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Lacunary Interpolation by Quartic Splines with Application to Quadratures

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Abstract

The aim of this work is to construct lacunary interpolation based on quartic C^3-spline and to apply this spline function for finding approximate values of smooth function and its continuous derivatives. Upper bounds for errors and convergence analysis of the presented lacunary interpolation studied. Also, we have solved numerically two examples, to show the validity of the prescribed method by depending on the $L\infty$-error estimation.

Keywords: *Lacunary interpolation model, Convergence analysis, Spline function, Quadrature, Algorithm.*

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1 Introduction

In this paper, we apply quartic C^3-spline interpolation to develop a numerical method for obtaining approximations to the value of integration, finding the error bounds and suitable assumptions with applications showed that this spline exists and is unique. The convergence analysis and the stability of the approximate solution is investigated and compared with the exact solution to illustrate practical usefulness of our approximation. El-Tarazi and Sallam (1987) and (1993) have constructed a quasi-Hermite interpolatory by quartic spline with periodic second derivatives, and also studied a quadratic interpolatory spline that matches the derivative values of a smooth function at arbitrary positions [4, 7]. Swardt and Villiers (1999) have derived Jackson-type estimate with respect to maximum norm for quadratic nodal spline interpolation error [12]. In 2000, Santos-Leon studied error bounds for interpolatory quadrature rules on the Unit Circle [3]. Other class of spline approaches based on function values given at the mesh points has been studied see [1, 14, 15, 16, 17 and 18]. Also, a spline function of quartic and quintic types has been used by many authors for solving boundary and initial value problems, see [2, 8, 9, 10, 11, 13 and 17] and their references.

This paper is organized as follows: In section 2, construction of lacunary interpolation based on quartic C^3-spline is studied with some theoretical results about the existence and uniqueness of the quartic C^3-spline. Error estimation and Convergence analysis of the spline function is introduced in section 3. Convergence analysis for some particular cases is studied and described the application of this quartic spline to quadratures in section 4. We have demonstrated the spline algorithm and numerical results in section 5. In the last section, we have prescribed the conclusions of the results.

2 Construction of a lacunary interpolation based on quartic spline

In this section, in a similar manner of Sallam and El-Tarazi(1993) [7], we have constructed quartic spline which interpolates function values, first derivative and the third derivative of the smooth function $f(x)$ defined on a closed interval I=[0,1].

Let $\Delta_n:\ 0 = x_0 < x_1 < x_2 < .... < x_n = 1$ denotes the uniform partition of I with $h = x_{i+1} - x_i$ for $i = 0, 1, 2, ..., n$, and it denotes by $S^3_{n,4}$ the linear space of quartic splines $S(x)$ such that $S \in C^3[0,1]$ is a quartic polynomial in each subinterval $[x_i, x_{i+1}]$. If g is a real-valued function defined in [0, 1] and $0 \le \lambda \le 1$ is a real parameter then $g_{\lambda+i}$ stands for $g(x_i + \lambda h)$ where i=0, 1, 2, …, n. Moreover, we state and prove our results:

Theorem 2.1 Given the real numbers $\{f_i'\}_{i=0}^{n+1}$, $\{f'''_{i+\lambda}\}_{i=0}^{n-1}$, f_0 and f'_{n+1}, then there exist a unique $S \in S^3_{n,4}$ such that :

$S_i' = f_i'$ (i=0, 1, 2, …, n+1)

$S'''_{i+\lambda} = f'''_{i+\lambda}$ (i=0, 1, 2,…, n-1)

(2.1)

and $S_0 = f_0$, $S_{n+1} = f_{n+1}$ whenever $0 \le \lambda \le 1$, $\lambda \ne \frac{1}{2}$.

Proof: It can easily be verified that $P(x)$ is a quartic polynomial in [0, 1] restricted to $[x_i, x_{i+1}]$ will be written as:

$$P_\lambda(x) = P_\lambda(0)A_\lambda(x) + P_\lambda(1)B_\lambda(x) + hP'_\lambda(0)C_\lambda(x) + hP'_\lambda(1)D_\lambda(x) + h^3 P'''_\lambda(\lambda)E_\lambda(x)$$

where

$$\left.\begin{aligned}
A_\lambda(x) &= 1 - \frac{6\lambda - 2}{2\lambda - 1}x^2 + \frac{4\lambda}{2\lambda - 1}x^3 - \frac{1}{2\lambda - 1}x^4, \\
B_\lambda(x) &= \frac{1}{2\lambda - 1}(6\lambda - 2)x^2 - \frac{4\lambda}{2\lambda - 1}x^3 + \frac{1}{2\lambda - 1}x^4, \\
C_\lambda(x) &= x + \frac{-8\lambda + 3}{4\lambda - 2}x^2 + \frac{2\lambda}{2\lambda - 1}x^3 - \frac{1}{4\lambda - 2}x^4, \\
D_\lambda(x) &= \frac{1}{2(2\lambda - 1)}\left[(-4\lambda + 1)x^2 + 4\lambda x^3 - x^4\right], \\
E_\lambda(x) &= \frac{1}{24\lambda - 12}x^2 - \frac{2\lambda}{24\lambda - 12}x^3 + \frac{1}{24\lambda - 12}x^4.
\end{aligned}\right\} \quad (2.2)$$

where $i \in \{1, 2, ..., n\}$; set $x = x_i + th,\ 0 \le t \le 1$ then the quartic spline S(x) which satisfies (2.1) in $[x_i, x_{i+1}]$ will be written as:

$$S_\lambda(x) = s_\lambda A_\lambda(x) + s_{\lambda+1}B_\lambda(x) + hf'_\lambda C_\lambda(x) + hf'_{\lambda+1}D_\lambda(x) + h^3 f'''_{\lambda+1}E_\lambda(x) \qquad (2.3)$$

We have a similar expression for s(x) in $[x_i, x_{i+1}]$. Since $s \in C^3[0,1]$, then $S'''(xi^+) = S'''(xi^-)$, (i= *1, 2, …, n*) leads to the following linear system of equations:

$$24(\lambda - 1)s_{i-1} - 24(2\lambda - 1)s_i + 24\lambda s_{i+1} =$$
$$-12h(\lambda - 1)f'_{i-1} + 12hf'_i + 12\lambda hf'_{i+1} - h^3 f'''_{i+\lambda} - h^3 f'''_{i+\lambda-1}. \qquad (2.4)$$

The coefficient matrix, say M_λ of the above system is nonsingular for all $0 \le \lambda \le 1$. This is clear for $\lambda = 0, \frac{1}{3}, \frac{2}{3}, 1$. The above system (2.4) and for all i=1, 2, …, n, has the following coefficient matrix

$$M_\lambda = \begin{bmatrix} -24(2\lambda-1) & 24\lambda & 0 & & 0 \\ 24(\lambda-1) & -24(2\lambda-1) & 24\lambda & & 0 \\ 0 & 24(\lambda-1) & -24(2\lambda-1) & 24\lambda & 0 \\ .... & & .... & & ..... \\ 0 & & & & \\ 0 & & 24(\lambda-1) & -24(2\lambda-1) & 24\lambda \\ 0 & & & 24(\lambda-1) & -24(2\lambda-1) \end{bmatrix}.$$

It is not difficult to show that the matrix M_λ is nonsingular and hence $S(x)$ is uniquely determined, (Howell and Varma (1989) [5]).

An interesting particular case is for $\lambda = 0$,(i.e. $f_0, f_{n+1}, f_i', f_{n+1}'$ and f_i''' , i=0, 1, 2, …, n-1are known); equation (2.4) is then reduced to the recurrence formula (i=1, 2,…, n).

$$-S_{i-1} + S_i = \frac{h}{2}[f_{i-1}' + f_i'] - \frac{h^3}{24}[f_{i-1}''' + f_i'''], \qquad (2.5)$$

with $S_0 = f_0$. Hence S_i (i=1, 2,…, n) can also be computed throughout the simple formula

$$S_i = f_0 + \frac{h}{2}\left[f_0' + f_1' + 2\sum_{j=1}^{i-1} f_i \right] - \frac{h^3}{24}[f_0''' + f_1''']. \qquad (2.6)$$

3 Error Bounds

In this section error bounds for the above quartic spline interpolation will be presented. The main result of this section is the following theorem and lemma:

Lemma 3.1: If $f \in C^l[0,1]$, $l = 3$ and $l = 5$ then we have

$$|e_{i+1} - e_i| \le \begin{cases} \left(\frac{|\beta|}{h|1-\lambda|} + \frac{1}{1-\lambda}\right)\frac{(5\lambda-2)}{12} h^3 W_3(h) & \text{where} l = 3 \\ \left(\frac{|\beta|}{h|1-\lambda|} + \frac{1}{1-\lambda}\right)\frac{\lambda|10\lambda-3|}{240} h^5 W_5(h) + \frac{|-10\lambda^2+4\lambda-2|}{240} h^5 \|f^{(5)}(h)\| & \text{where} l = 5 \end{cases}$$

where $W_l(h)$ denotes the modulus of continuity of $f^{(l)}$, when $\lambda \ne 0, 1$.

Proof: We will consider the cases l=3 and l=5. Set $e_0 = e_{N+1} = 0$ and $e_i = s(x_i) - f(x_i)$, for i=1, 2,…, n, then from equation (2.4) we have

$$24(\lambda-1)e_{i-1} - 24(2\lambda-1)e_i + 24\lambda e_{i+1} = -12h(\lambda-1) f_{i-1}' + 12hf_i' + 12\lambda hf_{i+1}'$$

$$-h^3 f'''_{i+\lambda} - h^3 f'''_{i+\lambda-1} - 24(\lambda-1) f_{i-1} + 24(2\lambda-1) f_i - 24\lambda\ f_{i+1} \quad , \tag{3.1}$$

for i=1, 2,…, n.
Now let

$$z_i = e_{i+1} - e_i \text{ for } i=1, 2,\ldots, \text{n}. \tag{3.2}$$

From (3.1) and (3.2), we obtain

$$\frac{2\lambda-1}{\lambda} z_{i-1} + z_i = \delta_i ,$$

Which can be written as

$$\beta z_{i-1} + z_i = \delta_i , \tag{3.3}$$

where $\beta = \dfrac{2\lambda-1}{\lambda}$, $\delta_i = \dfrac{d_i}{\lambda}$ and

$$d_i = -12\,h(\lambda-1)\, f'_{i-1} + 12\, hf_i' + 12\,\lambda h f'_{i+1} - h^3 f'''_{i+\lambda} - h^3 f'''_{i+\lambda-1} -$$
$$24\,(\lambda-1)\, f_{i-1} + 24\,(2\lambda-1)\, f_i - 24\,\lambda f_{i+1}.$$

Equation (3.3) is a first-order linear difference equation and its solution is

$$z_i = (-\beta)^i z_0 + \sum_{j=1}^{i} (-\beta)^{i-j} \delta_j \ , i=1, 2,\ldots, n. \tag{3.4}$$

From Sallam and El-Tarazi (1993) [7], we have

$$\sum_{i=0}^{n} z_i = -z_0 . \tag{3.5}$$

Now, from equations (3.4) and (3.5) we obtain

$$|z_0| \le \frac{n\delta_i}{1-\beta} = \frac{nd_i}{1-\lambda}, \text{ where } \quad \delta = \max_{1\le i\le n} |\delta_i| \text{ and } \ d = \max_{1\le i\le n} |d_i| \,.$$

Thus using (3.5), we have

$$|z_0| \le \frac{n\,\delta}{1-\beta} < \frac{d}{h|1-\lambda|} ,$$

and

$$|z_i| \le |\beta||z_0| + \delta \sum_{j=0}^{i-j} |\beta|^{i-j} ,$$

or

$|z_i| \le |\beta||z_0| + \dfrac{\delta}{1-|\beta|}$, for i=1, 2,…, n.

(3.6)

Consequently

$$|z_i| \le \frac{d\,|\beta|}{h|1-\lambda|} + \frac{d}{1-\lambda}.$$

(3.7)

Expanding the right-hand side of (3.1) about x_i using a Taylor's expansion of order 3 and 5, we have

$$-24(\lambda-1)f_{i-1} + 24(2\lambda-1)f_i - 24\lambda f_{i+1} = d_i$$

After applying Taylor expansion of order $l=3$, we get

$$d_i \le \frac{(5\lambda-2)}{12}h^3W_3(h).$$

(3.8)

We can show that, in a similar manner for $l=5$, that

$$d_i \le \frac{\lambda|10\lambda-3|}{240}h^5W_5(h) + \frac{|-10\lambda^2+4\lambda-2|}{240}h^5\|f^{(5)}(h)\|.$$

(3.9)

From (3.7), we obtain

$$|z_i| \le \begin{cases} \left(\dfrac{|\beta|}{h|1-\lambda|} + \dfrac{1}{1-\lambda}\right)\dfrac{(5\lambda-2)}{12}h^3W_3(h) & \text{where } l=3 \\ \left(\dfrac{|\beta|}{h|1-\lambda|} + \dfrac{1}{1-\lambda}\right)\dfrac{\lambda|10\lambda-3|}{240}h^5W_5(h) + \dfrac{|-10\lambda^2+4\lambda-2|}{240}h^5\|f^{(5)}(h)\| & \text{where } l=5 \end{cases}.$$

This completes the proof of the Lemma 3.1.

Theorem 3.1

Let $S(x)$ is a quartic spline defined in section 2. If $f \in C^l[0,1]$, for l=3 and 5, then for any $x \in [0,1]$ and $\lambda \ne 0, 1.$, we have

$$\|S^{(r)}(x) - f^{(r)}(x)\|_\infty \le \begin{cases} K_r + T_r h^l\|f^{(l)}\|_\infty + C_r W_3(h) & r=0,1,2,3 \text{ and } l=3 \\ K_r + P_r h^l\|f^{(l)}\|_\infty + R_r W_5(h) & r=0,1,2,3 \text{ and } l=5 \end{cases}.$$

Proof:

Letting $e_i = S(x_i) - f(x_i)$ and $x \in [x_i, x_{i+1}]$, i=0, 1, …, n and using (2.3) we have

$$s(x) - f(x) = e_i A(x) + e_{i+1} B(x) + hf_i' C(x) + hf_{i+1}' D(x) + h^3 f_{i+\lambda}'''' E(x) - f(x) + A(x) f_i + B(x) f_{i+1}.$$

Since we have (see Brownlee and Light (2004) [6])

$$A(x) + B(x) = 1, \quad 2D(x) + B(x) = 1, \quad C(x) + D(x) + B(x) = 1,$$

$$4D(x) + 24\lambda E(x) + B - 1 = x^4, \text{ and } \quad 3D(x) + 6E(x) + B(x) = x^3.$$

We get:

$$\|s(x) - f(x)\|_\infty \le \begin{cases} K_0 + T_0 h^3 \|f^{(3)}\|_\infty + C_0 h^3 W_3(h), & \text{for } l = 3 \\ K_0 + P_0 h^5 \|f^{(5)}\|_\infty + R_0 h^5 W_5(h), & \text{for } l = 5 \end{cases}$$

where $K_0 = |e_{i+1}|$, $T_0 = \frac{25}{48}$, $C_0 = \frac{1}{6}$, $P_0 = \frac{1}{120}$ and $R_0 = \frac{1}{24}$.

Now for first derivative

$$h(s'(x) - f'(x)) = e_i A'(x) + e_{i+1} B'(x) + hf_i' C'(x) + hf_{i+1}' D'(x) + h^3 f_{i+\lambda}'''' E'(x) - hf'(x) + A(x) f_i' + B(x) f_{i+1}'.$$

Since we have

$$A'(x) + B'(x) = 0, \quad 2D'(x) + B'(x) = 2, \quad 4D'(x) + 24\lambda E'(x) + B' - 2x^3 = 0$$

and $3D'(x) + 6E'(x) + B'(x) = 3x^2$

Hence,

$$\|s'(x) - f'(x)\|_\infty \le \begin{cases} K_1 + T_1 h^3 \|f^{(3)}\|_\infty + C_1 h^3 W_3(h), & \text{for } L = 3 \\ K_1 + P_1 h^5 \|f^{(5)}\|_\infty + R_1 h^5 W_5(h), & \text{for } l = 5, \end{cases}$$

where $K_1 = \frac{5}{2}|e_{i+1} - e_i|$, $T_1 = \frac{1}{2}$, $C_1 = \frac{1}{2}$, $P_1 = \frac{1}{16}$ and $R_1 = \frac{1}{24}$.

For the second derivative, we have

$$A''(x) + B''(x) = 0, \quad C''(x) + D''(x) + B''(x) = 0, \quad 2D''(x) + B''(x) = 2,$$

$$4D''(x) + 24\lambda E''(x) + B'' = 12x^2 \;,$$

and $3D''(x) + 6E''(x) + B''(x) = 6x.$

Hence,

$$\|s''(x)-f''(x)\|_\infty \le \begin{cases} K_2 + T_2\, h^3 \left\| f^{(3)} \right\|_\infty + C_2 h^3 W_3(h), & \text{for } l = 3 \\ K_2 + P_2\, h^5 \left\| f^{(5)} \right\|_\infty + R_2\, h^5 W_5(h), & \text{for } l = 5, \end{cases}$$

Where $K_2 = 16|e_{i+1} - e_i|$, $T_2 = \frac{27}{6}, C_2 = 1$, $P_2 = \frac{91}{120}$ and $R_2 = \frac{1}{6}$.

Finally for the third derivative, we have

$A'''(x) + B'''(x) = 0, \quad C'''(x) + D'''(x) + B'''(x) = 0, \quad 2D'''(x) + B'''(x) = 0,$
$4D''(x) + 24\lambda E''(x) + B'' = 24x,$

and $3D'''(x) + 6E'''(x) + B'''(x) = 6x.$

Hence,

$$\|s'''(x)-f'''(x)\|_\infty \le \begin{cases} K_3 + T_3 h^3 \left\| f^{(3)} \right\|_\infty + C_3 h^3 W_3(h), & \text{for } l = 3 \\ K_3 + P_3\, h^5 \left\| f^{(5)} \right\|_\infty + R_3 h^5 W_5(h), & \text{for } l = 5, \end{cases}$$

where $K_3 = 72|e_{i+1} - e_i|$, $T_3 = 32, C_3 = 1$, $P_3 = \frac{41}{10}$ and $R_3 = \frac{1}{2}$.

4 The particular cases $\lambda = 0$ and $\lambda = 1$

Suppose that $\lambda = 0$ (the case $\lambda = 1$ is very similar). Then equation (2.4) is reduced to the recurrence formula

$$-24\, S_{i-1} + 24\, S_i = 12\, h f'_{i-1} + 12\, h f_i' - h^3 f_i''' - h^3 f'''_{i-1},$$

(4.1)

and (2.3) takes the form

$$S(x) = S_i A_0(x) + S_{i+1} B_0(x) + h f_i' C_0(x) + h f'_{i+1} D_0(x) + h^3 f_i''' E_0(x)$$

$$S(x) = (1 - 2x^2 + x^4) S_i + (2x^2 - x^4) S_{i+1} + (x - \frac{3}{2}x^2 + \frac{1}{2}x^4) h f_i' + \frac{1}{2}(-x^2 + x^4) h f'_{i+1} + \frac{1}{12}(-x^2 + 2x^3 - x^4) h^3 f_i''', \quad (4.2)$$

with $h = \frac{x - x_i}{N+1}$. From equation (4.1) and Theorem 3.1 for $\lambda = 0$, $l = 3$ and $l = 5$, set $e_i = s(x_i) - f(x_i)$ then we obtain:

$$-e_{i-1} + e_i = \frac{h}{2}\left[f'_{i+1} + f_i' \right] - \frac{h^3}{24}\left[f'''_{i+1} + f'''_i \right] + f_{i-1} - f_i. \quad (4.3)$$

Since $f \in C^{l}[0,1]$, expand the right hand side of (4.3) about x_i by Taylor's expansions of order $l=3$ and $l=5$ to obtain

$$\left|-e_{i-1}+e\right|_i \le \begin{cases} \dfrac{h^3}{4}W_3(h), & \text{where } l=3 \\ \dfrac{h^5}{48}W_5(h)-\dfrac{h^5}{120}\left\|f^{(5)}\right\|_\infty, & \text{where } l=5 \end{cases} \tag{4.4}$$

Now from equation (2.6),

$$S_i = f_0 + \frac{h}{2}\left[f_0' + f_1' + 2\sum_{j=1}^{i-1} f_i' \right] - \frac{h^3}{24}\left[f_0''' + f_1''' \right].$$

Hence

$$S_i - f_i = \frac{h}{2}\left[f_0' + f_1' + 2\sum_{j=1}^{i-1} f_i' \right] - \frac{h^3}{24}\left[f_0''' + f_1''' \right] - \int_0^{x_i} f'(x)\,dx\,, \tag{4.5}$$

since $s_0 = f_0$, the first part of the right hand side of (4.5) is the fact of the classical trapezoidal rule applied to the function $f'(x)$ in $[0, x_i]$. Integrating once over $[x_i, x]$, using $s(0)-f(0)=0$ and $s(1)-f(1)=0$ and we notice that

$$f_i = f_0 + \int_0^{x_i} f'(t)dt \text{ where } i = 0, 1, 2, \ldots, n.$$

Moreover, in $[x_i, x_{i+1}]$, i= 0, 1, 2, …, n-1, we get

$$\left\|s(x)-f(x)\right\|_\infty \le \begin{cases} \left|e_{i+1}\right| + \dfrac{25h^3}{92}\left\|f^{(3)}\right\|_\infty + \dfrac{h^3}{6}W_3(h), & \text{for } l=3 \\ \left|e_{i+1}\right| + \dfrac{h^5}{192}\left\|f^{(5)}\right\|_\infty + \dfrac{h^5}{120}W_5(h) & \text{for } l=5 \end{cases},$$

$$\left\|s'(x)-f'(x)\right\|_\infty \le \begin{cases} \dfrac{3}{2}\left|e_{i+1}-e_i\right| + \dfrac{5h^3}{12}\left\|f^{(3)}\right\|_\infty + \dfrac{h^3}{6}W_3(h), & \text{for } l=3 \\ \dfrac{3}{2}\left|e_{i+1}-e_i\right| - \dfrac{h^5}{144}\left\|f^{(5)}\right\|_\infty + \dfrac{5h^5}{144}W_5(h), & \text{for } l=5 \end{cases},$$

$$\left\|s''(x)-f''(x)\right\|_\infty \le \begin{cases} 4\left|e_{i+1}-e_i\right| + 4h^3\left\|f^{(3)}\right\|_\infty + h^3W_3(h), & \text{for } l=3 \\ 4\left|e_{i+1}-e_i\right| + \dfrac{19h^3}{120}\left\|f^{(5)}\right\|_\infty + \dfrac{h^5}{6}W_5(h), & \text{for } l=5 \end{cases},$$

and

$$\|s'''(x)-f'''(x)\|_\infty \le \begin{cases} 24\,|e_{i+1}-e_i| + 14h^3\|f^{(3)}\|_\infty + h^3 W_3(h), & \text{for } l=3 \\ 24\,|e_{i+1}-e_i| + \dfrac{h^5}{5}\|f^{(5)}\|_\infty + \dfrac{h^5}{2} W_5(h), & \text{for } l=5. \end{cases}$$

5 Numerical results

In this section, we are going to present some numerical results to demonstrate the convergence and computational interpolation of quartic spline collection method. From equations (2.3) and (2.4) constitute a new numerical quadrature rule which allow us to approximate

$$f(x)=\int_a^x f'(t)dt \quad \text{in } [a,b]$$

(5.1)

One may look upon this as a generalization of the traditional trapezoidal rule for integration, which corresponds to the cases $\lambda = 0$ or $\lambda = 1$.

The results show that the various optimal error bounds obtained in Theorem 3.1 when $\lambda = 0$ and indicate a complete agreement between the analytical and numerical behaviors of the method when we integrate $f'(t)$ belongs to $C^3[a,b]$ and $C^5[a,b]$, i.e. f belongs to $C^3[a,b]$ and $C^5[a,b]$. Thus we can compute their actual errors. For these examples the following outlines is of the new quartic spline Algorithm:

Step 1: Partition the interval $[a,b]$ into n subintervals I.

Step 2: Set

$S'_i = f'_i$ (i=0, 1, 2, …, n)

$S'''_{i+\lambda} = f'''_{i+\lambda}$ (i=0, 1, 2, …, n-1)

and $S_0 = f_0$, $S_1 = f_1$.

Step 3: Use (2.4) to find S_i, $i = 1, 2, \ldots, n$.

Step 4: Use (2.4) to find S_i and S'_i for n equally spaced points in each subinterval $x \in [x_{i-1}, x_i]$ go to step 5, else *i*=*i*+1 and repeat this iteration to find a proper *i*.

Step 5: Stop.

Example 5.1: Consider the function $f(x) = \frac{1}{1+x^2}$ where $x \in [0, 1]$. Table 1 gives the maximum error bounds for $s - f$ and its continuous derivatives for the values of $h = \frac{1}{1+n}$ for $N = 9, 19,\ 29$ and 39, for the values of $\lambda \in [0, 1],\ h = 0.2$.

Example 5.2: Consider the function $f(x) = x^3 + 1$ where $x \in [0, 1]$. Table 2 gives the maximum error bounds for $s - f$ and its continuous derivatives for the values of $h = \frac{1}{1+n}$ for $N = 9, 19,\ 29$ and 39, for the values of $\lambda \in [0, 1],\ h = 0.2$ considered.

Table 1: Absolute maximum errors for $s(x)$ and it's derivatives with different values of h for Example 5.1

Error bounds For	N	$\lambda = 0.0$	$\lambda = 0.2$	$\lambda = 0.4$
$\|s - f\|_\infty$	9	10.02×10^{-1}	10.02×10^{-1}	10.03×10^{-1}
	19	9.9×10^{-1}	9.9×10^{-1}	9.9×10^{-1}
	29	9.915×10^{-1}	9.917×10^{-1}	9.924×10^{-1}
$\|s' - f'\|_\infty$	9	18.5×10^{-2}	18.24×10^{-2}	16.97×10^{-2}
	19	18×10^{-2}	18.33×10^{-2}	16.9×10^{-2}
	29	18.59×10^{-2}	18.31×10^{-2}	18.93×10^{-2}
$\|s'' - f''\|_\infty$	9	19.6×10^{-1}	19.5×10^{-1}	18.6×10^{-1}
	19	19.7×10^{-1}	19.6×10^{-1}	18.69×10^{-1}
	29	19.76×10^{-1}	19.58×10^{-1}	18.69×10^{-1}
$\|s''' - f'''\|_\infty$	9	23.4×10^{-1}	19.7×10^{-1}	95×10^{-3}
	19	23.53×10^{-1}	21.8×10^{-1}	13.29×10^{-1}
	29	23.51×10^{-1}	21.73×10^{-1}	12.83×10^{-1}

Error bounds For	N	$\lambda = 0.6$	$\lambda = 0.8$	$\lambda = 1.0$
$\|s-f\|_\infty$	9	10.01×10^{-1}	10.0186×10^{-1}	10.019×10^{-1}
	19	9.9×10^{-1}	9.9088×10^{-1}	9.9098×10^{-1}
	29	9.9014×10^{-1}	9.908×10^{-1}	9.91×10^{-1}
$\|s'-f'\|_\infty$	9	20.7×10^{-2}	19.5×10^{-2}	19.25×10^{-2}
	19	21.15×10^{-2}	19.74×10^{-2}	19.46×10^{-2}
	29	19.69×10^{-2}	19.69×10^{-2}	19.42×10^{-2}
$\|s''-f''\|_\infty$	9	21.1×10^{-1}	20.3×10^{-1}	20.14×10^{-1}
	19	21.4×10^{-1}	20.5×10^{-1}	20.3×10^{-1}
	29	21.34×10^{-1}	20.46×10^{-1}	20.28×10^{-1}
$\|s'''-f'''\|_\infty$	9	57×10^{-2}	38.48×10^{-1}	34.47×10^{-1}
	19	38.89×10^{-1}	30.36×10^{-1}	28.6×10^{-1}
	29	39.5×10^{-1}	30.63×10^{-1}	28.8×10^{-1}

Table 2: Absolute maximum errors for $s(x)$ and its derivatives with different values of h for Example 5.2.

Error bounds for	N	$\lambda = 0.0$	$\lambda = 0.2$	$\lambda = 0.4$
$\|s-f\|_\infty$	9	99.96×10^{-2}	99.96×10^{-2}	99.96×10^{-2}
	19	10.0077×10^{-1}	10.007×10^{-1}	10.0069×10^{-1}
	29	10.0089×10^{-1}	10.00875×10^{-1}	10.00845×10^{-1}
$\|s'-f'\|_\infty$	9	26.37×10^{-3}	26.63×10^{-3}	26.37×10^{-3}
	19	29.37×10^{-3}	28.79×10^{-3}	28.07×10^{-3}
	29	29.69×10^{-3}	29.26×10^{-3}	28.73×10^{-3}
$\|s''-f''\|_\infty$	9	59.72×10^{-2}	60.09×10^{-2}	61.60×10^{-2}
	19	60.64×10^{-2}	60.42×10^{-2}	60.15×10^{-2}
	29	60.54×10^{-2}	60.32×10^{-2}	60.03×10^{-1}
$\|s'''-f'''\|_\infty$	9	59.98×10^{-1}	61.3×10^{-1}	67.86×10^{-1}
	19	59.18×10^{-1}	60.22×10^{-1}	61.52×10^{-1}
	29	59.61×10^{-1}	60.09×10^{-1}	60.69×10^{-1}

Error bounds For	N	$\lambda = 0.6$	$\lambda = 0.8$	$\lambda = 1.0$
$\|s - f\|_\infty$	9	99.96×10^{-2}	99.96×10^{-2}	99.96×10^{-2}
	19	10.008×10^{-1}	10.0077×10^{-1}	10.007×10^{-1}
	29	10.00935×10^{-1}	10.009×10^{-1}	10.0092×10^{-1}
$\|s' - f'\|_\infty$	9	26.37×10^{-3}	26.37×10^{-3}	26.37×10^{-3}
	19	30.21×10^{-3}	29.51×10^{-3}	29.37×10^{-3}
	29	30.33×10^{-3}	29.79×10^{-3}	29.37×10^{-3}
$\|s'' - f''\|_\infty$	9	57.07×10^{-2}	58.58×10^{-2}	59.72×10^{-2}
	19	60.96×10^{-2}	60.69×10^{-2}	60.64×10^{-2}
	29	60.89×10^{-1}	60.68×10^{-2}	60.61×10^{-2}
$\|s''' - f'''\|_\infty$	9	48.18×10^{-1}	54.74×10^{-1}	56.053×10^{-1}
	19	57.62×10^{-1}	58.92×10^{-1}	59.18×10^{-1}
	29	58.89×10^{-1}	59.49×10^{-1}	59.97×10^{-1}

6 Conclusion

The theoretical and numerical constructions for the lacunary quartic $C^{(3)}$-spline interpolation in this paper with, show that the lacunary interpolation suitable for finding approximate values of the definite integrals.

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