) $a, b \in I$ implies that $a \vee b \in I$ and (ii) $a \in I$ implies that $x \wedge a \in I$, for each $x \in A$. The set $\{x \wedge a \mid x \in A\}$ is the principal ideal generated by an element a in A, and is denoted by $< a >$. It is observed that $y \in < a >$ if and only if $y = y \wedge a$ and if I is an ideal of a C-algebra A then $x \wedge x' \in I$ for all $x \in A$. $I_0 = \{x \wedge x' \mid x \in A\}$ is the smallest ideal of A. If I and J be any two ideals of a C-algebra A then $I \vee J = < I \cup J > = \{\bigvee_{i=1}^{n} a_i \mid a_i \in I \cup J\}$ is the smallest ideal containing I and J. Finally it is proved that the set of ideals of A, $\Im(A)$ is an algebraic distributive lattice with respect to the inclusion ordering $\subseteq$ in which, for any set $\{I_\alpha\}_{\alpha\in\Delta}$ of ideals, Inf $\{I_\alpha\}_{\alpha\in\Delta} = \bigcap_{\alpha\in\Delta} I_\alpha$ and Sup $\{I_\alpha\}_{\alpha\in\Delta} = < \bigcup_{\alpha\in\Delta} I_\alpha >$.

Now first we start with the definition of ideal congruence.

Definition 4.1. *For any ideal I of a C-algebra A, we define*
$\phi_I = \{(a, b) \mid x \vee a = x \vee b$ *for some* $x \in I\}$.

Theorem 4.2. *ϕ_I is a congruence on a C-algebra A for any ideal I of A.*

Proof. It is known that if a congruence class is directed above then the union of any congruence sub class is again a congruence. Now, consider $\mathcal{C} = \{\phi_x \mid x \in I\}$. Since each ϕ_x is a congruence on A (by Lemma 3.2), $\mathcal{C}$ is a class of congruences on A. Also, for any $x, y \in I$, we have $x \vee y \in I$ and, by Lemma 3.3, $\phi_x, \phi_y \subseteq \phi_{x\vee y} \in \mathcal{C}$. Therefore $\mathcal{C}$ is a directed above class of congruences and hence $\bigcup\limits_{x\in I} \phi_x (= \phi_I)$ is a congruence on A. □

Remark 4.3. *If $< x >$ is the principal ideal generated by an element x in a C-algebra A, then clearly $\phi_x \subseteq \phi_{<x>}$. However, equality does not hold as in the case of distributive lattices. For, consider the three-element C-algebra $C = \{T, F, U\}$, $< F >= \{F, U\}$ and $\phi_F = \triangle_C$, $\phi_{<F>} = C \times C$. Here $\phi_{<F>} \nsubseteq \phi_F$.*

Theorem 4.4. *Let I be an ideal of a C-algebra A. Then ϕ_I is the smallest congruence on A containing $I \times I$.*

Proof. We have already proved in Theorem 4.2 that ϕ_I is a congruence on A. Also, for any $a, b \in I$, we have $a \vee b \in I$ and $(a \vee b) \vee a = a \vee b = (a \vee b) \vee b$ and hence $(a, b) \in \phi_I$. Therefore $I \times I \subseteq \phi_I$.

Now, let ϕ be any congruence on A such that $I \times I \subseteq \phi$. Then

$$\begin{aligned}
(a,b) \in \phi_I &\Rightarrow x \vee a = x \vee b \text{ for some } x \in I\\
&\Rightarrow (a \wedge a', x) \in \phi \qquad (\text{since } a \wedge a' \in I, x \in I)\\
&\Rightarrow ((a \wedge a') \vee a, x \vee a) \in \phi \ (\text{ since } \phi \text{ is a congruence })\\
&\Rightarrow (a, x \vee a) \in \phi \text{ and, for similar reasons, } (b, x \vee b) \in \phi\\
&\Rightarrow (a,b) \in \phi \qquad (\text{since, } \phi \text{ is transitive and } x \vee a = x \vee b).
\end{aligned}$$

Therefore $\phi_I \subseteq \phi$. Thus ϕ_I is the smallest congruence on A containing $I \times I$. □

Lemma 4.5. *For any ideals I and J of a C-algebra A, the following hold.*
(1) $I \subseteq J \Rightarrow \phi_I \subseteq \phi_J$
(2) $\phi_I \cap \phi_J = \phi_{I\cap J}$
(3) $\phi_I \vee \phi_J = \phi_{I\vee J}$.

Proof. Let I and J be ideals of A.(1) is clear.

(2) Since $I \cap J \subseteq I$ and $I \cap J \subseteq J$, we get that $\phi_{I\cap J} \subseteq \phi_I \cap \phi_J$.

$$\begin{aligned}
\text{Also, } (a,b) \in \phi_I \cap \phi_J &\Rightarrow x \vee a = x \vee b \text{ and } y \vee a = y \vee b \text{ for some } x \in I \text{ and } y \in J.\\
&\Rightarrow x \wedge y \in I \cap J \text{ and } (x \wedge y) \vee a = (x \vee a) \wedge (x' \vee y \vee a)\\
&\qquad = (x \vee b) \wedge (x' \vee y \vee b) = (x \wedge y) \vee b\\
&\Rightarrow (a,b) \in \phi_{I\cap J}
\end{aligned}$$

Thus $\phi_I \cap \phi_J = \phi_{I\cap J}$.

(3) Since $I, J \subseteq I \vee J$, we have $\phi_I \subseteq \phi_{I\vee J}$ and $\phi_J \subseteq \phi_{I\vee J}$ and hence $\phi_I \vee \phi_J \subseteq \phi_{I\vee J}$. On the other hand,

$$\begin{aligned}
(a,b) \in \phi_{I\vee J} &\Rightarrow (a,b) \in \phi_z \text{ for some } z \in I \vee J\\
&\Rightarrow z = \bigvee_{i=1}^{n} x_i, \text{ for some } x_i \in I \cup J \text{ and } (a,b) \in \phi_{\bigvee\limits_{i=1}^{n} x_i} = \bigvee_{i=1}^{n} \phi_{x_i}\\
&\qquad (\text{ by Lemma 3.3})\\
&\Rightarrow (a,b) \in \bigvee_{i=1}^{n} \phi_{x_i} \subseteq \phi_I \vee \phi_J
\end{aligned}$$

(since each $x_i \in I$ or J. If $x_i \in I$ then $\phi_{x_i} \subseteq \phi_I$ similarly $x_i \in J$ then $\phi_{x_i} \subseteq \phi_J$)

Thus $\phi_I \vee \phi_J = \phi_{I \vee J}$. □

Let us recall that the set Con(A) of all congruences on any algebra A is an algebraic lattice under the inclusion ordering in which the g.l.b and l.u.b of any subset $\mathfrak{C}$ of Con(A) are given by g.l.b $\mathfrak{C} = \bigcap_{\theta \in \mathfrak{C}} \theta$ and l.u.b $\mathfrak{C} = \bigcup \{\theta_1 \circ \theta_2 \circ ... \circ \theta_n \mid \theta_i \in \mathfrak{C}\}$. Also, from [6], it is known that the set $\Im(A)$ of all ideals of a C-algebra A forms an algebraic lattice under the inclusion ordering. Now we have the following.

Theorem 4.6. *Let $\Im(A)$ be the lattice of all ideals of a C-algebra A. Then $I \mapsto \phi_I$ is a homomorphism of the lattice $\Im(A)$ into the lattice Con(A) of all congruences on A.*

Proof. From Lemma 4.5, it follows that $I \mapsto \phi_I$ is a lattice homomorphism of $\Im(A)$ into the lattice Con(A). □

The above map $I \mapsto \phi_I$ need not be an injection, in general. However, we have the following.

Theorem 4.7. *For any C-algebra A, the map $I \mapsto \phi_I$ of $\Im(A)$ into Con(A) is an injection if and only if A is a Boolean algebra.*

Proof. Suppose that $I \mapsto \phi_I$ is an injection. Then, for any $x, y \in A$, we shall prove that $\phi_{<x \vee y>} = \phi_{<y \vee x>}$. Let $a, b \in A$. Then

$$
\begin{array}{rcl}
(a,b) \in \phi_{<x\vee y>} & \Rightarrow & z \vee a = z \vee b \text{ for some } z \in < x \vee y > \\
& \Rightarrow & (s \wedge (x \vee y)) \vee a = (s \wedge (x \vee y)) \vee b \text{ for some } s \in A \\
& \Rightarrow & (s \wedge x) \vee (s \wedge y) \vee a = (s \wedge x) \vee (s \wedge y) \vee b, \quad s \in A \\
& \Rightarrow & (s \wedge y) \vee (s \wedge x) \vee (s \wedge y) \vee a = (s \wedge y) \vee (s \wedge x) \vee \\
& & \qquad (s \wedge y) \vee b, \ s \in A \\
& \Rightarrow & (s \wedge y) \vee (s \wedge x) \vee a = (s \wedge y) \vee (s \wedge x) \vee b, \quad s \in A \\
& \Rightarrow & (s \wedge (y \vee x)) \vee a = (s \wedge (y \vee x)) \vee b, \quad s \in A \\
& \Rightarrow & (a,b) \in \phi_{s \wedge (y \vee x)} \subseteq \phi_{<y \vee x>}.
\end{array}
$$

Therefore $\phi_{<x\vee y>} \subseteq \phi_{<y\vee x>}$. By symmetry, we have $\phi_{<x\vee y>} = \phi_{<y\vee x>}$.
Since $I \mapsto \phi_I$ is an injection, we get that $< x \vee y >=< y \vee x >$ and hence

$$
\begin{array}{rcll}
x \vee y & = & (x \vee y) \wedge (y \vee x) & (\text{since } x \vee y \in < y \vee x >) \\
& = & (y \vee x) \wedge (x \vee y) & (\text{by } (7)' \text{ definition of 2.1}) \\
& = & y \vee x & (\text{since } y \vee x \in < x \vee y >)
\end{array}
$$

Therefore $x \vee y = y \vee x$ for all $x, y \in A$. In[7] it is proved that a C-algebra A is a Boolean algebra if and only if $a \vee b = b \vee a$ for all $a, b \in A$ thus A is a Boolean algebra. Conversely suppose that A is a Boolean algebra and I, J are ideals of A such that $\phi_I = \phi_J$. Then for any $a \in I$ and $b \in J$, $(a \vee b, b) \in \phi_a \subseteq \phi_I = \phi_J$ and hence $x \vee a \vee b = x \vee b$ for some $x \in J$ which implies that $a = a \wedge (x \vee a \vee b) = a \wedge (x \vee b) \in J$ (since, $x \vee b \in J$). Therefore $I \subseteq J$ and, similarly $J \subseteq I$ and hence $I = J$. Thus $I \mapsto \phi_I$ is an injection. □

Even though the set I_0 of left zeros for $\wedge$ forms the smallest ideal in a C-algebra A, ϕ_{I_0} may not be the smallest congruence (the diagonal) on A. For consider the three element C-algebra $C = \{T, F, U\}$, in which $I_0 = \{F, U\}$ and $\phi_{I_0} = C \times C \neq \triangle_C$.

However, if ϕ_{I_0} is the smallest congruence then the C-algebra becomes a Boolean algebra. This is proved in the following.

Theorem 4.8. *Let I_0 be the set of all left zeros for $\wedge$ in a C-algebra A. Then A is a Boolean algebra if and only if $\phi_{I_0} = \triangle_A$, the diagonal on A.*

Proof. If A is a Boolean algebra, then the smallest element 0 is the only (left) zero for $\wedge$, so that $I_0 = \{0\}$ and clearly, for any $a, b \in A$,
$(a, b) \in \phi_{I_0}$ if and only if $0 \vee a = 0 \vee b$ if and only if $a = b$ and hence $\phi_{I_0} = \triangle_A$, the diagonal on A. Conversely, if $\phi_{I_0} = \triangle_A$, then, for any $a \in A$ and $z \in I_0$, we have $(a, z \vee a) \in \phi_{I_0} = \triangle_A$ and $(a, a \vee z) \in \phi_{I_0} = \triangle_A$ and hence $a = z \vee a$ and $a = a \vee z$. Therefore z is the identity for $\vee$. This implies that every element of I_0 is the identity for $\vee$ and therefore I_0 has only one element and hence $x \wedge x' = y \wedge y'$ for all $x, y \in A$. It is well known from [7] that a C-algebra A is a Boolean algebra if and only if $x \wedge x' = y \wedge y'$ for all $x, y \in A$. Therefore A is a Boolean algebra. It is well known that every congruence on a Boolean algebra is of the form θ_I for some ideal I. This is not true in the case of C-algebras,unless it is a Boolean algebra. □

Theorem 4.9. *The following are equivalent for any C-algebra A*
(1) $\phi_I = \triangle_A$ for some ideal I of A
(2) I_0 is a singleton set; that is, there is only one left zero for $\wedge$
(3) $\phi_{I_0} = \triangle_A$
(4) A is a Boolean algebra
(5) Every congruence of A is of the form ϕ_I for some ideal I of A.

Proof. (1) $\Rightarrow$ (2): Suppose that I is an ideal of A such that $\phi_I = \triangle_A$.
Then, for any $a \in A$ and $x \in I$, we have $(a, x \vee a)$ and $(a, a \vee x) \in \phi_I = \triangle_A$ and hence $x \vee a = a = a \vee x$, so that x is the identity for $\vee$. Therefore every element of I is identity for $\vee$. Thus I is a singleton. Also, $x = x \wedge (x \vee a) = x \wedge a$, for any $x \in I$ and $a \in A$. Therefore x is a left zero for $\wedge$. Hence, we have $I \subseteq I_0$ and therefore $I_0 = I = \{z\}$. (2) $\Rightarrow$ (3) follows from the fact that ϕ_{I_0} is the smallest congruence containing $I_0 \times I_0$. (3) $\Rightarrow$ (4) is proved in the above Theorem.
(4) $\Rightarrow$ (5): Let A be a Boolean algebra and ϕ be a congruence on A.
Let $I = \{a \in A \mid (a, 0) \in \phi\}$. Then, for any $a, b \in I$, $(a, 0)$, $(b, 0) \in \phi$ and hence $(a \vee b, 0) \in \phi$ and therefore $a \vee b \in A$. Also, for any $a \in I$ and $x \in A$, $(x, x) \in \phi$ and $(a, 0) \in \phi$ and hence $(x \wedge a, x \wedge 0) \in \phi$. Thus $x \wedge a \in I$. Therefore I is an ideal of A.
We shall now prove that $\phi = \phi_I$. For any $a, b \in A$,

$$
\begin{aligned}
(a,b)\in\phi &\Rightarrow (a\wedge b', b\wedge b'),\ (a\wedge a', a'\wedge b)\in\phi\\
&\Rightarrow (a\wedge b', 0),\ (0, a'\wedge b)\in\phi\\
&\Rightarrow a\wedge b',\ a'\wedge b\in I\\
&\Rightarrow (a\wedge b')\vee(a'\wedge b)\in I \text{ and}\\
&\qquad (a\wedge b')\vee(a'\wedge b)\vee a = a\vee b = (a\wedge b')\vee(a'\wedge b)\vee b\\
&\Rightarrow (a,b)\in\phi_I.
\end{aligned}
$$

and, on the other hand,

$$
\begin{aligned}
(a,b)\in\phi_I &\Rightarrow x\vee a = x\vee b \quad \text{for some } x\in I\\
&\Rightarrow (x,0)\in\phi \ \text{ and } x\vee a = x\vee b\\
&\Rightarrow (x\vee a, 0\vee a),\ (x\vee b, 0\vee b)\in\phi\\
&\Rightarrow (a, x\vee a),\ (x\vee b, b)\in\phi \text{ and } x\vee a = x\vee b\\
&\Rightarrow (a,b)\in\phi.
\end{aligned}
$$

Thus $\phi = \phi_I$.
(5) $\Rightarrow$ (1) is trivial, since the diagonal $\triangle_A$ is a congruence on A. □

Corollary 4.10. *The following are equivalent for any C-algebra A.*
(1) $I \mapsto \phi_I$ *is an injection of* $\Im(A)$ *into Con (A)*
(2) $I \mapsto \phi_I$ *is a surjection of* $\Im(A)$ *onto Con (A)*
(3) A is a Boolean algebra
(4) $\Im(A)$ *and Con (A) are isomorphic under the map* $I \mapsto \phi_I$.

Proof. This is a direct consequence of Theorems 4.7 and 4.9. □

5 Open Problems

1. It is known that in a C-algebra, congruences need not be permutable. It is under investigation that when two ideal congruences are permutable.

2. It is proved that the set of all ideal congruences $\{\phi_I \,/\, I$ is an Ideal of a C-algebra A$\}$ forms a lattice (by Lemma 4.5). It is under investigation that in which circumstances the set of all ideal congruences forms a Boolean algebra.

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References

[1] Birkhoff,G.: *Lattice theory*, Amer.Math.Soc.Colloquium publications, Vol. 24 (1967).

[2] Guzman,F. and Squier,C.C.: *The algebra of Conditional Logic*, Algebra Universalis 27, 88-110 (1990)

[3] Kalesha Vali.S., Sundarayya,P.and Swamy.U.M.,: *Ideals of C-algebras,* accepted for publication in the Journal Asian-European Journal of Mathematics, World Scientific.

[4] Rao.G.C.and Sundarayya,P.: *C-algebra as a Poset*, International Journal of Mathematical sciences, Dec 2005, Vol 4, No. 2, 225-236.

[5] Rao.G.C.and Sundarayya,P.: *Decompositions of a C-algebra*, International Journal of Mathematics and Mathematical Sciences, Vol. 2006, Article ID 78981, 1-8.

[6] Stanley Burris and Sankappanavar.H.P.: *A Course in Universal Algebra,* The Millennium edition.

[7] Swamy,U.M., Rao,G.C. and RaviKumar,R.V.G.: *Centre of a C-algebra,* Southeast Asian Bulletin of Mathematics $27, 357 - 368$(2003).