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Adjoint of Sublinear Operators

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Abstract

Let $SB(X,Y)$ be the set of the bounded sublinear operators from a Banach space X into a complete Banach lattice Y. Let $T \in SB(X,Y)$. In the present paper, we will introduce the concept of adjoint sublinear operator of T, we proof that the adjoint operator $T^*$is also sublinear operator and we give some properties.

Keywords: *Adjoint operator, Banach lattice, factorization, Köthe space, quasi-linear operator, sublinear operator.*

1 Introduction

L. Mezrag and A. Tiaiba in [13] give the generalization of Maurey's theorem of factorization to sublinear operators which is a simple application of [10, Theorem 2] (also, by an other method Defant in [3] has generalized this type of factorization to homogeneous operators) and study the following equivalence. Let $0 < p \leq q \leq \infty$. Let $T : X \longrightarrow L_p(\Omega, \mu)$ be a bounded sublinear operator.

$$\forall u \in \nabla T,\ u \text{ factors through } L_q(\Omega, \mu) \Longleftrightarrow T \text{ factors through } L_q(\Omega, \mu),$$

where $\nabla T = \{\text{linear operators } u : X \longrightarrow L_p(\Omega, \mu) \text{ such that } u \leq T\}$.
A. Tiaiba in [17] introduce the concept of l_p-summing sublinear operators in the non commutative case and characterize this class of operators by giving the extension of the Pietsch domination theorem.
Pietsch has shown in [14, p. 338] that the identity from l_1 into l_2 is $2-$absolutely summing but the adjoint operator is not $2-$absolutely summing. For this, the

concept of strongly $p-$summing linear operators ($1 \leq p < \infty$) was introduced by J. S. Cohen [4] as a characterization of the conjugates of absolutely p^*-summing linear operators. Cohen deduced the domination theorem simply from the adjoint operator which is p^*-summing. That is not the case for sublinear operators because we do not know the adjoint of a sublinear operator. In [1] D. Achour, L. Mezrag and A. Tiaiba show it directly by using Ky Fan's lemma.
This problem is the first idea given the construction of the adjoint of sublinear operators which is the operators positively homogeneous and subadditive.
Let $SB(X,Y)$ be the set of the bounded sublinear operators from a Banach space X into a complete Banach lattice Y. Let $T \in SB(X,Y)$. We define T^* (adjoint of T) from Y^* into X^* by

$$T^*(y^*) = \sup_{u \in \nabla T} u^*(y^*).$$

The operator T^* is sublinear. When we take X a (Köthe function space) Banach lattice, Y be a (complete Köthe function space) complete Banach lattice. A Köthe function space (which is Banach lattice represented as spaces of measurable functions) is an abstract Banach lattice and from this theorem an abstract Banach lattice (with a weak unit and order continuous) is isomorphic as a Banach space and as a lattice to a Köthe function space. Thus it can be assumed to a Köthe function space on a suitable probability space. To simplify this presentation, we take in the sequel a Banach lattice of equivalent classes of locally integrable measurable functions on a complete $\sigma-$finite measure space $(\Omega, \mathcal{T}, \mu)$.

2 Preliminaries

In this section we introduce some terminology concerning the (quasi-) Banach lattice and the (quasi-) Köthe function spaces. For more details, the interested reader can consult the references [3, 8, 11, 18].

We recall the abstract definition of (quasi-) Banach lattice. A real (quasi-) Banach space X is a complete metrizable real vector space whose topology is given by a (quasi-) norm. If in addition X is a vector lattice and $\|x\| \leq \|y\|$ whenever $|x| \leq |y|$ ($|x| = \sup\{x, -x\}$) we say that X is a (quasi-) Banach lattice. Note that this implies obviously that for any $x \in X$ the elements x and $|x|$ have the same (quasi-) norm.
Recall that a (quasi-) norm on a real vector space X is a function $x \longrightarrow \|x\|$ from X to $\mathbb{R}_+$ which satisfies

(1) $\|x\| > 0$ for all $x \neq 0$.
(2) $\|tx\| = |t| \, \|x\|$ for all $t \in \mathbb{R}$ and $x \in X$.
(3) $\exists C_X \geq 1 : \|x + y\| \leq C_X(\|x\| + \|y\|)$ for all $x, y \in X$.
C_X is called the modulus of concavity of $\|.\|$.

If (3) is substituted by (3) $\|x + y\|^p \leq \|x\|^p + \|y\|^p$ for all $x, y \in X$ and for some fixed $p \in]0, 1]$, then $\|.\|$ is called a $p-$norm on X. Note that $1-$norm is the usual norm. A (quasi-) Banach space is isomorphic to a Banach space if and only if it is locally convex. Every $p-$norm is a quasi-norm with $C = 2^{\frac{1}{p}-1}$. Also, for every quasi-Banach space X there is a number $0 < p < 1$ and an equivalent $p-$norm satisfying

$$\|x + y\|^p \leq \|x\|^p + \|y\|^p$$

for all $x, y \in X$.
If $|||.|||$ denote the original quasi-norm on X with the constant C in the quasi-triangle inequality, then the $p-$norm ($C = 2^{\frac{1}{p}-1}$) can be defined as follows

$$\|x\| = \inf \left\{ \left(\sum_{i=1}^{n} |||x_i|||^p \right)^{\frac{1}{p}} : n > 0, \quad x = \sum_{i=1}^{n} x_i \right\}.$$

This assertation is due to Aoki and Rolewicz [7].

Definition 2.1. *A quasi- Banach lattice X is said to be order continuous ($\sigma-$order continuous) if, every downward directed set (sequence) $(x_i)_{i \in I}$ in X verified that* $\inf_{i \in I} x_i = 0, \quad \lim_i \|x_i\| = 0.$

For more details about the following. The interested reader can consult the references [1, 2, 3, 5, 6, 8, 9].

Proposition 2.2. *Let X be a Banach lattice. Then the following assertions are equivalent.*
(1)- X is $\sigma-$order complete and $\sigma-$order continuous.
(2)- Every order bounded increasing sequence in X converge in the norm topology of X.
(3)- X is order continuous.
(4)- X is order complete and order continuous.

Let now $(\Omega, \mathcal{T}, \mu)$ be a complete $\sigma-$finite measure space. A quasi-Banach space X consisting of equivalence classes, modulo equality almost everywhere, of locally integrable real valued functions on Ω is called quasi-Köthe function space if the following conditions hold:
(1) If $|f| \leq |g| \quad \mu - a.e.$ on Ω, with $f \in L_0(\Omega, \mathcal{T}, \mu, \mathbb{R})$ and $g \in X$. Then $f \in X$ and $\|f\| \leq \|g\|$.

(2) There is some $0 < t < \infty$ such that for all $x, y \in X$

$$\left\| \left(|x|^t + |y|^t \right)^{\frac{1}{t}} \right\|_X \leq \left(\|x\|^t + \|y\|^t \right)^{\frac{1}{t}}.$$

(3) For every $A \in \mathcal{T}$ with $\mu(A) < \infty$ the characteristic function χ_A of A belong to X.
In condition (2) if $t = 1$, we say Köthe function space. Every Köthe function space is a Banach lattice in the obvious order ($f \geq 0$ if $f(\omega) \geq 0 \quad \mu - a.e.$). This lattice is $\sigma-$order complete. For example the L_p, the Lorentz spaces $L_{p,q}$ and the Orlicz spaces over Ω are all quasi-Köthe function spaces. In general $C(K)$ is not a Köthe function space.

Definition 2.3. *We say that a quasi-Köthe function space X is $\sigma-$order continuous if whenever*
$f_n \in X : \quad 0 \leq f_n \searrow 0 \quad \mu - a.e.$ *then* $\|f_n\|_X \searrow 0$.

Definition 2.4. *The Köthe dual X' of X is the Köthe function space of all $g \in L_0(\Omega, \mathcal{T}, \mu, \mathbb{R})$ such that $f.g \in L_1(\Omega, \mathcal{T}, \mu, \mathbb{R})$.*
The space X' equipped with the norm induced by the norm of X^*

$$\|g\|_{X'} = \sup_{\|f\|_X = 1} \left| \int_\Omega f(\omega) g(\omega) d\mu(\omega) \right|$$

is a Köthe function space. The Köthe dual X'can be regarded as a closed subspace of the dual X^* (X' is an ideal of X^*).
We have also, X is $\sigma-$order continuous if and only if $X' = X^*$. We say now that:
X satisfies the Fatou property if, for $x_n, x \in X_+$ such that $x_n \nearrow x \quad \mu - a.e.$ then $\|x_n\| \longrightarrow \|x\|$.
X' is a norming subspace of X^* if,

$$\|x\|_X = \sup_{\|x^*\|_{X'} = 1} |\langle x^*, x \rangle|$$

for every x in X.
After we have given these definitions, we can announced the following property: The space X' is a norming subspace of X^* if and only if X satisfies the Fatou property.
The dual X^* of a Banach lattice space X is Banach complete lattice space with the natural order see [8]

$$x_1^* \leq x_2^* \Longleftrightarrow \langle x_1^*, x \rangle \leq \langle x_2^*, x \rangle, \quad \forall x \in X^+. \tag{2.1}$$

If we consider X as a sub-lattice of X^{**} we have [8], for all $x_1, x_2 \in X$

$$x_1 \leq x_2 \Longleftrightarrow \langle x_1, x^* \rangle \leq \langle x_2, x^* \rangle, \quad \forall x^* \in X_+^*. \tag{2.2}$$

The following theorem establishes the tie between the abstract Banach lattice and the Köthe function space.

Theorem 2.5. *Let X be an order continuous Banach lattice which has a weak unit. Then there exists a probability space $(\Omega, \mathcal{T}, \mu)$, an ideal $\widetilde{X}$ of $L_1(\Omega, \mathcal{T}, \mu)$ and a lattice norm $\|.\|_{\widetilde{X}}$ on $\widetilde{X}$ so that*
(1)- X is order isometric to $\left(\widetilde{X}, \|.\|_{\widetilde{X}}\right)$.
(2)- $\widetilde{X}$ is dense in $L_1(\Omega, \mathcal{T}, \mu)$ and $L_\infty(\Omega, \mathcal{T}, \mu)$ is dense in $\widetilde{X}$.
(3)- $\|f\|_1 \leq \|f\|_{\widetilde{X}} \leq \|f\|_\infty, \forall f \in L_\infty$.
(4)- The dual of the isometry given in (1) maps onto the Banach lattice.

3 Sublinear and Quasilinear Operators

For the convenience of the reader, we recall in this section some elementary definitions and fundamental properties relative to sublinear operators. For more details see [1, 2, 13].

Definition 3.1. *A mapping T from a Banach space X into a Banach lattice Y is said to be sublinear if for all x, y in X and λ in $\mathbb{R}_+$, we have*
(1) $T(\lambda x) = \lambda T(x)$ (i.e. positively homogeneous),
(2) $T(x + y) \leq T(x) + T(y)$ (i.e. subadditive).

Note that the sum of two sublinear operators is a sublinear operator and the multiplication by a positive number is also a sublinear operator.
Let us denote by

$$\mathcal{SL}(X, Y) = \{\text{sublinear mappings } T : X \longrightarrow Y\}$$

and we equip it with the natural order induced by Y

$$T_1 \leq T_2 \Longleftrightarrow T_1(x) \leq T_2(x), \quad \forall x \in X \tag{3.1}$$

and

$$\nabla T = \{u \in L(X, Y) : u \leq T \,(\text{i.e. } \forall x \in X,\, u(x) \leq T(x))\}.$$

The set ∇T is not empty by Proposition 3.3 in the next page. As a consequence

$$u \leq T \Longleftrightarrow -T(-x) \leq u(x) \leq T(x), \quad \forall x \in X \tag{3.2}$$

and

$$\lambda T(x) \leq T(\lambda x), \quad \forall x \in \mathbb{R}. \tag{3.3}$$

Also we say that a sublinear operator T:
is symmetrical if for all x in X, $T(x) = T(-x)$,
is positive if for all x in X, $T(x) \geq 0$,
is increasing if for all x, y in X, $T(x) \leq T(y)$ when $x \leq y$.
The symmetry implies the positivity, the converse is false. Also, there is no relation between positivity and increasing like the linear case (a linear operator $u \in \mathcal{L}(X,Y)$ is positive if $u(x) \geq 0$ for $x \geq 0$).

Let us now define the definition of quasilinear operators. An operator T from a Banach space X into a quasi-Banach lattice Y is called to be quasilinear if for all x, y in X and λ in $\mathbb{R}$, we have

(1) $|T(\lambda x)| = |\lambda| \, |T(x)|$,

(2) $|T(x+y)| \leq |T(x)| + |T(y)|$.

If we put $\varphi(x) = |T(x)|$ then φ is a symmetrical sublinear operator. T sublinear and symmetrical implies T quasilinear.

Note that in general the sum of two quasilinear operators is not a quasilinear operator but the multiplication by a scalar is a quasilinear operator.
Let T be a sublinear or quasilinear operator from a Banach space X into a quasi-Banach lattice Y. Then we have,
T is continuous iff there is $C > 0$ such that for all $x \in X$, $\|T(x)\| \leq C \|x\|$.
In this case we also say that T is bounded and we put

$$\|T\| = \sup\{\|T(x)\| : \|x\|_{B_X} = 1\}.$$

We will denote by $SB(X,Y)$ (resp. $\mathcal{S}B(X,Y)$) the set of all bounded sublinear (quasilinear) operators from X into Y and by $B(X,Y)$ the Banach space of all bounded linear operators from X into Y.

Examples

1- If $u : X \to Y$ is a linear operator from a Banach space X into a Banach lattice Y, then $T(x) = |u(x)|$ is a symmetrical sublinear operator.

2- Let X be a Banach space, Y be a Banach lattice. Consider T in $\mathcal{SL}(X,Y)$. If we put $\varphi(x) = \sup\{T(x), T(-x)\}$ then φ is a symmetrical sublinear operator.

Let T be a sublinear operator from a Banach space X into a Banach lattice Y. Then we have,
T is continuous if and only if there is $C > 0$ such that for all $x \in X$, $\|T(x)\| \leq C \|x\|$.

In this case we also say that T is bounded and we put

$$\|T\| = \sup\{\|T(x)\| : \|x\|_{B_X} = 1\}$$

where B_X denotes the closed unit ball of X.
We will need the following remark.

Remark 3.2. *Let X be an arbitrary Banach space. Let Y, Z be Banach lattices.*
(1) Consider T in $\mathcal{SL}(X,Y)$ and u in $\mathcal{L}(Y,Z)$. Assume that u is positive (i.e., $u(x) \geq 0$ for every $x \in X_+$). Then, $u \circ T \in \mathcal{SL}(X,Z)$.
(2) Consider u in $\mathcal{L}(X,Y)$ and T in $\mathcal{SL}(Y,Z)$. Then, $T \circ u \in \mathcal{SL}(X,Z)$.
(3) Consider S in $\mathcal{SL}(X,Y)$ and T in $\mathcal{SL}(Y,Z)$. Assume that S is increasing. Then, $S \circ T \in \mathcal{SL}(X,Z)$.

The following proposition, will be used implicitly in the sequel. For the proof see [1, Proposition 2.3].

Proposition 3.3. *Let X be a Banach space and let Y be a complete Banach lattice. Let $T \in \mathcal{SL}(X,Y)$. Then, for all x in X there is $u_x \in \nabla T$ such that, $T(x) = u_x(x)$, (i.e. the supremum is attained, $T(x) = \sup\{u(x) : u \in \nabla T\}$).*

As an immediate consequence of Proposition 3.3, we have for all $x \in X$

$$\|T(x)\| \leq \sup_{u \in \nabla T} \|u(x)\| \leq \|T(x)\| + \|T(-x)\| \tag{3.4}$$

and consequently

$$\|T\| \leq \sup_{u \in \nabla T} \|u\| \leq 2\,\|T\|\,. \tag{3.5}$$

Hence, the operator T is bounded if and only if for all $u \in \nabla T$, u is bounded.

Proposition 3.4. *Let T be a quasilinear operator from a Banach space X into a complete (quasi-) Banach lattice Y. Then for all x in X there is $u_x \in L(X,Y)$ such that, $|T(x)| = |u_x(x)|$, (i.e. the supremum is attained, $|T(x)| = \sup\{|u(x)| : u_x \in L(X,Y),\ |u_x| \leq |T|\}$).*

Proof.

We have $\varphi(x) = |T(x)|$ which is a symmetrical sublinear operator. For all x in X there is $u_x \in L(X,Y)$ such that, $|T(x)| = u_x(x)$. We have also $|T(-x)| = u_{-x}(-x)$ and this implies $|T(x)| = -u_{-x}(x)$ by the symmetrization of φ. Hence $|T(x)| = |T(-x)|$ and consequently $|T(x)| = \sup\{|u(x)| : u_x \in L(X,Y),\ |u_x| \leq |T|\}$.

4 Adjoint Operator of Sublinear Operator

In this section we define the adjoint operator of the sublinear operator. We also give some properties.

Proposition 4.1. *Let X be a (Köthe function space) Banach lattice, Y be a (complete Köthe function space) complete Banach lattice. Let T be a bounded sublinear operator from X into Y. Then the set $\{u^*(y^*)\}_{u\in\nabla T}$ is simply bounded in X^*.(i.e.*

$$\forall y^* \in Y^*, \exists C_{y^*} \in X^* : \forall u \in \nabla T, \quad u^*(y^*) \leq C_{y^*}). \tag{4.1}$$

Proof.
Let X be a Köthe function space on $(\Omega, \mathcal{T}, \mu, \mathbb{R})$. Suppose that $\{u^*(y^*)\}_{u\in\nabla T}$ is not simply bounded in Y^*. Namely, there is Ω_0 $(\mu(\Omega_0) > 0) \in \mathcal{T}$, and $y^* \in Y^*$ such that for all $C \in X^*$, there is $u \in \nabla T$ and $u^*(y^*) > C$. We can choose C in X^*_+. Then we have

$$u^*(y^*) > C \text{ on } \Omega_0. \Longrightarrow \|u^*(y^*)\| > \|C\|.$$

(Because X^* a complete Banach lattice as dual of Banach lattice). Hence

$$\|C\| < \|u^*(y^*)\| \leq \|u\| \|y^*\|.$$

We take

$$C = 2\|T\| \|y^*\| x^*, \quad x^* \in B_{X^*},\ x^* > 0 \text{ and } \|x^*\| = 1$$

we have

$$2\|T\| \|y^*\| \|x^*\| < \|u\| \|y^*\|$$

this implies

$$2\|T\| < \|u\|.$$

We have in contradiction with (3.5) which says $2\|T\| \geq \|u\|$,
and consequently $\{u^*(y^*)\}_{u\in\nabla T}$ is simply bounded.

Definition 4.2. *Let X be a (Köthe function space) Banach lattice, Y be a (complete Köthe function space) complete Banach lattice. Let T be a bounded* *sublinear operator from X into Y. We define T^* (adjoint of T) from Y^* into X^* by*

$$T^*(y^*) = \sup_{u \in \nabla T} u^*(y^*). \tag{4.2}$$

Remark 4.3. *Let X be a (Köthe function space) Banach lattice, Y be a (complete Köthe function space) complete Banach lattice. Let T be a bounded sublinear operator from X into Y. The operator T^* defined by proposition 4.1 is sublinear operator.*

Proof.

Let y_1^* and y_2^* in Y^*. We have

$$\begin{aligned} T^*(y_1^* + y_2^*) &= \mathop{sup}_{u \in \nabla T} u^*(y_1^* + y_2^*) \\ &= \mathop{sup}_{u \in \nabla T}(u^*(y_1^*) + u^*(y_2^*)) \\ &\leq \mathop{sup}_{u \in \nabla T} u^*(y_1^*) + \mathop{sup}_{u \in \nabla T} u^*(y_2^*) \\ &\leq T^*(y_1^*) + T^*(y_2^*). \end{aligned}$$

Let $\lambda > 0$ and $y \in Y^*$, we have

$$\begin{aligned} T^*(\lambda y^*) &= \mathop{sup}_{u \in \nabla T} u^*(\lambda y^*) \\ &= \lambda \mathop{sup}_{u \in \nabla T} u^*(y^*) \\ &= \lambda T^*(y^*). \end{aligned}$$

We have by proposition 3.3

$$T^*(y^*) = \sup_{u \in \nabla T^*} v(y^*) \tag{4.3}$$

where

$$\nabla T^* = \{v \in B(Y^*, X^*) : v(y^*) \leq T^*(y^*)\}.$$

Corollary 4.4. *Let X be a (Köthe function space) Banach lattice, Y be a (complete Köthe function space) complete Banach lattice. Let T be a bounded sublinear operator from X into Y. Then, for all y^* in Y^* there is v_{y^*} in ∇T such that $T^*(y^*) = v_{y^*}^*(y^*)$.*

Proof.

We have by proposition 3.3

$$\forall y^* \in Y^* \quad \exists v_{y^*} \in \nabla T^* : T^*(y^*) = v_{y^*}(y^*).$$

Remark 4.5. *Of course, every linear is sublinear operator. So, the terminology "adjoint" is already used for linear operator is justify because the definition 4.2 coincide with the the adjoint of a linear operator. We know that $\mathcal{L}(X,Y)$ is not lattice space for this if T is a linear operator from X to Y then,*

$$\nabla T = \{u : X \to Y / \ u(x) \leq T(x) \text{ for all } x \in X\}.$$

With ∇T have only one element $u = T$ or $\nabla T = \{T\}$ this implies that

$$\begin{aligned} T^*(y^*) &= \sup_{u \in \nabla T} u^*(y^*) \\ &= \sup_{u \in \nabla T} T^*(y^*) \\ &= T^*(y^*). \end{aligned}$$

Examples

1- Let the sublinear application T "we can see T as convex function see [16]" defined from $\mathbb{R}$ to $\mathbb{R}$ by

$$\begin{array}{rccc} T: & \mathbb{R} & \to & \mathbb{R} \\ & x & \to & T(x) = |x|. \end{array}$$

We know that $\mathcal{L}(\mathbb{R},\mathbb{R}) = \{u(x) = \alpha x \ / \ \alpha \in \mathbb{R}\}$ and we simply deduce that

$$\nabla T = \left\{ \begin{array}{c} u : \mathbb{R} \to \mathbb{R} / \ u(x) = \alpha x, \ \alpha \in [-1,1] \\ / \ u(x) \leq T(x), \ \forall \ x \in \mathbb{R} \end{array} \right\};$$

and we have $|x| = T(x) = \sup\limits_{u \in \nabla T} u(x) = \sup\limits_{\alpha \in [-1,1]} \alpha x.$

We remember that $\mathbb{R}^* = \{y^* : \mathbb{R} \to \mathbb{R} / \ y^*(x) = yx \text{ " linear form "}\}.$

We have also $u^*(y^*) = \alpha y^* \ \forall u \ (u(x) = \alpha x) \in \nabla T$ then, for all $y^* \in \mathbb{R}^*$.

We have

$$\begin{aligned} \langle u(x), y^* \rangle &= \langle \alpha x, y^* \rangle, \\ &= \alpha \langle x, y^* \rangle, \\ &= \langle x, \alpha y^* \rangle, \\ &= \langle x, u^*(y^*) \rangle. \end{aligned}$$

Hence

$$\begin{aligned} T^*(y^*) &= \sup_{u \in \nabla T} u^*(y^*), \\ &= \sup_{\alpha \in [-1,1]} \alpha y^*, \\ &= \sup\{-y^*, y^*\}, \\ &= |y^*|. \end{aligned}$$

T^* is also a sublinear operator from $\mathbb{R}^*$ to $\mathbb{R}^*$.
2- Let the sublinear application T defined from $\mathbb{H}$ Hilbert space to $\mathbb{R}$ by

$$\begin{array}{rccc} T: & \mathbb{H} & \rightarrow & \mathbb{R} \\ & x & \rightarrow & T(x) = \|x\| . \end{array}$$

We know by Riesz representation that $\forall u \in \mathcal{L}(\mathbb{H}, \mathbb{R})$, $\exists a \in \mathbb{H}$ / $u(x) = ax$. We simply deduce that

$$\nabla T = \left\{ \begin{array}{c} u : \mathbb{H} \rightarrow \mathbb{R}/ \ u(x) = ax, \ \ \|a\| \leq 1 \\ / \ u(x) \leq T(x), \forall x \in \mathbb{H} \end{array} \right\},$$

and we have $\|x\| = T(x) = \sup\limits_{u \in \nabla T} u(x) = \sup\limits_{\|a\| \leq 1} ax$.
We have also $u^*(y^*) = ay^*$ $\forall u$ $(u(x) = ax) \in \nabla T$.
Then, for all $y^* \in \mathbb{R}^*$ we have

$$\begin{array}{rcl} \langle u(x), y^* \rangle & = & \langle ax, y^* \rangle, \\ & = & axy^* \\ & = & xay^* \\ & = & \langle x, ay^* \rangle \\ & = & \langle x, u^*(y^*) \rangle . \end{array}$$

Hence

$$\begin{array}{rcl} T^*(y^*) & = & \sup\limits_{u \in \nabla T} u^*(y^*), \\ & = & \sup\limits_{\|a\| \leq 1} \alpha y^*, \\ & = & |y^*| = |y| . \end{array}$$

T^* is also a sublinear operator from $\mathbb{R}^* = \mathbb{R}$ to $\mathbb{H}^* = \mathbb{H}$.
3- We can consider the open problem 5.2 a example when we illustrated the greatest importance of the introduction of adjoint sublinear operator in some theory demonstrations.

Proposition 4.6. *Let X be a (Köthe function space) Banach lattice, Y be a (complete Köthe function space) complete Banach lattice. Let T be a bounded sublinear operator from X into Y. We can see that:*
1- We have for all x in X^+ and y^* in Y^*

$$|\langle T(x), y^* \rangle| \leq |\langle x, T^*(y^*) \rangle| + |\langle x, T^*(-y^*) \rangle| .$$

And for all x in X^+ and y^* in Y^*_+

$$|\langle T(x), y^* \rangle| \ \ \leq \ \ |\langle x, T^*(y^*) \rangle| .$$

2- For all y^* in Y^* there is u_{y^*} in $SB(X,Y)$ with u_{y^*} in $\nabla(T^{**}/X)$ such that

$$T^*(y^*) = u^*_{y^*}(y^*).$$

Proof.
1- Let x in X^+. By proposition 3.3, for all x in X there is $u_x \in \nabla T$ such that, $T(x) = u_x(x)$, $u_x \in \nabla T$.
Let $y^* \in Y^*$. We have

$$\begin{array}{rcl} \langle T(x), y^* \rangle & = & \langle u_x(x), y^* \rangle, \\ & = & \langle x, u_x^*(y^*) \rangle, \\ \text{(by (2.1))} & \leq & \langle x, T^*(y^*) \rangle. \end{array}$$

This implies (we replace y^* by $-y^*$) for all x in X^+ and y^* in Y^*

$$\langle T(x), -y^* \rangle \leq \langle x, T^*(-y^*) \rangle.$$

Then

$$\begin{array}{rcl} |\langle T(x), y^* \rangle| & \leq & \sup \{ \langle x, T^*(y^*) \rangle, \langle x, T^*(-y^*) \rangle \}, \\ & \leq & |\langle x, T^*(y^*) \rangle| + |\langle x, T^*(-y^*) \rangle|. \end{array}$$

By (2.2) for all y^* in Y^*

$$\begin{array}{rcl} \langle -T(x), y^* \rangle & \leq & \langle T(-x), y^*) \rangle, \\ & \leq & \langle x, u_x^*(y^*) \rangle, \\ \text{(by (2.1))} & \leq & |\langle x, T^*(y^*) \rangle|. \end{array}$$

and

$$\langle T(x), y^* \rangle \leq \langle x, T^*(-y^*) \rangle.$$

This implies that for all x in X^+ and y^* in Y^*_+

$$|\langle T(x), y^* \rangle| \leq |\langle x, T^*(y^*) \rangle|.$$

For $x < 0$,

$$\langle -T(x), y^* \rangle \leq \langle T(x), y^* \rangle \leq |\langle x, T^*(y^*) \rangle|.$$

2- We have by proposition 3.3

$$\forall y^* \in Y^* \quad \exists v_{y^*} \in \nabla T^* : T^*(y^*) = v_{y^*}(y^*).$$

We search

$$u_{y^*} : X \to Y \text{ such that } T^*(y^*) = u^*_{y^*}(y^*).$$

Let y^* be in Y^* and v_{y^*} be in ∇T^* such that

$$T^*(y^*) = v_{y^*}(y^*)$$

Let $v^*_{y^*} : X^{**} \to Y^{**}$ and put $u_{y^*} = (v^*_{y^*})_{/X}$ which is a bounded linear operator from X into Y. We have $u^*_{y^*} = v_{y^*}$. Indeed, let z^* be in Y^* and $x \in X$. We have

$$\begin{aligned} \langle u^*_{y^*}(z^*), x\rangle &= \langle z^*, u_{y^*}(x)\rangle, \\ &= \langle z^*, (v^*_{y^*})_{/X}(x)\rangle, \\ &= \langle z^*, v^*_{y^*}(x)\rangle, \\ &= \langle v_{y^*}(z^*), x\rangle. \end{aligned}$$

This implies that $u^*_{y^*} = v_{y^*}$. We don't know if $u_{y^*} \in \nabla T$ or not.

Remark 4.7. *Let E be a finite dimensional Banach lattice, Y be a complete Banach lattice and $T : E \longrightarrow Y$ be a bounded sublinear operator. We have by proposition 3.3, for all $y^* \in Y^*$ there is $v_{y^*} \in \nabla T^*$ such that $T^*(y^*) = v_{y^*}(y^*)$, where $v_{y^*} : Y^* \longrightarrow E^*$. First we notice that $B(E, Y^{**}) \equiv B(E, Y)^{**} \equiv B(E^*, Y^*)$ isometrically. Since $v_{y^*} : Y^* \longrightarrow E^*$ is in $B(E, Y)^{**}$, then by Goldstine's theorem, there is a set of operators $u^*_i : Y^* \longrightarrow E^*$ which are w^*-continuous with $\|u_i\| \leq \|v_{y^*}\|$ for all i and $\{u^*_i(y^*)\}$ converges to $v_{y^*}(y^*)$ in w^*-topology of $B(E, Y)^{**}$.*

Theorem 4.8. *Let X be a (Köthe function space) Banach lattice, Y be a (complete Köthe function space) complete Banach lattice. Let T be a bounded sublinear operator from X into Y. Then we have*

$$\textit{1)-} \sup_{u \in \nabla T} \|u^*\| \leq 2 \|T^*\|, \tag{4.4}$$

and consequently

$$\textit{2)}-\frac{1}{2}\|T\| \leq \|T^*\|. \tag{4.5}$$

Proof.

1)- Let y^* be in Y^* and u be in ∇T ($\Longrightarrow u^*$ in ∇T^*). We have

$$|u^*(y^*)| \leq |T^*(y^*)| + |T^*(-y^*)|$$

hence

$$\|u^*(y^*)\| \leq 2\|T^*\|\|y^*\|$$

and consequently

$$\sup_{u\in\nabla T}\|u^*\| \leq 2\|T^*\|.$$

(2)- We use (1), we have

$$\begin{aligned}\|T\| &\leq \sup_{u\in\nabla T}\|u\| \text{ by (3.5)}\\ &\leq \sup_{u\in\nabla T}\|u^*\|\\ &\leq 2\|T^*\|.\end{aligned}$$

Proposition 4.9. *Let X be a (Köthe function space) Banach lattice, Y be a (complete Köthe function space) complete Banach lattice. Let T be a bounded sublinear operator from X into Y. Then, we have*

$$T \leq T^{**}_{/X}. \tag{4.6}$$

Proof.
Consider z in X^{**}. We have by definition

$$T^{**}(z) = \sup_{v\in\nabla T^*} v^*(z).$$

If we take x in X we have also

$$T^{**}(x) = \sup_{v\in\nabla T^*} v^*(x).$$

As the net $\{u^*\}_{u\in\nabla T} \subset \nabla T^*$, we obtain

$$\sup_{u\in\nabla T} u^{**}(x) \leq \sup_{v\in\nabla T^*} v^*(x)$$

and hence

$$\sup_{u\in\nabla T} u(x) \leq \sup_{v\in\nabla T^*} v^*(x).$$

Consequently

$$T(x) \leq \sup_{v\in\nabla T^*} v^*(x) = T^{**}(x), \quad \forall x \in X,$$

then the result.

Corollary 4.10. *Let X be a (Köthe function space) Banach lattice, Y be a (complete Köthe function space) complete Banach lattice. Let T be a bounded sublinear operator from X into Y. Then, we have*

$$\nabla T \subset \nabla T^{**}_{/X}. \tag{4.7}$$

Proof.
Let $u \in \nabla T$. This implies by the above proposition that $u(x) \leq T(x) \leq T^{**}(x)$, for all $x \in X$.

5 Open Problems

Problem 5.1. Let X be a Banach lattice, Y and Z be two complete Banach lattice spaces , T be a bounded sublinear operator from X into Y. We don't know yet if T is bounded T^* is it bounded.

Problem 5.2. The concept of strongly $p-$summing linear operators ($1 \leq p < \infty$) was introduced by J. S. Cohen [4] as a characterization of the conjugates of absolutely p^*-summing linear operators. Cohen deduced the domination theorem simply from the adjoint operator which is p^*-summing. If the answer of problem 4.1 is positive " we think with addition of some conditions on spaces X and Y " it possible to use it in the proof of the sublinear case.

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