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Optimal Strategy Analysis of an N-policy Two-phase $M^X/M/1$ Gated Queueing System with Server Startup and Breakdowns

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Abstract

This paper deals with the optimal operation of a single removable and unreliable server in an N- policy two-phase $M^X/M/1$ queueing system with gating, server startups and unpredictable breakdowns. Arrivals occur in batches according to a compound Poisson process and waiting customers receive batch service all at a time in the first phase and proceed to the second phase to receive individual service. Customers who arrive during the batch service are not allowed to enter the same batch. As soon as the system becomes empty the server leaves for a vacation of random length. When the queue length reaches a threshold value N ($N \geq 1$) the server is immediately turned on but is temporarily unavailable to serve the waiting batch of customers. The server needs a startup time before providing batch service in the first phase. The server is subject to breakdowns during individual service according to a Poisson process and repair times of the server follow an exponential distribution. Explicit expressions for the steady state distribution of the number of customers in the system and hence the expected system length is derived. The total expected cost function is developed to determine the optimal threshold of N at a minimum cost. Numerical experiment is performed to validate the analytical results. The sensitivity analysis has been carried out to examine the effect of different parameters in the system.

Key words: *Vacation, N-policy, Queueing System, Two-phase, Startup, Breakdowns, Gating.*

1 Introduction

Queueing models with two phases of service and server vacations have several applications in many areas such as in computer network administration, in telecommunication systems – messages are processed in two phases by a single server and in inventory control processes – due date, quantity and quality are analyzed in batch mode followed by individual service of the batch. Out of several such situations, some of them which have wider applicability in manufacturing and in pharmaceutical sector are:

1. Consider a manufacturing system where items are produced in bulk. There after these items need individual attention, such as fine polishing, providing with individual facilities. After individual service to all items in the batch, the bulk production process commences again if there is at least one order, failing which the system remains idle and restarts only when the order level reaches a specific limit. Before each production cycle, the manufacturing process may need startup time and the facility may need repair.

2. The case of manufacturing bulk drugs in pharmaceutical industry is another classical example where the production cannot commence until the order quantity reaches specific requirement. Once the specific requirement condition is met, the drug is manufactured in the first phase and then it is tested to the specifications in the second phase well before it is released to the customer. After the second phase, the production process commences even if there is a single order, failing which the system remains in idle or no production state. The production cannot commence until the ordered quantity reaches specific limit. The production process needs startup period after an idle state.

The examples are many. However, the above said examples are only few out of many that are encountered in several industrial and real life situations. All such examples provide an insight for the optimal control policy at every stage of processing. Several attempts have been made by many investigators to provide a valid and a meaningful solution so as to estimate the optimal control policy. Out of several those who made an attempt to study the system, Krishna and Lee [20], Doshi [8] studied distributed systems where all customers waiting in the queue receive batch service in the first phase followed by individual service in the second phase. Selvam and Sivasankaran [25] introduced two-phase queueing system with server vacation.

For the control policy of vacation queues it is usually assumed that the server becoming available or unavailable depends completely on the number of customers in the system. Every time when the system is empty, the server goes on a vacation. The instance at which the server comes back from a vacation and finds at least N customers in the system, it begins serving immediately and exhaustively. This type of control policy is also called N-policy queueing system with vacations. Kim and Chae [19] analyzed the two-phase queueing system with N-policy. Vasanta Kumar and Chandan [29] and [30] presented the optimal control policy of two-phase M/M/1 and $M^X/E_K/1$ queueing systems with N-policy.

The server startup corresponds to the preparatory work of the server before starting the service. In some of the practical situations that are encountered in the real life situation involves the server often requires a startup before startup of each service period. Concerning to queueing systems with startup time, Baker [3] first proposed the N-policy M/M/1 queueing system with startup time. Later, Borthakur et al. [4] extended Baker's result to the general startup time. The N-policy M/G/1 queueing system with startup time was first studied by Minh [24] and was investigated by several researchers. Medhi and Templeton [23] extended Minh's result to the general startup time. Lee and Park [22] introduced the early startup for the N-policy system, i.e., the server starts his setup when $m < N$ customers are waiting. If there are still less than N customers after setup, he waits until N customers accumulate in the system. Krishna Reddy et al. [21] have analyzed a bulk queueing model and multiple vacations with setup time. They derived the expected number of customers in the queue at an arbitrary epoch and obtained other measures. Hur and Paik [11] examined the operating characteristics of M/G/1 queueing system under N-policy with server startup and explained how the system's optimal policy and cost parameters behave for various arrival rates. Ke [13] presented the optimal control of an $M^X/G/1$ queue with server startup and two vacation types. Arumuganathan and Jayakumar [2] presented the steady state analysis of $M^X/G(a,b)/1$ queueing system with multiple vacations, closedown times and setup time with N-policy.

Queueing models with server breakdowns and vacations have also been investigated in different frame works in recent past. The unreliable server is commonly found in computer systems, communication systems and in manufacturing systems and other day to day realistic queueing problems. Jayaraman et al. [12] considered a single-server queueing system with arrival rate dependent on server breakdowns and with general bulk service. The duration of the operating periods and repair periods follow exponential and phase-type distributions respectively. Wang [31] first proposed an Markovian queueing system under the N-policy with server breakdowns and was studied by several researchers. Wang [32], Wang et al. [33] and Wang et al. [35] examined the model proposed by Wang [31] to $M/E_k/1$, $M/H_2/1$and $M/H_K/1$ queueing systems respectively. They developed closed form solutions and provided a sensitivity analysis. Gray et al. [9] studied a multiple vacation queueing model with unreliable server and the arrivals follow Poisson process with rates depending upon the system state, namely, vacation or service or breakdown state. Wang and Ke [36] analyzed the control policies for the M/G/1 queueing system with an unreliable server. Ke [14] presented the control policy of a removable and unreliable server for an $M^X/M/1$ queueing system, where the removable server operates an N-policy and takes a sequence of exponentially distributed vacations until the queue length reaches a predetermined number. Ke [15] studied an $M^X/G/1$ queue under bi-level control policy, where an unreliable server operates N-policy with an early startup. Ke [16] studied a variant T-policy for an M/G/1 queueing system with an unreliable server may take at most J vacations repeatedly until at least one customer appears in the queue upon returning from a vacation, and the server needs a startup time before starting each of his service periods. Ke [17] presented the control policies of an unreliable server, in which the length of the vacation period is controlled either by the number of arrivals during the idle period, or by a timer. Sherman and Kharoufeh [26] analyzed an un-reliable M/M/1 retrial queue with orbit and normal queue having infinite storage capacity. Customers join the retrial orbit if and only if they are interrupted by a server breakdown. Arrival customers do not rejoin the normal queue, but rather attempt to access the server directly at random intervals independently of arrivals or other retrial customers. However, these interrupted customers can regain access to the server only when it is operational and idle and repeat service until they have been successfully processed. Wang et al. [38] presented the optimal control of the N-policy M/G/1 queueing system with a single removable and unreliable server. Service times, repair times and startup times are assumed to be generally distributed. Anantha Lakshmi et al. [1] presented the optimal control strategy of an N-policy bulk arrival queueing system with server startup and breakdowns. Ke and Lin [18] studied the $M^X/G/1$ queueing systems under N-policy with an unreliable server and single vacation. The maximum entropy approach is used to examine the steady-state probability distribution, because the exact probability distributions of the variant vacation queueing system are difficult to be obtained. Wang and Huang [37] used a maximum entropy approach to study a single removable and unreliable server in an M/G/1 queue operating under the (p, N)-policy. Wang et al. [39] investigated the T-policy M/G/1 queue with server breakdowns and startup times. Tadj and Choudhury [28] analyzed a bulk service quorum queueing system with an unreliable server, Poisson input and general service and repair times.

Wang [34] considered an M/G/1 model with an additional second phase of optional service with the assumption that the server is subject to breakdowns and repairs in which he assumed that second optional service times follows an exponential distribution. Choudhury and Deka [5] generalized this model by introducing the concept of repeated attempts. Choudhury and Tadj [6] generalized this type of model by introducing the concept of a delayed repair. Later Choudhury et al. [7] investigated such a type of model, where concept of N-policy is also investigated along with a delayed repair for batch arrival queueing system.

Existing research works for the N-policy two-phase systems including those mentioned above, have never investigated cases involving server vacations, startups and breakdowns.

In the present investigation we develop an N-policy $M^X/M/1$ gated queueing model with two-phases of service, setup times and unreliable server. Customers arrive in batches of random size according to a Poisson process and waiting customers receive batch service all at a time in the first phase and are served individually in the second phase. By gating we mean that the customers who arrive during batch service are not allowed to enter the batch which is in service, but are served during the next visit of the server to the batch service. The server is turned off each time the system empties, as and when the queue length reaches or exceeds N, batch service starts. Before the batch service, the system requires a random startup time for pre-service. As soon as the startup period is over the server starts the batch service followed by individual service to all customers in the bath. Whenever the server is working in the individual queue, it is assumed that the server can breakdown at any time.

The four main objectives for which the analysis has been carried out in this paper for the optimal control policy are:

(i) To establish state equations to obtain the steady state probability distribution of the number of customers in the system.
(ii) To derive expected number of customers in the system when the server is in vacation, in startup, in batch service, in individual service and breakdown states respectively for the three batch size distributions viz. Deterministic, Geometric and positive Poisson [27].
(iii) To formulate the total expected cost function for the system, and determine the optimum value of the control parameter N.
(iv) To carryout a sensitivity analysis of the optimal value of N and the minimum expected cost for various system parameters through numerical illustrations.

2 The System and Assumptions

In the fitness of realistic situation and under the assumptions stated above, it is more appropriate to consider that the customer arrivals occur according to a compound Poisson process with arrival rate λ and arriving customers form the batch queue. The sizes of successive arriving batches are X_1, X_2, . . . ; where X_1, X_2, . . . is a sequence of independent and identically distributed random variables with probability mass function $\{a_n = P_r(X = n);\ n \geq 1\}$. The first phase of service is a batch service to all customers waiting in the queue and the second phase of service is individual to each customer in the batch. On completion of individual service, the server returns to the batch queue to serve the customers who have arrived. If the customers are waiting, the server restarts the cycle by providing them batch service followed by individual service. If no customer is waiting, the server takes a vacation and return from vacation only after N customers accumulate in the batch queue and will spend a random time 't' for pre-service, which is assumed to follow an exponential distribution with mean $1/\theta$. As soon as the startup period is over, the server begins batch service to all customers waiting in the queue in the first phase. Arrivals during batch service are not served in the batch which is in service, but are served in the next visit of the server to the batch queue. The batch service time is assumed to be exponentially distributed with mean $1/\beta$ and is independent of batch size. Individual service is also assumed to be exponentially distributed with mean $1/\mu$. While serving in individual queue, the server may fail at any time with a Poisson breakdown rate α. When the server fails, it is immediately repaired, where the repair times are exponentially distributed with mean $1/\gamma$. After repair the server immediately resumes service in individual queue.

3 Steady – State Results

In steady – state the following notations are used.

$P_{0,i,0}$ = The probability that there are i customers in the batch queue when the server is on vacation, where i = 0,1,2,3, … , (N – 1).

$P_{1,i,0}$ = The probability that there are i customers in the batch queue when the server is doing pre-service (startup work), where i = N, N+1, N+2,…

$P_{2,i,0}$ = The probability that there are i customers in the batch queue when the server is in batch service where i = 1,2,3,…

$P_{3,i,j}$ = The probability that there are i customers in the batch queue and j customers in individual queue when the server is in individual service, where i = 0,1,2,… and j = 1,2,3,…

$P_{4,i,j}$ = The probability that there are i customers in the batch queue and j customers in individual queue when the server is in individual service but found to be broken down, where i = 0,1,2,… and j = 1,2,3, …

The steady-state equations satisfied by the system size probabilities are as follows :

$$\lambda\, P_{0,0,0} = \mu\, P_{3,0,1}\,. \quad (1)$$

$$\lambda\, P_{0,i,0} = \lambda \sum_{k=1}^{i} a_k P_{0,i-k,0}\,,\ 1 \le i \le N-1\,. \quad (2)$$

$$(\lambda+\theta)\, P_{1,N,0} = \lambda \sum_{k=1}^{N} a_k P_{0,N-k,0} \quad (3)$$

$$(\lambda+\theta)\, P_{1,i,0} = \lambda \sum_{k=1}^{i-N} a_k P_{1,i-k,0} + \lambda \sum_{k=i-(N-1)}^{i} a_k P_{0,i-k,0}\,,\ i > N \quad (4)$$

$$\beta\, P_{2,i,0} = \mu\, P_{3,i,1},\ \ 1 \le i \le N-1 \quad (5)$$

$$\beta\, P_{2,i,0} = \mu\, P_{3,i,1} + \theta\, P_{1,i,0},\ \ i \ge N \quad (6)$$

$$(\lambda+\alpha+\mu)\, P_{3,0,j} = \mu\, P_{3,0,j+1} + \pi_0\, \beta\, P_{2,j,0} + \gamma\, P_{4,0,j},\ \ j \ge 1 \quad (7)$$

$$(\lambda+\alpha+\mu)\, P_{3,i,j} = \mu\, P_{3,i,j+1} + \gamma\, P_{4,i,j} + \pi_i\, \beta\, P_{2,j,0} + \lambda \sum_{k=1}^{i} a_k\, P_{3,i-k,j},$$
$$i \ge 1,\ j \ge 1 \quad (8)$$

$$(\lambda+\gamma)\, P_{4,0,j} = \alpha\, P_{3,0,j}\,,\ \ j \ge 1\,. \quad (9)$$

$$(\lambda+\gamma)\, P_{4,i,j} = \alpha\, P_{3,i,j} + \lambda \sum_{k=1}^{i} a_k\, P_{4,i-k,j}\,,\ \ i \ge 1,\ \ j \ge 1\,. \quad (10)$$

where $\pi_i = \dfrac{(\lambda A'(1))^i \beta}{(\lambda A'(1)+\beta)^{i+1}}$.

The following probability generating functions are defined:

$$G_0(z) = \sum_{i=0}^{N-1} P_{0,i,0}\, z^i,\ G_1(z) = \sum_{i=N}^{\infty} P_{1,i,0}\, z^i,\ G_2(z) = \sum_{i=1}^{\infty} P_{2,i,0}\, z^i,$$

$$G_3(z,y) = \sum_{i=0}^{\infty}\sum_{j=1}^{\infty} P_{3,i,j}\, z^i y^j,\ G_4(z,y) = \sum_{i=0}^{\infty}\sum_{j=1}^{\infty} P_{4,i,j}\, z^i y^j,\ R_j(z) = \sum_{i=0}^{\infty} P_{3,i,j}\, z^i,$$

$$S_j(z) = \sum_{i=0}^{\infty} P_{4,i,j}\, z^i \text{ and } A(z) = \sum_{i=1}^{\infty} a_i\, z^i,\ \text{ where } |z| \le 1 \text{ and } |y| \le 1.$$

$A(z)$ is the probability generating function of the number of customers in the arriving batch of size X. Then the expected size of the number of customers in an arriving batch is $E(X) = A'(1)$. The second factorial moment of X is $E(X(X-1)) = A''(1)$, where A' and A'' denote the first and second order derivatives of $A(z)$ with respect to z.

Using equation (2), we get

$$P_{0,i,0} = y_i\, P_{0,0,0}\,,$$

where y_i's are defined as $y_0 = 1$ and $y_i = \sum_{k=1}^{i} a_k\, y_{i-k}$. (11)

$$G_0(z) = \sum_{i=0}^{N-1} P_{0,i,0}\, z^i = P_{0,0,0} \sum_{i=0}^{N-1} y_i\, z^i = P_{0,0,0}\, Y_N(z),$$

where $Y_N(z) = \sum_{i=0}^{N-1} y_i\, z^i$ with $Y_N(1) = \sum_{i=0}^{N-1} y_i$ and $Y'_N(1) = \sum_{i=1}^{N-1} i\, y_i$. (12)

Multiplication of equation (3) by z^N and equation (4) by z^i and summing over i ($i \geq N$) yields

$$[\lambda\,(1-A(z)) + \theta]\, G_1(z) = \lambda\, P_{0,0,0} + [\lambda\,(A(z)-1)]\, G_0(z) \qquad (13)$$

Multiplication of equations (5) and (6) by z^i and adding over i ($i \geq N$) yields

$$\beta\, G_2(z) = \mu\, R_1(z) + \theta\, G_1(z) - \lambda\, P_{0,0,0} \qquad (14)$$

Multiplication of equation (8) by z^i ($i \geq 1$) and using (7) yiedlds

$$[\lambda\,(1-A(z)) + \alpha + \mu]\, R_j(z) = \mu\, R_{j+1}(z) + \gamma\, S_j(z) + \beta\, P_{2,j,0}\, \pi(z)$$

Multiplication of this equation by y^j and adding over j ($j \geq 1$) yields

$$[\lambda y\,(1-A(z)) + \alpha y + \mu(y-1)]\, G_3(z, y) = \gamma y\, G_4(z, y) + \beta y\, \pi(z)\, G_2(y) - \mu y\, R_1(z) \qquad (15)$$

Multiplication of equation (10) by z^i ($i \geq 1$) and using (9) yields

$$[\lambda\,(1-A(z)) + \gamma]\, S_j(z) = \alpha\, R_j(z)$$

Multiplication of this equation by y^j and adding over j ($j \geq 1$) yields

$$[\lambda\,(1-A(z)) + \gamma]\, G_4(z, y) = \alpha\, G_3(z, y) \qquad (16)$$

The total probability generating function G (z, y) is given by

$$G(z, y) = G_0(z) + G_1(z) + G_2(z) + G_3(z, y) + G_4(z, y)\,.$$

The normalizing condition is

$$G(1, 1) = G_0(1) + G_1(1) + G_2(1) + G_3(1, 1) + G_4(1, 1) = 1. \qquad (17)$$

From equations (12) to (16)

$$G_0(1) = y_N(1)\, P_{0,0,0}\,, \qquad (18)$$

$$G_1(1) = (\lambda/\theta)\, P_{0,0,0}\,, \qquad (19)$$

$$G_2(1) = \mu\, R_1(1)/\beta\,, \qquad (20)$$

$$G_3(1,1) = \frac{[\theta G_1'(1) + \lambda A'(1) G_2(1)]\gamma}{[\mu\gamma - \lambda A'(1)(\alpha+\gamma)]\beta} \tag{21}$$

and $G_4(1,1) = (\alpha/\gamma)\, G_3(1,1)$, (22)

where $P_{0,0,0} = \left[1 - \left(\left(\frac{\lambda}{\beta} + \frac{\lambda}{\mu}\left(1+\frac{\alpha}{\gamma}\right)\right) A'(1)\right)\right] \div \left(\lambda/\theta + y_N(1)\right)$.

Normalizing condition (17) yields

$$R_1(1) = \lambda A'(1)/\mu .$$

Substituting the value of $R_1(1)$ in (20) and (21) yields

$$G_3(1,1) = \lambda\, A'(1)/\mu$$

and $G_4(1,1) = (\alpha/\gamma)[\lambda A'(1)/\mu]$. .

Under steady state conditions, let P_0, P_1, P_2, P_3 and P_4 be the probabilities that the server is in vacation, in startup, in batch service, in individual service and breakdown states respectively. Then,

$$P_0 = G_0(1) = y_N(1)\, P_{0,0,0} \tag{23}$$

$$P_1 = G_1(1) = \lambda P_{0,0,0}/\theta \tag{24}$$

$$P_2 = G_2(1) = \lambda\, A'(1)/\beta \tag{25}$$

$$P_3 = G_3(1,1) = \lambda\, A'(1)/\mu \tag{26}$$

and $P_4 = G_4(1,1) = (\alpha/\gamma)(\lambda A'(1)/\mu)$ (27)

3.1 Expected number of customers in the system

Using the probability generating functions expected number of customers in the system at different states are presented below.

Let L_0, L_1, L_2, L_3 and L_4 be the expected number of customers in the system when the server is in idle, startup, in batch service, in individual service and breakdown states respectively.

Then $L_0 = \sum_{i=0}^{N-1} i\, P_{0,i,0} = G_0'(1) = Y'_N(1)\, P_{0,0,0}$. (28)

$$L_1 = \sum_{i=N}^{\infty} i\, P_{1,i,0} = G_1'(1) = \frac{\lambda(\lambda + Y_N(1)\theta)\, A'(1)}{\theta^2}\, P_{0,0,0}\,. \quad (29)$$

$$L_2 = \sum_{i=1}^{\infty} i\, P_{2,i,0} = G_2'(1) = \lambda\, A'(1)/\beta\,. \quad (30)$$

$$L_3 = \sum_{i=0}^{\infty}\sum_{j=1}^{\infty} (i+j)\, P_{3,i,j} = G_3'(1,1)$$

$$= \rho\left[\frac{1}{1-\rho_1} + \frac{\lambda A'(1)\left(\lambda + \theta y_N(1)\right)}{\theta^2(1-\rho_1)} P_{0,0,0} + \frac{\lambda\alpha\rho A'(1)}{\gamma^2(1-\rho_1)} + \frac{y_N'(1)}{(1-\rho_1)} P_{0,0,0}\right.$$

$$\left. + \frac{A''(1)}{2(1-\rho_1)A'(1)} + \frac{\lambda A'(1)}{\beta(1-\rho_1)} + \frac{(\lambda A'(1))^2}{\beta^2(1-\rho_1)} - \frac{\lambda A''(1)}{2\beta(1-\rho_1)}\right] \quad (31)$$

and $L_4 = \sum_{i=0}^{\infty}\sum_{j=1}^{\infty} (i+j)\, P_{4,i,j} = G_4'(1,1) = \dfrac{\alpha}{\gamma}\left[G_3'(1,1) + \dfrac{\lambda\rho A'(1)}{\gamma}\right],$ (32)

where $\rho = \lambda A'(1)/\mu$, $\rho_1 = \rho(1 + a/\gamma)$

and $P_{0,0,0} = \left[1 - \rho_1 - \dfrac{\lambda A'(1)}{\beta}\right]\dfrac{\theta}{(\lambda + \theta y_N(1))}$.

The expected number of costumers in the system

$$L(N) = L_0 + L_1 + L_2 + L_3 + L_4$$

$$= \frac{y_N'(1)}{(1-\rho_1)} P_{0,0,0} + \frac{\lambda A'(1)(\lambda + \theta y_N(1))}{\theta^2(1-\rho_1)} P_{0,0,0} + \frac{\rho_1}{1-\rho_1} + \frac{\lambda A'(1)\alpha\rho}{\gamma^2(1-\rho_1)}$$

$$+ \frac{\rho_1 A''(1)}{2(1-\rho_1)A'(1)} + \frac{\lambda A'(1)}{\beta(1-\rho_1)} + \frac{(\lambda A'(1))^2\rho_1}{\beta^2(1-\rho_1)} - \frac{\lambda A''(1)\rho_1}{2\beta(1-\rho_1)}\,. \quad (33)$$

4 Characteristic Features of the System

Let E_0, E_1, E_2, E_3 and E_4 denote the expected length of idle period, startup period, batch service period, individual service period and breakdown period respectively. Then the expected length of a cycle is given by

$$E_C = E_0 + E_1 + E_2 + E_3 + E_4 \quad (34)$$

The long-run fractions of time the server is idle, in startup, in batch service, in individual service and breakdown states are respectively

$$\frac{E_0}{E_C} = P_0 = G_0(1) = y_N(1)\, P_{0,0,0} \quad (35)$$

$$\frac{E_1}{E_C} = P_1 = G_1(1) = \lambda P_{0,0,0} / \theta \tag{36}$$

$$\frac{E_2}{E_C} = P_2 = G_2(1) = \lambda A'(1) / \beta \tag{37}$$

$$\frac{E_3}{E_C} = P_3 = G_3(1) = \lambda A'(1) / \mu = \rho \tag{38}$$

and $$\frac{E_4}{E_C} = P_4 = G_4(1,1) = (\alpha / \gamma)\rho . \tag{39}$$

Expected length of idle period = $E_0 = y_N(1) / \lambda$.

Substituting this in equation (35)

$$E_C = \left(\frac{\lambda + \theta\, y_N(1)}{\theta} \right) \frac{1}{\lambda\left(1 - \rho_1 - \lambda A'(1)/\beta\right)} . \tag{40}$$

5 Determination of the Optimal Policy

We develop a steady state total expected cost function per unit time for the N-policy, two-phase $M^X/M/1$ queueing system with startup and server breakdowns, in which N is a decision variable. With the cost structure being considered, the objective is to determine the optimal operating N-policy so as to minimize this function.

Let

$C_h \equiv$ holding cost per unit time for each customer present in the system,

$C_o \equiv$ cost per unit time for keeping the server on and in operation,

$C_m \equiv$ startup cost per unit time per cycle,

$C_s \equiv$ setup cost per cycle,

$C_b \equiv$ breakdown cost per unit time for the unreliable server and

$C_r \equiv$ reward per unit time for the server being on vacation and doing secondary work.

The total expected cost function per unit time is given by

$$T(N) = C_h L(N) + C_0 \left(\frac{E_2 + E_3}{E_C} \right) + C_m \left(\frac{E_1}{E_C} \right) + C_b \left(\frac{E_4}{E_C} \right) + C_s \left(\frac{1}{E_C} \right) - C_r \left(\frac{E_0}{E_C} \right) . \tag{41}$$

From (37) to (39), it is observed that E_2 / E_C, E_3 / E_C and E_4 / E_C are not functions of decision variable N. Hence for determination of the optimal operating N-policy, minimizing T(N) in (41) is equivalent to minimizing

$$T_1(N) = C_h\left[\frac{y'_N(1)}{1-\rho_1}P_{0,0,0}\right] + C_m\left(\frac{\lambda}{\theta}P_{0,0,0}\right) + C_s\ (\lambda P_{0,0,0}) - C_r\ (y_N(1)P_{0,0,0})$$

$$= \frac{[1-\rho_1-\lambda A'(1)/\beta]}{y_N(1)+\lambda/\theta}\left[\frac{C_h\, y'_N(1)+(1-\rho_1)\{C_m(\lambda/\theta)+\lambda C_s - C_r\, y_N(1)\}}{(1-\rho_1)}\right]. \quad (42)$$

It is hard to prove that T_1 (N) is convex. But we now present a procedure that makes it possible to calculate the optimal threshold N^*.

Result

Under the long run expected average cost criterion, the optimal threshold N^* for the model is given by

$$N^* = \min.\left\{k \geq 1 / \sum_{j=0}^{k}(k-j)\,y_j + \frac{k\lambda}{\theta} > \frac{\lambda(1-\rho_1)}{C_h}\left(\frac{C_m - C_r}{\theta} + C_s\right)\right\}.$$

Proof

Let $P = \left((1-\rho_1-\lambda A'(1)/\beta\right)/\left(1-\rho_1\right)$, $J_k = \sum_{i=0}^{k} y_i + \frac{\lambda}{\theta}$ and $M_k = \sum_{i=0}^{k} i\, y_i$,

where k = 1, 2, 3, . . .

Then we have

$$T_1(k+1) - T_1(k)$$

$$= C_h P\left(\frac{M_k}{J_k} - \frac{M_{k-1}}{J_{k-1}}\right)$$

$$+ (1-\rho_1)P\left[C_m\frac{\lambda}{\theta} + \lambda C_s\left(\frac{1}{J_k} - \frac{1}{J_{k-1}}\right) - C_r\left(\frac{J_k - \lambda/\theta}{J_k} - \frac{J_{k-1}-\lambda/\theta}{J_{k-1}}\right)\right]$$

$$= C_h P\left[\frac{y_k(k J_k - M_k)}{J_k J_{k-1}} + (1-\rho_1)\left(C_m\frac{\lambda}{\theta} + \lambda C_s\right)\left(\frac{-y_k}{J_k J_{k-1}}\right) - \frac{C_r(\lambda/\theta)(y_k)}{J_k J_{k-1}}\right]$$

$$= \frac{y_k}{J_k J_{k-1}}(h_k),$$

where $H(k) = C_h(k J_k - M_k) - \lambda(1-\rho_1)\{(C_m - C_r)/\theta + C_s\}$.

Since $C_h(k J_k - M_k) > 0$ and $\frac{y_k}{J_k J_{k-1}} > 0$, the sign of $H(k)$ determines whether $T_1(K)$ increases or decreases.

Let m be the first k such that $H(k) > 0$. Then

$$H(m+1) = C_h P\big((m+1)J_{m+1} - M_{m+1}\big) - \lambda P(1-\rho_1)\big((C_m - C_r)/\theta + C_s\big)$$
$$= C_h P\big(mJ_m - M_m\big) - \lambda P(1-\rho_1)\big((C_m - C_r)/\theta + C_s\big) > 0\,.$$

Thus we see that $T_1(n) > T_1(m)$ for $n > m$.

Finally we have

$$N^* = \text{first } k \text{ such that } H(k) > 0$$
$$= \min\left\{ k \ge 1 \,/\, \sum_{j=0}^{k}(k-j)\,y_j + \frac{k\lambda}{\theta} > \frac{\lambda(1-\rho_1)}{C_h}\left(\frac{C_m - C_r}{\theta} + C_s\right) \right\}.$$

6 Specific Batch Size Distributions

To make a comparative study, when the arrival batch size follows Deterministic, Geometric and positive Poisson distributions the expected number of customers in the system are obtained below:

(i) For the Deterministic batch size distribution with parameter *m*, the generating function is $A(z) = z^m$.

This gives $A'(1) = m,\ A''(1) = m(m-1)$.

Expected number of costumers in the system

$$L(N) = \frac{m\,k\,(k+1)}{2(1-\rho_1)} P_{0,0,0} + \frac{\lambda\, m\,(\lambda + \theta\, k)}{\theta^2\,(1-\rho_1)} P_{0,0,0} + \frac{\rho_1}{1-\rho_1} + \frac{\lambda\, m\, \alpha\, \rho}{\gamma^2 (1-\rho_1)} + \frac{\rho_1 (m-1)}{2(1-\rho_1)}$$
$$+ \frac{\lambda\, m}{\beta(1-\rho_1)} + \frac{\lambda^2 m^2 \rho_1}{\beta^2 (1-\rho_1)} - \frac{\lambda m(m-1)\rho_1}{2\beta(1-\rho_1)}$$

where *k* = Integral part of $\left(\dfrac{N-1}{m}\right)$, $\rho = \dfrac{\lambda\, m}{\mu}$, $\rho_1 = \rho\left(1 + \dfrac{\alpha}{\gamma}\right)$, *m* is the mean size of the arrival batch and $P_{0,0,0} = \dfrac{\left[1 - \left(\rho_1 + \lambda m/\beta\right)\right]}{\left(k + \lambda/\theta\right)}$.

(ii) For the Geometric batch size distribution with parameter *p*, the generating function is

$$A(z) = p\left(z^{-1} - (1-p)\right)^{-1}.$$

This gives $A'(1) = 1/p$ and $A''(1) = 2(1-p)/p^2$.

Expected number of costumers in the system

$$L(N)=\frac{\sum_{i=1}^{N-1} i\,y_i}{(1-\rho_1)}P_{0,0,0}+\frac{\lambda\left(\lambda+\theta\sum_{i=0}^{N-1} y_i\right)}{\theta^2\,(1-\rho_1)\,p}P_{0,0,0}+\frac{\rho_1}{(1-\rho_1)}+\frac{\lambda\,\alpha\,\rho}{\gamma^2\,(1-\rho_1)p}+\frac{\rho_1(1-p)}{(1-\rho_1)p}$$
$$+\frac{\lambda}{\beta(1-\rho_1)p}+\frac{\lambda^2\rho_1}{p^2\beta^2(1-\rho_1)}-\frac{\lambda(1-p)\rho_1}{p^2\beta(1-\rho_1)}.$$

where $y_i=\sum_{k=1}^{i} a_k\,y_{i-k}$, $y_0=1$, $a_k=p(1-p)^{k-1}$, $\rho=\lambda/(p\mu)$, $1/p$ is the mean size of arrival batch and $P_{0,0,0}=\left[1-\left(\rho_1+\lambda/(\beta p)\right)\right]/\left(\sum_{i=0}^{N-1} y_i+\lambda/\theta\right)$.

(iii) For the positive Poisson batch size distribution with no batch size of zero, let $\{a_n\}$ represent a probability sequence that governs the batch size. The probability a_n is defined as

$$a_n=P(n;\delta)/\left[1-P(0;\delta)\right]=\frac{\delta^n e^{-\delta}/n!}{1-e^{-\delta}}.$$

where δ is a parameter. The average batch size m in this case will be given by,

$$E(X)=\delta/(1-e^{-\delta})=m.$$

The generating function is

$$A(z)=\frac{m\,e^{-\delta}}{\delta}\left(e^{\delta z}-1\right).$$

This gives $A'(1)=m$ and $A''(1)=m\delta$.

Expected number of costumers in the system

$$L(N)=\frac{\sum_{i=1}^{N-1} i\,y_i}{(1-\rho_1)}P_{0,0,0}+\frac{\lambda\,m\left(\lambda+\theta\sum_{i=0}^{N-1} y_i\right)}{\theta^2\,(1-\rho_1)}P_{0,0,0}+\frac{\rho_1}{(1-\rho_1)}+\frac{\lambda\,m\,\alpha\,\rho}{\gamma^2\,(1-\rho_1)}+\frac{\rho_1\,\delta}{2(1-\rho_1)}$$
$$+\frac{\lambda\,m}{\beta(1-\rho_1)}+\frac{\lambda^2 m^2\rho_1}{\beta^2(1-\rho_1)}-\frac{\lambda\,m\,\delta\,\rho_1}{2\beta(1-\rho_1)}$$

.

where $y_i = \sum_{i=0}^{k} a_k y_{i-k}$, $y_0 = 1$, $a_k = \left(\dfrac{e^{-\delta}}{\left(1-e^{-\delta}\right)}\right)\dfrac{\delta^k}{k!}$, $\rho = \dfrac{\lambda m}{\mu}$, m is the mean size of arrival batch and $P_{0,0,0} = \left[1-\left(\rho_1 + \lambda m/\beta\right)\right] / \left(\sum_{i=0}^{N-1} y_i + \lambda/\theta\right)$.

7 Sensitivity Analysis

A numerical illustration of the sensitivity analysis on the optimum thresholds N* and T(N*) based on changes in the specific values of system parameters for the three arrival batch size distributions viz. Deterministic, Geometric and positive Poisson is conducted in the following different cost cases:
Case 1: $C_h = 5$, $C_o = 100$, $C_m = 300$, $C_b = 125$, $C_r = 25$, $C_s = 500$
Case 2: $C_h = 5$, $C_o = 200$, $C_m = 500$, $C_b = 250$, $C_r = 50$, $C_s = 1250$
Case 3: $C_h = 5$, $C_o = 400$, $C_m = 800$, $C_b = 500$, $C_r = 100$, $C_s = 2500$
Case 4: $C_h = 10$, $C_o = 400$, $C_m = 800$, $C_b = 500$, $C_r = 100$, $C_s = 2500$
Case 5: $C_h = 50$, $C_o = 400$, $C_m = 800$, $C_b = 500$, $C_r = 200$, $C_s = 2500$

The joint optimum threshold values N* and the minimum expected cost for the above five cost cases are summarized in figure 1 for the three arrival batch size distributions viz. Deterministic, positive Poisson and Geometric and for various values of λ. As expected it is noticed from figure1 that N* shows increasing trend and T(N*) increase as λ increases.

The joint optimum threshold values N* and the minimum expected cost for the above four cost cases are summarized in figures 2, 3 and 4 for all the three arrival batch size distributions and for various values of μ, β and θ. It is evident from figures 2, 3 and 4 that N* shows an increasing trend and T(N*) decrease with increase in the values of μ, and N* is constant and T(N*) decrease with increase in the values of β and θ.

The joint optimum threshold value N* and the minimum expected cost T(N*) for the above five cost cases are summarized in figures 5 and 6 for all the three batch size distributions and for various values of γ and α. It is seen from figures 5 and 6 that, N* shows increasing trend in positive Poisson distribution and constant in the other arrival batch size distributions, where as T(N*) decreases slightly in all the three arrival batch size distributions with increase in the values of γ. N* is constant and T(N*) increases slightly with increase in the value of α.

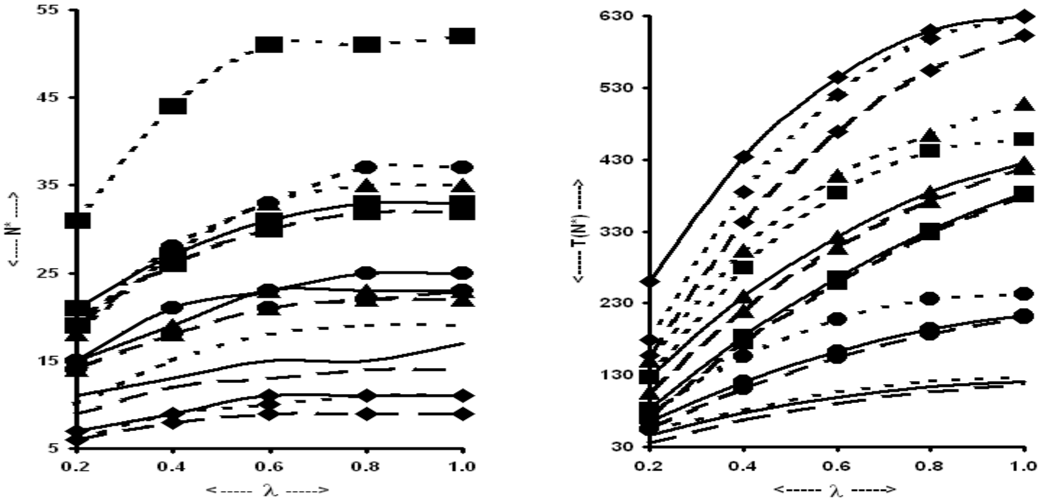


Fig. 1: The effect of λ on the optimum threshold value N* and the minimum expected cost for the five cost cases.
($\mu = 4, \beta = 7, \theta = 2, \alpha = 0.1, \gamma = 4, A'(1) = 2$)

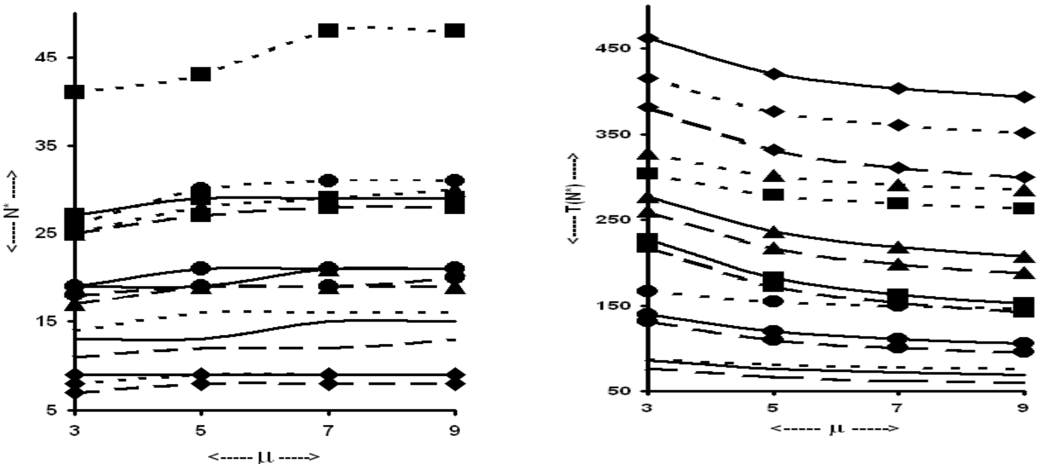
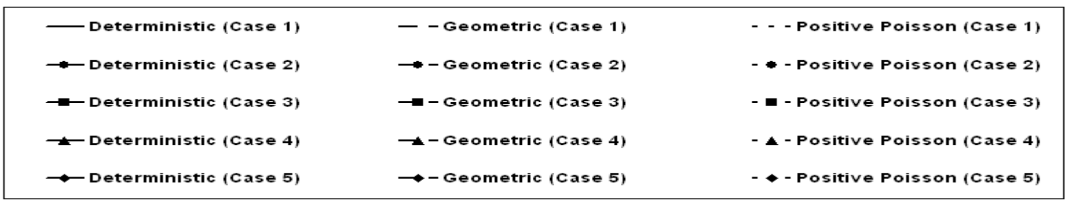


Fig. 2: The effect of μ on the optimum threshold value N* and the minimum expected cost for the five cost cases.($\lambda = 0.4, \beta = 5, \theta = 2, \alpha = 0.1, \gamma = 4, A'(1) = 2$)



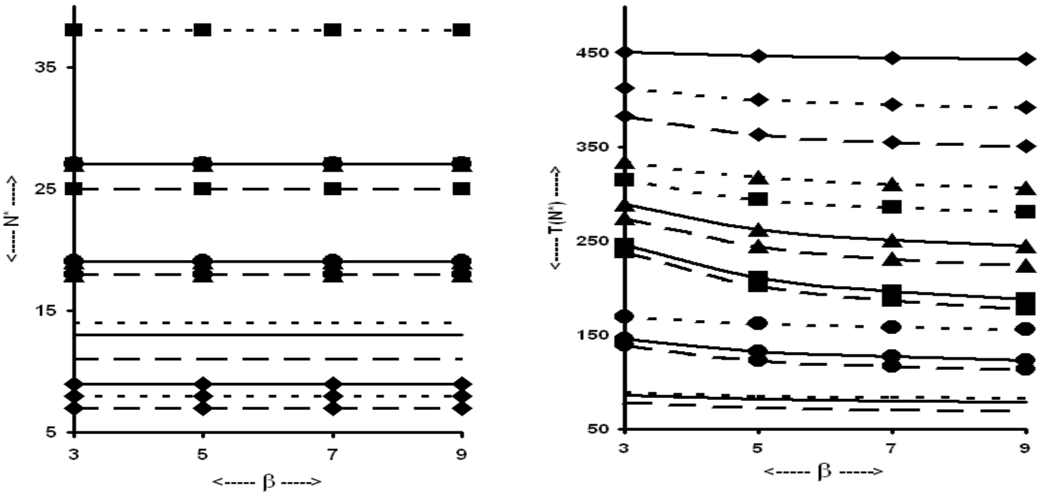


Fig. 3: The effect of β on the optimum threshold value N*and the minimum expected cost for the five cost cases.
($\lambda = 0.4$, $\mu = 3.5$, $\theta = 2$, $\alpha = 0.1$, $\gamma = 4$, $A'(1) = 2$)

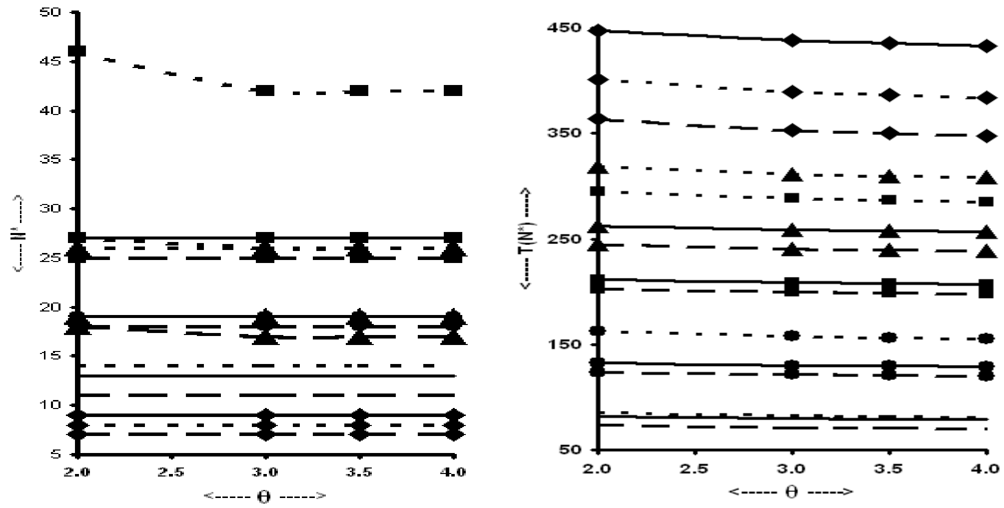
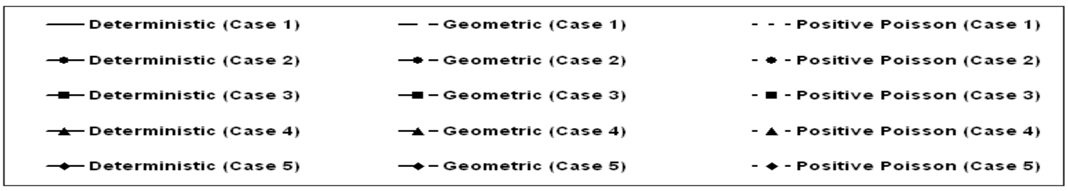


Fig. 4: The effect of θ on the optimum threshold value N* and the minimum expected cost for the five cost cases.
($\lambda = 0.4$, $\mu = 3.5$, $\beta = 5$, $\alpha = 0.1$, $\gamma = 4$, $A'(1) = 2$)



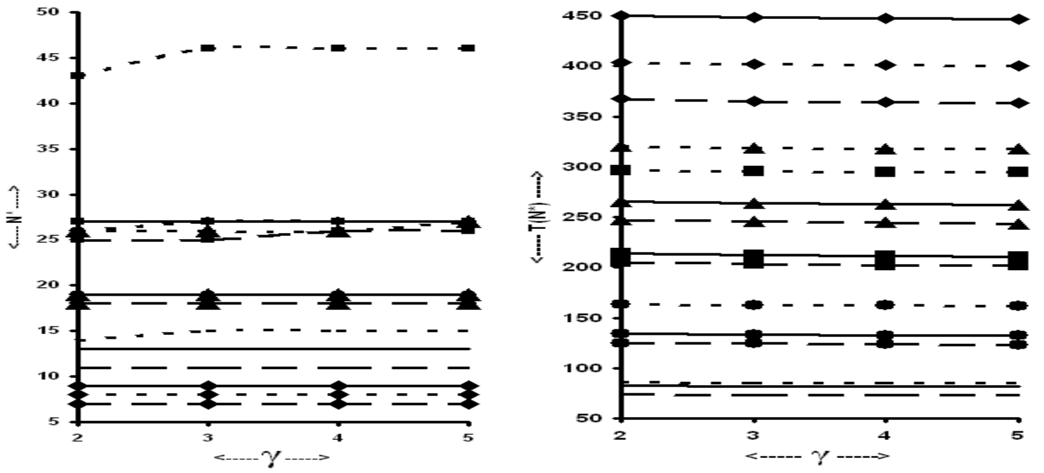


Fig. 5: The effect of γ on the optimum threshold value N* and the minimum expected cost for the five cost cases.
($\lambda = 0.4, \mu = 3.5, \beta = 5, \theta = 2, \alpha = 0.1, A'(1) = 2$)

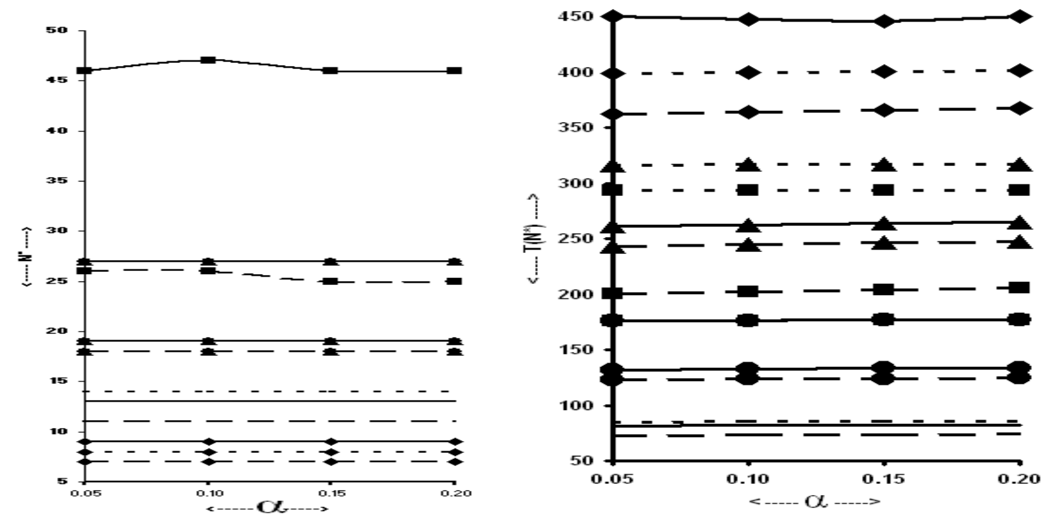
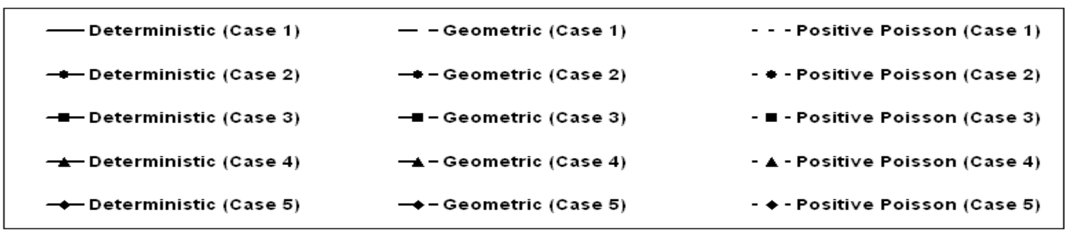


Fig. 6: The effect of α on the optimum threshold value N* and the minimum expected cost for the five cost cases.
$\lambda = 0.4, \mu = 3.5, \beta = 5, \theta = 2, \gamma = 4, A'(1) = 2$)



Overall we conclude that

- The minimum expected cost shows increasing and decreasing trends with $(\lambda, \alpha, C_o, C_m, C_b, C_s)$ and $(\mu, \beta, \theta, \gamma, C_r)$ in all the three batch size distributions.
- The optimum threshold value N* and minimum expected cost are least in Geometric distribution and greatest in positive Poisson distribution.
- With increase in the values of C_o, C_b, C_m, C_s and C_r, N* is convex.

8 Conclusions

Two-phase N-policy $M^X/M/1$ queueing system with startup times and server breakdowns is studied. Some of the system performance measures are derived. Cost function is formulated to determine the optimal value of N. Sensitivity analysis is performed for the optimum value of N, expected system length and expected cost with various system parameters and the cost elements. The optimal value of N, expected system length and expected cost are presented graphically for three specific batch size distributions viz. Deterministic, Geometric and positive Poisson. Many queueing systems of this type are special cases of our system. We look forward to the analysis of queueing systems with various vacation policies in addition to the N-policy.

9 Open Problem

This model can be extended to study other related models such as individual service in *k* exponential phases, general service and delayed repair etc.

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