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Quarter-Sweep Iteration for First Kind Linear Fredholm Integral Equations

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Abstract

The main aim of this paper is to investigate the application of the quarter-sweep iteration in solving linear Fredholm integral equations of the first kind. The effectiveness Quarter-Sweep Gauss-Seidel (QSGS) method is examined by solving dense linear system generated from the discretization of the first kind linear Fredholm integral equations. In addition, the formulation and implementation of the proposed method are also presented. Some numerical simulations are carried out to show that the proposed method is superior compared to the standard method.

Keywords: *Linear Fredholm equations, Quarter-sweep iteration, Quadrature, Gauss-Seidel*

1 Introduction

Presently, the theory and application of integral equations is an important subject within applied mathematics. Integral equations are used as mathematical models for many and varied physical circumstances and also occur as reformulations of other mathematical models. Particularly, linear Fredholm integral equations of the first kind appear in the mathematical formulation of various and important inverse problems such as seismology, gravity surveying, computerized tomography and image deblurring [3].

The above-mentioned inverse problems, as well as others, can be formulated as a first kind linear integral equations, which has the generic form as follows

$$\int_{\Gamma} K(x,t)y(t)dt = f(x), \quad \Gamma = [a,b] \tag{1}$$

where the kernel function $K \in L^2(\Gamma \times \Gamma)$ and the function $f \in L(\Gamma)$ are given, and $y \in L(\Gamma)$ is the unknown function to be determined. $K(x,t)$ is called Fredholm kernel if the kernel in Eq. (1) is continuous on the square $S = \{a \le x \le b, a \le t \le b\}$ or at least square integrable on this square. Then, Eq. (1) with constant integration limits and Fredholm kernel are called first kind linear Fredholm integral equations [17]. Meanwhile, Eq. (1) also can be rewritten in the operator form as follows

$$\kappa : S \to T \quad \kappa(y(t)) = \int_a^b K(x,t)y(t)dt. \tag{2}$$

Definition 2.1 *Let $\kappa : S \to T$ be an operator from normed space S into a normed space T, the equation $\kappa y = f$ is called well-posed if κ is onto, one to one and $\kappa^{-1} : T \to S$ is continuous. Otherwise the equation is called ill-posed* [9].

In many application areas, numerical approaches were used widely to solve Fredholm integral equations. To solve Eq. (1) numerically, we either seek to determine an approximation solution in a chosen finite dimensional space by using projection method [4, 6, 7, 9, 11, 12, 18] or the quadrature method [8, 10, 17]. Such discretizations of integral equations lead to dense linear systems and can be prohibitively expensive to solve as the order of the linear systems increases. For large systems, iterative methods are preferred than direct methods because iterative methods often yield a solution within an acceptable error with fewer operations and round-off error are dumped out as the process evolves. Rounding errors due to floating-point arithmetic are frequently become the main problem of direct methods when dealing with large and / or ill conditioned systems [5]. Thus, iterative methods are the natural options for efficient solutions.

Consequently, the concept of the half-sweep iteration method has been introduced by Abdullah [1] via the Explicit Decoupled Group (EDG) iterative method to solve two-dimensional Poisson equations. Following to that, applications of the half-sweep iteration iterative methods have been discussed in [2, 16, 21, 22, 24, 27]. Meanwhile, Othman and Abdullah [13] extended the half-sweep iteration concept by introducing quarter-sweep iterative method via the Modified Explicit Group (MEG) iterative method to solve two-dimensional Poisson equations. Further studies to verify the effectiveness of the quarter-sweep iterative methods have been carried out; see [14, 15, 19, 23, 25, 26]. The basic idea of the half- and quarter-sweep iterative methods is to reduce the computational complexities during iteration process, since the implementation of the half- and quarter-sweep iterations will only consider nearly half and quarter of all interior node points in a solution domain respectively. In this paper, we examined the applications of the half- and quarter-sweep iteration concepts with Gauss-Seidel (GS) iterative method by using approximation equation based on quadrature scheme for solving problem (1). The standard GS iterative method is also called as the Full-Sweep Gauss-Seidel (FSGS) method. Meanwhile, combinations of the GS method with half- and quarter-sweep iterations are called as Half-Sweep Gauss-Seidel (HSGS) and Quarter-Sweep Gauss-Seidel (QSGS) methods respectively.

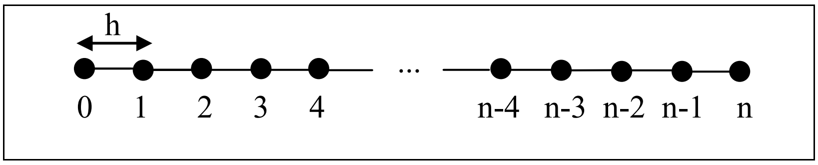
The outline of this paper is organized in following way. In Section 2, the formulation of the full-, half- and quarter-sweep quadrature approximation equations will be elaborated. The latter section of this paper will discuss the formulations of the FSGS, HSGS and QSGS methods, and some numerical results will be shown in fourth section to assert the performance of the iterative methods. Besides that, analysis on computational complexity is mentioned in Section 5. Meanwhile, conclusions and open problems are given in Section 6 and 7 respectively.

2 Quadrature Approximation Equations

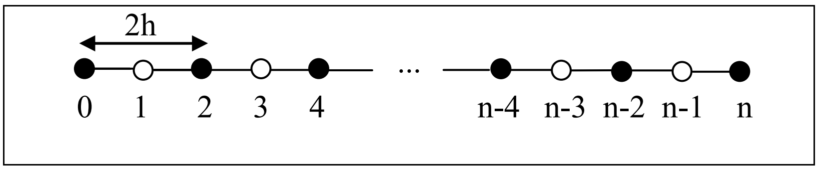
As afore-mentioned, a discretization scheme based on method of quadrature was used to construct approximation equations for problem (1) by replacing the integral to finite sums. Generally, quadrature method can be defined as follows

$$\int_a^b y(t)dt = \sum_{j=0}^{n} A_j y(t_j) + \varepsilon_n(y) \tag{3}$$

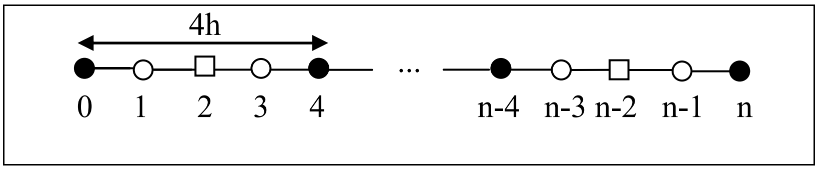
where t_j $(j = 0,1,2,\cdots,n)$ is the abscissas of the partition points of the integration interval $[a,b]$, A_j $(j = 0,1,2,\cdots,n)$ is numerical coefficients that do not depend on the function $y(t)$ and $\varepsilon_n(y)$ is the truncation error of Eq. (3). Meanwhile, Fig. 1 shows the finite grid networks in order to form the full-, half- and quarter-sweep quadrature approximation equations.



a)



b)



c)

Fig. 1: a), b) and c) show distribution of uniform node points for the full-, half- and quarter-sweep cases respectively.

Based on Fig. 1, the full-, half- and quarter-sweep iterative methods will compute approximate values onto node points of type ● only until the convergence criterion is reached. Then, other approximate solutions at remaining points (points of the different type) can be computed using the direct method [1, 13].

By applying Eq. (3) into Eq. (1) and neglecting the error, $\varepsilon_n(y)$, a system of linear equations can be formed for approximation values of $y(t)$. The following linear system generated using quadrature method can be easily shown in matrix form as follows

$$M \underset{\sim}{y} = \underset{\sim}{f} \tag{4}$$

where

$$M = \begin{bmatrix} A_0 K_{0,0} & A_p K_{0,p} & A_{2p} K_{0,2p} & \cdots & A_n K_{0,n} \\ A_0 K_{p,0} & A_p K_{p,p} & A_{2p} K_{p,2p} & \cdots & A_n K_{p,n} \\ A_0 K_{2p,0} & A_p K_{2p,p} & A_{2p} K_{2p,2p} & \cdots & A_n K_{2p,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_0 K_{n,0} & A_p K_{n,p} & A_{2p} K_{n,2p} & \cdots & A_n K_{n,n} \end{bmatrix}_{\left(\left(\frac{n}{p}\right)+1\right)\times\left(\left(\frac{n}{p}\right)+1\right)},$$

$$\underset{\sim}{y} = \begin{bmatrix} y_0 & y_p & y_{2p} & \cdots & y_{n-2p} & y_{n-p} & y_n \end{bmatrix}^T,$$

and

$$\underset{\sim}{f} = \begin{bmatrix} f_0 & f_p & f_{2p} & \cdots & f_{n-2p} & f_{n-p} & f_n \end{bmatrix}^T.$$

In order to facilitate the formulation of the full-, half- and quarter-sweep quadrature approximation equations for problem (1), further discussion will be restricted onto repeated trapezoidal (RT) scheme, which is based on linear interpolation formula with equally spaced data. Based on RT scheme, numerical coefficient A_j will satisfy the following relation

$$A_j = \begin{cases} \frac{1}{2} ph, & j = 0, n \\ ph, & otherwise \end{cases} \tag{5}$$

where the constant step size, h is defined as follows

$$h = \frac{b-a}{n} \tag{6}$$

and n is the number of subintervals in the interval $[a,b]$. Meanwhile, the value of p, which corresponds to 1, 2 and 4, represents the full-, half- and quarter-sweep cases respectively.

3 Formulation of the GS Iterative Methods

As mentioned above, FSGS, HSGS and QSGS iterative methods will be applied to solve linear systems generated from the discretization of the problem (1), as shown in Eq. (4). Let coefficient matrix, M be decomposed into

$$M = D - L - U \tag{7}$$

where D, $-L$ and $-U$ are diagonal, strictly lower triangular and strictly upper triangular matrices respectively. Thus, the general scheme for FSGS, HSGS and QSGS iterative methods can be written as

$$\underset{\sim}{y}^{(k+1)} = (D-L)^{-1}\left(U \underset{\sim}{y}^{(k)} + \underset{\sim}{f}\right). \tag{8}$$

Actually, the iterative methods attempts to find a solution to the system of linear equations by repeatedly solving the linear system using approximations to the vector $\underset{\sim}{y}$. Iterations for FSGS, HSGS and QSGS methods continue until the solution is within a predetermined acceptable bound on the error. By determining values of matrices D, $-L$ and $-U$ as stated in Eq. (7), the general algorithm for FSGS, HSGS and QSGS iterative methods to solve problem (1) would be generally described in Algorithm 1.

Algorithm 1: FSGS, HSGS and QSGS methods

For $i = 0, p, 2p, \cdots, n-2p, n-p, n$ and $j = 0, p, 2p, \cdots, n-2p, n-p, n$

Calculate

$$y_i^{(k+1)} \leftarrow \begin{cases} \left(f_i - \sum_{j=p}^{n} A_j K_{i,j} y_j^{(k)} \right) \Big/ A_i K_{i,i} & i = 0 \\ \left(f_i - \sum_{j=0}^{n-p} A_j K_{i,j} y_j^{(k+1)} \right) \Big/ A_i K_{i,i} & i = n \\ \left(f_i - \sum_{j=0}^{i-p} A_j K_{i,j} y_j^{(k+1)} - \sum_{j=i+p}^{n} A_j K_{i,j} y_j^{(k)} \right) \Big/ A_i K_{i,i} & i = otherwise \end{cases}$$

4 Numerical Simulations

In order to compare the performances of the iterative methods described in the previous section, several experiments were carried out on the following two Fredholm integral equations problems. In this paper, we will only consider well-posed equations and the case where $a = 0$ and $b = 1$.

Example 1 [20]

$$\int_0^1 K(x,t)\,y(t)\,dt = \frac{1}{6}(x^3 - x),\; 0 < x < 1 \tag{9}$$

with kernel

$$K(x,t) = \begin{cases} t(x-1), & t < x \\ x(t-1), & x \le t \end{cases}.$$

The exact solution of the problem is

$$y(x) = x\,.$$

Example 2 [20]

$$\int_0^1 K(x,t)\,y(t)\,dt = e^x + (1-e)x - 1,\; 0 < x < 1 \tag{10}$$

with kernel

$$K(x,t) = \begin{cases} t(x-1), & t \le x \\ x(t-1), & x < t \end{cases}.$$

The exact solution of the problem is

$$y(x) = e^x\,.$$

There are three parameters considered in numerical comparison such as number of iterations, execution time and maximum absolute error. Throughout the simulations, the convergence test considered the tolerance error, $\varepsilon = 10^{-10}$ and carried out on several different values of n. Results of numerical simulations, which were obtained from implementations of the FSGS, HSGS and QSGS iterative methods for Examples 1 and 2, have been recorded in Tables 1 and 2 respectively. Meanwhile, reduction percentage of the number of iterations and execution time for the HSGS and QSGS methods compared with FSGS method have been summarized in Table 3.

Table 1: Comparison of a number of iterations, execution time (seconds) and maximum absolute error for the iterative methods (Example 1)

Methods	**Number of iterations**				
	n				
	512	**1024**	**2048**	**4096**	**8192**
FSGS	381	461	550	646	753
HSGS	309	381	461	550	646
QSGS	243	309	381	461	550
	Execution time (seconds)				
Methods	***n***				
	512	**1024**	**2048**	**4096**	**8192**
FSGS	6.36	30.70	143.14	665.59	3177.57
HSGS	2.58	7.03	32.18	179.75	796.18
QSGS	1.27	2.98	8.42	52.67	264.39
	Maximum absolute error				
Methods	***n***				
	512	**1024**	**2048**	**4096**	**8192**
FSGS	6.823 E-10	8.343 E-10	8.445 E-10	9.714 E-10	9.797 E-10
HSGS	6.444 E-10	6.823 E-10	8.343 E-10	8.445 E-10	9.714 E-10
QSGS	6.642 E-10	6.444 E-10	6.823 E-10	8.343 E-10	8.445 E-10

Table 2: Comparison of a number of iterations, execution time (seconds) and maximum absolute error for the iterative methods (Example 2)

Methods	**Number of iterations**				
	n				
	512	**1024**	**2048**	**4096**	**8192**
FSGS	394	479	568	667	728
HSGS	321	394	479	568	667
QSGS	253	321	394	479	568
	Execution time (seconds)				
Methods	***n***				
	512	**1024**	**2048**	**4096**	**8192**
FSGS	4.77	20.36	91.35	423.66	2034.36
HSGS	2.25	6.24	33.27	140.56	595.66
QSGS	1.05	3.45	8.68	49.11	223.57
	Maximum absolute error				
Methods	***n***				
	512	**1024**	**2048**	**4096**	**8192**
FSGS	8.624 E-7	2.157 E-7	5.589 E-8	2.571 E-8	4.255 E-8
HSGS	8.591 E-6	2.154 E-6	5.391 E-7	1.354 E-7	3.844 E-8
QSGS	3.416 E-5	8.591 E-6	2.154 E-6	5.391 E-7	1.354 E-7

Table 3: Reduction percentage of the number of iterations and execution time for the HSGS and QSGS methods compared with FSGS method

Methods	Example 1	
	Number of iterations	**Execution time**
HSGS	14.20 - 18.90%	59.43 - 77.52%
QSGS	26.95 - 36.23%	80.03 - 94.12%
Methods	**Example 2**	
	Number of iterations	**Execution time**
HSGS	8.37 - 18.53%	63.57 - 70.73%
QSGS	21.97 - 35.79%	83.05 - 90.50%

5 Computational Complexity Analysis

In order to measure the computational complexity of the FSGS, HSGS and QSGS iterative methods, an estimation of the amount of the computational work required has been conducted. The computational work is estimated by considering the arithmetic operations performed per iteration for each iterative method. Based on Algorithm 1, it can be observed that there are $\frac{n}{p}$ additions/subtractions (ADD/SUB) and $2\left(\frac{n}{p}+1\right)$ multiplications/divisions (MUL/DIV) in computing a value for each node point in the solution domain. From the order of the coefficient matrix, M in Eq. (4), the total number of arithmetic operations per iteration for the FSGS, HSGS and QSGS iterative methods has been summarized in Table 4.

Table 4: Total number of arithmetic operations per iteration for FSGS, HSGS and QSGS methods

Methods	Arithmetic Operation	
	ADD/SUB	**MUL/DIV**
FSGS	n^2+n	$2(n+1)^2$
HSGS	$\frac{n^2}{4}+\frac{n}{2}$	$2\left(\frac{n}{2}+1\right)^2$
QSGS	$\frac{n^2}{16}+\frac{n}{4}$	$2\left(\frac{n}{4}+1\right)^2$

6 Conclusion

In this paper, we present an application of the quarter-sweep iterative method for solving dense linear systems arising from the discretization of the first kind linear

Fredholm integral equations by using RT scheme. Through numerical solutions obtained in Tables 1 and 2, it clearly shows that applications of the half- and quarter-sweep iteration concepts reduce number of iterations and computational time of the iterative method (refer Table 3). Overall, the numerical results show that the QSGS method is a better method compared to the FSGS and HSGS methods in the sense of number of iterations and execution time. This is mainly because of computational complexity of the QSGS method which is approximately 50% and 75% less than HSGS and FSGS methods respectively as shown in Table 4.

7 Open Problem

From the observation of the numerical results, it shows that application of the quarter-sweep iteration concept reduces the accuracy of the numerical solutions for certain Fredholm integral equations of the first kind (refer Table 2). Decrement of the accuracy for quarter-sweep iterative method is mainly due to the computational technique to calculate remaining points. Thus, new approach to calculate the remaining points can be proposed. Besides that, effectiveness of the quarter-sweep iteration with other iterative methods in order to solve linear systems generated from Eq. (1) can be investigated.

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