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Lie Ideals and Generalized Derivations of Rings With Involution

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Abstract

Let $(R, *)$ ***be a ring with involution. The main purpose of this paper is to investigate generalized derivations satisfying certain identities on Lie ideals of*** R***. Furthermore, some well known results for generalized derivations in*** $*$***-prime rings as well as in prime rings are extended to Lie ideals. Finally, examples are given to demonstrate that the restrictions imposed on the hypothesis of the various theorems were not superfluous.***

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1 Introduction

Throughout R will denote an associative ring with center $Z(R)$, not necessarily with an identity element. For any $x, y \in R$, the symbol $[x, y]$ stands for the commutator $xy - yx$. We shall make extensive use of the following basic commutator identities without any specific mention: $[xy, z] = x[y, z] + [x, z]y$, $[x, yz] = y[x, z] + [x, y]z$. Recall that R is prime if $aRb = 0$ implies $a = 0$ or $b = 0$. If R admits an involution $*$, then R is $*$-prime if $aRb = aRb^* = 0$ yields $a = 0$ or $b = 0$. Clearly, a prime ring admitting an involution $*$ is $*$-prime but the converse is in general not true. Indeed, if R^o denotes the opposite ring of a prime ring R, then $R \times R^o$ equipped with the exchange involution τ_{ex},

defined by $\tau_{ex}(x,y) = (y,x)$, is τ_{ex}-prime but not prime. This example shows that every prime ring can be injected in a $*$-prime ring and from this point of view $*$-prime rings constitute a more general class of prime rings.
An additive subgroup U of R is said to be a Lie ideal of R if $[u,r] \in U$ for all $u \in U$ and $r \in R$. We will say that a Lie ideal U is a 2-Lie ideal if $2uv \in U$ for all $u,v \in U$. The fact that $2r[u,v] = 2[ru,v] - 2[r,v]u$ implies that $2r[u,v] \in U$ for all $u,v \in U$ and $r \in R$. If U is a Lie (resp. 2-Lie) ideal such that $U^* = U$ then U is called a $*$-Lie (resp. $(*,2)$-Lie) ideal.
An additive mapping $d: R \to R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all x,y in R. An additive mapping $F: R \to R$ is called a generalized derivation if there exists a derivation $d: R \to R$ such that $F(xy) = F(x)y + xd(y)$ holds for all $x,y \in R$. Hence generalized derivation covers both the concepts of derivation and generalized inner derivation. Furthermore, generalized derivation with $d = 0$ covers the concept of left multipliers.
Many analysts have studied generalized derivations in the context of algebras on certain normed spaces (see [6] for references). Moreover, there has been considerable interest concerning the relationship between the commutativity of a prime ring R and the behavior of generalized derivations of R. In [12], it is proved that if R is a 2-torsion free $*$-prime ring and I is a nonzero $*$-ideal of R and F is a generalized derivation of R, associated with a derivation d which commutes with $*$, such that one of the following conditions holds, then R is commutative:

(1) $F(xy) - xy \in Z(R)$ (2) $F(xy) + xy \in Z(R)$ (3) $F(xy) - yx \in Z(R)$

(4) $F(xy) + yx \in Z(R)$ (5) $F(x)F(y) - xy \in Z(R)$ (6) $F(x)F(y) + xy \in Z(R)$

for all $x,y \in I$. In this paper we extend some results of [1] and those of [12] to Lie ideals. Moreover, examples are given to prove that the $*$-primeness hypothesis in the various theorems were not superfluous.

2 The results

Throughout, $(R,*)$ will be a 2-torsion free ring with involution and $Sa_*(R) := \{r \in R / r^* = \pm r\}$ the set of symmetric and skew symmetric elements of R. We begin with the following lemmas which are essential for developing the proof of our results.

Lemma 2.1 ([7], Lemma 4) *If $U \not\subseteq Z(R)$ is a $*$-Lie ideal of a 2-torsion free $*$-prime ring R and $a,b \in R$ such that $aUb = a^*Ub = 0$, then $a = 0$ or $b = 0$.*

Lemma 2.2 ([8], Lemma 2.3) *Let $0 \neq U$ be a $*$-Lie ideal of a 2-torsion free $*$-prime ring R. If $[U,U] = 0$, then $U \subseteq Z(R)$.*

Lemma 2.3 ([9], Lemma 2.2) *Let R be a 2-torsion free $*$-prime ring and U a nonzero $*$-Lie ideal of R. If d is a derivation of R such that $d(U) = 0$, then either $d = 0$ or $U \subseteq Z(R)$.*

Theorem 2.4 *Let U be a $(*, 2)$-Lie ideal of R and F a generalized derivation associated with a nonzero derivation d such that For each $x, y \in U$ either $F(xy) - xy \in Z(R)$ or $F(xy) + xy \in Z(R)$. If R is $*$-prime, then $U \subseteq Z(R)$.*

Proof. Assume that $U \not\subseteq Z(R)$. Let U_1 and U_2 be the subgroups of U defined by $U_1 = \{x \in U/F(xy) - xy \in Z(R)$ for all $y \in U\}$ and $U_2 = \{x \in U/F(xy) + xy \in Z(R)$ for all $y \in U\}$. Since $U = U_1 \cup U_2$, by hypothesis, and as a group cannot be a union of two of its proper subgroups, then $U = U_1$ or $U = U_2$. Thus $F(xy) - xy \in Z(R)$ for all $x, y \in U$ or $F(xy) + xy \in Z(R)$ for all $x, y \in U$. Suppose that $F(xy) - xy \in Z(R)$ for all $x, y \in U$; then we obtain

$$F(x)y + xd(y) - xy \in Z(R) \quad \text{for all } \ x, y \in U. \tag{1}$$

Writing $2xz$ instead of x in (1), we get

$$F(x)zy + xd(z)y + xzd(y) - xzy \in Z(R),$$

so that $[(F(x)z + xd(z) - xz)y + xzd(y), y] = 0$ for all $x, y, z \in U$. Therefore,

$$[xzd(y), y] = 0 \ \text{ for all } \ x, y, z \in U,$$

whence it follows that

$$xz[d(y), y] + x[z, y]d(y) + [x, y]zd(y) = 0 \quad \text{for all } \ x, y, z \in U. \tag{2}$$

Replacing z by $2wz$ in (2) and using (2), we get $[x, y]wzd(y) = 0$ and thus

$$[x, y]Uzd(y) = 0 \quad \text{for all } \ x, y, z \in U. \tag{3}$$

Let $y \in U \cap Sa_*(R)$; using Lemma 1 together with (3) we get $[x, y] = 0$ for all $x \in U$ or $zd(y) = 0$ for all $z \in U$ which leads to $d(y) = 0$. Hence

$$[y, U] = 0 \quad \text{or} \quad d(y) = 0 \quad \text{for all } \ y \in U \cap Sa_*(R).$$

Let $u \in U$; as $u^* - u \in U \cap Sa_*(R)$, then either $[u^* - u, U] = 0$ or $d(u^* - u) = 0$. If $[u^* - u, U] = 0$, then $[x, u] = [x, u^*]$ for all $x \in U$ thereby (3) yields

$$[x, u]^* Rzd(u) = 0 \ \text{ and } \ [x, u]Rzd(u) = 0 \quad \text{for all } \ x, z \in U;$$

whence it follows, because of Lemma 1, that $[u, U] = 0$ or $d(u) = 0$.
Assume that $d(u^* - u) = 0$; substituting u^* for y in (3), we arrive at

$$[x, u]^* Uzd(u) = 0 \quad \text{for all } \ x, z \in U.$$

As $[x,u]Uzd(u)=0$ by (3), applying Lemma 1, then we get $[u,U]=0$ or $d(u)=0$. In conclusion, we find that

$$[y,U]=0 \quad \text{or} \quad d(y)=0 \quad \text{for all} \quad y\in U. \tag{4}$$

Let $W_1=\{y\in U/[y,U]=0\}$ and $W_2=\{y\in U/d(y)=0\}$. Clearly, W_1 and W_2 are additive subgroups of U such that $U=W_1\cup W_2$, by (4). Accordingly, either $U=W_1$ or $U=W_2$. If $U=W_1$, then $[U,U]=0$ which when compared with Lemma 2 contradicts $U\not\subseteq Z(R)$. Therefore, $U=W_2$ so that $d(U)=0$ which, because of $d\neq 0$, contradicts Lemma 3. Hence, necessarily $U\subseteq Z(R)$. Finally, to prove the case $F(xy)+xy\in Z(R)$, it suffices to replace F by $-F$. ∎

The following example shows that in Theorem 2.4 the $*$-primeness hypothesis can not be omitted.

Example 1. Let $S=\mathbb{R}[X]\times\mathbb{R}[X]$; if we define an addition on S by componentwise and multiplication by $(P_1,P_2)(Q_1,Q_2)=(P_1Q_2-P_2Q_1,0)$. Clearly, S is a noncommutative ring in which $s^2=0$ and $st=-ts$ for all $s,t\in S$.
Let us consider $R=\left\{\begin{pmatrix} x & y\\ 0 & x\end{pmatrix} : x,\, y\in S\right\}$ and $U=\left\{\begin{pmatrix} 0 & y\\ 0 & 0\end{pmatrix} : y\in S\right\}$.
Define $F:R\longrightarrow R$ by $F\begin{pmatrix} x & y\\ 0 & x\end{pmatrix}=\begin{pmatrix} -x & 0\\ 0 & -x\end{pmatrix}$. It is easy to see that F is a generalized derivation associated to the nonzero derivation d defined on R by $d\begin{pmatrix} x & y\\ 0 & x\end{pmatrix}=\begin{pmatrix} 0 & y\\ 0 & 0\end{pmatrix}$. Furthermore, the map $*:R\longrightarrow R$ defined by $\begin{pmatrix} u & v\\ 0 & u\end{pmatrix}^*=\begin{pmatrix} -u & -v\\ 0 & -u\end{pmatrix}$ is an involution and if we set $r=\begin{pmatrix} s & 0\\ 0 & s\end{pmatrix}$, where $s\neq 0$, then using $sus=0$ for all $u\in S$ we find that $aRa=0=aRa^*$ proving that R is a non $*$-prime ring. Moreover, U is a $(*,2)$-Lie ideal of R such that $F(xy)-xy\in Z(R)$ and $F(xy)+xy\in Z(R)$ for all $x,y\in U$; but $U\not\subseteq Z(R)$. Indeed, if $r=\begin{pmatrix} s & 0\\ 0 & s\end{pmatrix}$ and $u=\begin{pmatrix} 0 & w\\ 0 & 0\end{pmatrix}$, with $sw\neq 0$, then $[u,r]\neq 0$. Hence in Theorem 2.4 the hypothesis of $*$-primeness is crucial.

It is worthwhile to note that a $*$-prime ring admitting a nonzero central $*$-ideal must be commutative. Using this fact together with Theorem 2.4, the following result proves that Theorem 2.1 of [12] remains valid without supposing that d commutes with $*$.

Theorem 2.5 *Let R be a 2-torsion free $*$-prime ring and I a nonzero $*$-ideal of R. If R admits a generalized derivation F associated with a nonzero derivation d such that for each $x,y\in I$ either $F(xy)-xy\in Z(R)$ or $F(xy)+xy\in Z(R)$, then R is commutative.*

As an application of Theorem 2.4, we extend Theorem 2.1 of [1] to Lie ideals.

Theorem 2.6 *Let R be a 2-torsion free prime ring and U a 2-Lie ideal of R. If R admits a generalized derivation F associated with a nonzero derivation d such that $F(xy) - xy \in Z(R)$ for all $x, y \in U$, then $U \subseteq Z(R)$.*

Proof. Let $\mathcal{F}$ be the additive mapping defined on $\mathcal{R} = R \times R^0$ by $\mathcal{F}(x, y) = (F(x), y)$. Clearly, $\mathcal{F}$ is a generalized derivation associated with the nonzero derivation $\mathcal{D}$ defined by $\mathcal{D}(x, y) = (d(x), 0)$. Let $W = U \times U$; it is easy to verify that W is a $(*_{\text{ex}}, 2)$-Lie ideal of $\mathcal{R}$. Moreover, $\mathcal{F}(xy) - xy \in Z(\mathcal{R})$ for all $x, y \in W$. Since $\mathcal{R}$ is a $*_{\text{ex}}$-prime ring, in view of Theorem 2.4 we deduce that $W \subseteq Z(\mathcal{R})$ and therefore $U \subseteq Z(R)$. ∎

As another application of Theorem 2.4, we extend Theorem 2.2 of [1] to Lie ideals.

Theorem 2.7 *Let R be a 2-torsion free prime ring and U a 2-Lie ideal of R. If R admits a generalized derivation F associated with a nonzero derivation d such that $F(xy) + xy \in Z(R)$ for all $x, y \in U$, then $U \subseteq Z(R)$.*

Proof. Let $\mathcal{R} = R \times R^0$ and $W = U \times U$. If we define $\mathcal{F} : \mathcal{R} \longrightarrow \mathcal{R}$ by $\mathcal{F}(x, y) = (F(x), -y)$, then $\mathcal{F}$ is a generalized derivation associated with the nonzero derivation $\mathcal{D}$ defined by $\mathcal{D}(x, y) = (d(x), 0)$. Furthermore, $\mathcal{F}(xy) + xy \in Z(R)$ for all $x, y \in W$. Since $\mathcal{R}$ is $*_{\text{ex}}$-prime and W is a $(*_{\text{ex}}, 2)$-Lie ideal of $\mathcal{R}$, then Theorem 2.4 yields $W \subseteq Z(\mathcal{R})$ which forces $U \subseteq Z(R)$. ∎

Theorem 2.8 *Let U be a $(*, 2)$-Lie ideal of R and F a generalized derivation associated with a nonzero derivation d such that for each $x, y \in U$ either $F(xy) - yx \in Z(R)$ or $F(xy) + yx \in Z(R)$. If R is $*$-prime, then $U \subseteq Z(R)$.*

Proof. Assume that $U \not\subseteq Z(R)$. Reasoning as in Theorem 2.4 we arrive at $F(xy) - yx \in Z(R)$ for all $x, y \in U$ or $F(xy) + yx \in Z(R)$ for all $x, y \in U$. Suppose that $F(xy) - yx \in Z(R)$ for all $x, y \in U$; then we get

$$F(x)y + xd(y) - yx \in Z(R) \quad \text{for all} \;\; x, y \in U.$$

Hence $[F(x)y + xd(y) - yx, t] = 0$ for all $t \in U$ so that

$$[F(x), t]y + F(x)[y, t] + [x, t]d(y) + x[d(y), t] = y[x, t] + [y, t]x. \tag{5}$$

Replacing y by $2yt$ in (5), we conclude that

$$[y, t]xt + y[x, t]t + [x, t]yd(t) + xy[d(t), t] + x[y, t]d(t) - [y, t]tx - yt[x, t] = 0$$

and therefore

$$[y,t][x,t]+y[[x,t],t]+[x,t]yd(t)+xy[d(t),t]+x[y,t]d(t)=0. \tag{6}$$

Substituting $2xy$ for y in (6) and employing (6) we see that

$$[x,t]y[x,t]+[x,t]xyd(t)=0 \quad \text{for all } \ t,x,y\in U. \tag{7}$$

Writing $x+t$ instead of x in (7), because of (7), we find that

$$[x,t]tyd(t)=0 \quad \text{for all } \ t,x,y\in U. \tag{8}$$

Replacing x by $2xz$ in (8), we are forced to $[x,t]ztyd(t)=0$ and thereby

$$[x,t]Rtyd(t)=0 \quad \text{for all } \ t,x,y\in U. \tag{9}$$

If $t\in U\cap \in Sa_*(R)$; then (9) yields $tyd(t)=0$ for all $y\in U$, in which case $tUd(t)=0$ and thus $t=0$ or $d(t)=0$, or $[x,t]=0$ for all $x\in U$. Accordingly,

$$[U,t]=0 \ \text{ or } \ d(t)=0 \quad \text{for all } \ t\in U\cap Sa_*(R).$$

Let $u\in U$, as $u+u^*\in U\cap Sa_*(R)$, then either $[U,u+u^*]=0$ or $d(u+u^*)=0$. If $[U,u+u^*]=0$, then $[x,u]=-[x,u^*]$ for all $x\in U$ and from (9) this yields

$$[x,u]^*Ruyd(u)=0 \quad \text{for all } \ x,y\in U. \tag{10}$$

Using (9) together with (10), we arrive at $[U,u]=0$ or $uUd(u)=0$.
If $[U,u]\neq 0$ hence $uUd(u)=0$ and as $u-u^*\in U\cap Sa_*(R)$, then $d(u)=d(u^*)$. Writing u^* instead of t in (9) we find that

$$[x,u]^*Uu^*yd(u^*)=0 \quad \text{for all } \ x,y\in U. \tag{11}$$

Since $[x,u]=-[x,u^*]$ for all $x\in U$, from (11) it follows that

$$[x,u]Uu^*yd(u^*)=0 \quad \text{for all } \ x,y\in U. \tag{12}$$

Combining (11) and (12) we obtain $[x,u]=0$ for all $x\in U$ or $u^*yd(u^*)=0$ for all $y\in U$. Since $[U,u]\neq 0$, then $u^*Ud(u^*)=0$ and the fact that $d(u)=d(u^*)$ assures that

$$u^*Ud(u)=0.$$

As $uUd(u)=0$, then applying Lemma 1, we arrive at $d(u)=0$.
Now suppose $d(u+u^*)=0$. If $d(u-u^*)=0$, then $2d(u)=0$ so that $d(u)=0$. If $[U,u-u^*]=0$ then reasoning as above we obtain $[U;u]=0$ or $d(u)=0$.
In conclusion we find that

$$[U,u]=0 \quad \text{or} \ \ d(u)=0 \quad \text{for all } \ u\in U. \tag{13}$$

Since equation (13) is the same as equation (4), arguing as in the proof of Theorem 2.4, we get a contradiction. Thus, necessarily $U \subseteq Z(R)$.
Finally, to prove the case $F(xy) + yx \in Z(R)$, it suffices to replace F by $-F$. ∎

Example 2. In hypothesis of Theorem 2.8 the $*$-primeness condition is necessary. Indeed, in example 1 it is easy to verify that $F(xy) - yx \in Z(R)$ and $F(xy) + yx \in Z(R)$ for all $x, y \in U$, but $U \not\subseteq Z(R)$.

As an application of Theorem 2.8, the following theorem gives an improved version of ([12], Theorem 2.2).

Theorem 2.9 *Let R be a 2-torsion free $*$-prime ring and I a nonzero $*$-ideal of R. If R admits a generalized derivation F associated with a nonzero derivation d such that for each $x, y \in I$ either $F(xy) - yx \in Z(R)$ or $F(xy) + yx \in Z(R)$, then R is commutative.*

Theorem 2.10 *Let U be a $(*, 2)$-Lie ideal of R and F a generalized derivation associated to a nonzero derivation d, commuting with $*$, such that for each $x, y \in U$ either $F(x)F(y) - xy \in Z(R)$ or $F(x)F(y) + xy \in Z(R)$. If R is $*$-prime, then $U \subseteq Z(R)$.*

Proof. Assume that $U \not\subseteq Z(R)$. Similarly, as in Theorem 2.4, we have $F(x)F(y) - xy \in Z(R)$ for all $x, y \in U$ or $F(x)F(y) - xy \in Z(R)$ for all $x, y \in U$.
Suppose that

$$F(x)F(y) - xy \in Z(R) \text{ for all } x, y \in U. \tag{14}$$

Replacing y by $2yt$ in (14), we obtain

$$(F(x)F(y) - xy)t + F(x)yd(t) \in Z(R) \text{ for all } x, y, t \in U,$$

so that

$$[(F(x)F(y) - xy)t + F(x)yd(t), t] = 0 \text{ for all } x, y, t \in U,$$

which reduces to

$$F(x)y[d(t), t] + F(x)[y, t]d(t) + [F(x), t]yd(t) = 0. \tag{15}$$

Since U is a 2-Lie ideal, then $2r[u, v] \in U$ for all $u, v \in U$ and $r \in R$.
Substituting $2F(y)r[u, v]$ for y in (15), where $u, v \in U$ and $r \in R$, we get

$$F(x)(F(y)z[d(t), t] + F(y)[z, t]d(t) + [F(y), t]zd(t)) + [F(x), t]F(y)zd(t) = 0,$$

where $z = 2r[u, v]$. In view of (15), the least equation becomes $[F(x), t]F(y)zd(t) = 0$ and the fact that $\text{char}R \neq 2$ yields

$$[F(x), t]F(y)r[u, v]d(t) = 0 \text{ for all } t, u, v, x, y \in U, r \in R. \tag{16}$$

Replacing v by $2vw$ in (16), where $w \in U$, and using (16) we find that $[F(x), t]F(y)r[u, v]wd(t) = 0$ and thus

$$[F(x), t]F(y)r[u, v]Ud(t) = 0 \text{ for all } t, u, v, x, y \in U, r \in R. \quad (17)$$

Since d commutes with $*$, then Lemma 1 together with (17) assure that for all $t \in U \cap Sa_*(R)$ either $d(t) = 0$ or $[F(x), t]F(y)r[u, v] = 0$.
Suppose that $[F(x), t]F(y)r[u, v] = 0$ for all $u, v, x, y \in U, r \in R$; then

$$[F(x), t]F(y)R[u, v] = 0 \text{ for all } u, v, x, y \in U. \quad (18)$$

As U is invariant under $*$, then (18) assures that $[F(x), t]F(y)R[u, v]^* = 0$ and the $*$-primeness of R, implies that either $[F(x), t]F(y) = 0$ or $[u, v] = 0$. Since $U \not\subseteq Z(R)$, then Lemma 2 forces

$$[F(x), t]F(y) = 0 \text{ for all } x, y \in U. \quad (19)$$

Substituting $2yt$ for y in (19) we arrive at $[F(x), t]yd(t) = 0$ ant thus

$$[F(x), t]Ud(t) = 0 \text{ for all } x \in U. \quad (20)$$

Since $t \in U \cap Sa_*(R)$ and d commutes with $*$, in view of (20), Lemma 2 assures that $d(t) = 0$ or $[F(x), t] = 0$ for all $x \in U$. Accordingly, for all $t \in U \cap Sa_*(R)$

$$d(t) = 0 \text{ or } [F(x), t] = 0 \text{ for all } x \in U. \quad (21)$$

Let $w \in U$; we have $d(w - w^*) = 0$ or $[F(x), w - w^*] = 0$ for all $x \in U$.
If $d(w - w^*) = 0$, then $d(w) \in Sa_*(R)$ and (17) implies that $d(w) = 0$ or $[F(x), w]F(y)r[u, v] = 0$ for all $u, v, x, y \in U, r \in R$ in which case we arrive, as above, at $d(w) = 0$ or $[F(x), w] = 0$ for all $x \in U$.
Suppose that $[F(x), w - w^*] = 0$ for all $x \in U$; if $[F(x), w + w^*] = 0$, then $[F(x), w] = 0$. If $d(w + w^*) = 0$, then $d(w) \in Sa_*(R)$ and reasoning as above we arrive at $d(w) = 0$ or $[F(x), w] = 0$ for all $x \in U$. In conclusion,

$$\text{for all } w \in U, \text{ either } d(w) = 0 \text{ or } [F(x), w] = 0 \text{ for all } x \in U.$$

Consequently, U is a union of the additive subgroups G and H, where $G = \{u \in U \,/\, d(u) = 0\}$ and $H = \{u \in U \,/\, [F(x), u] = 0 \text{ for all } x \in U\}$ and thus $U = G$ or $U = H$. If $U = G$, then $d(U) = 0$ which when compared with Lemma 3 contradicts the fact that $d \neq 0$ and $U \not\subseteq Z(R)$. Accordingly, $U = H$ so that

$$[F(x), u] = 0 \text{ for all } u, x \in U. \quad (22)$$

Replacing u by $2r[u, v]$ in (22), where $r \in R$ and $v \in U$, and using (22) we arrive at

$$[F(x), r][u, v] = 0 \text{ for all } u, v, x \in U, r \in R. \quad (23)$$

Writing rs instead of r in (23) we obtain $[F(x), r]s[u, v] = 0$ and hence

$$[F(x), r]R[u, v] = 0 \text{ for all } u, v, x \in U, r \in R. \tag{24}$$

Since $[F(x), r]R[u, v]^* = 0$, because of Lemma 1, (24) yields $[F(x), r] = 0$ or $[u, v] = 0$. As $U \nsubseteq Z(R)$, hence Lemma 2 gives $[F(x), r] = 0$ which yields

$$F(x) \in Z(R) \text{ for all } x \in U. \tag{25}$$

In view of (25), the hypothesis $F(x)F(y) - xy \in Z(R)$ implies that

$$xy \in Z(R) \text{ for all } x, y \in U,$$

which when combined with (25) forces

$$F(xy) - xy \in Z(R) \text{ for all } x, y \in U.$$

Therefore, the required result follows immediately from Theorem 2.4.
Finally, one can prove the case $F(x)F(y)+xy \in Z(R)$, by a slight modification in the proof of the first case. ∎

Example 3. In hypothesis of Theorem 2.10 the $*$-primeness condition cannot be omitted. Indeed, in example 1 we have $F(x)F(y) - xy \in Z(R)$ and $F(x)F(y)+xy \in Z(R)$ for all $x, y \in U$ and d commutes with $*$; but $U \nsubseteq Z(R)$.

In ([12], Theorem 2.3), author supposes that F as well as d commutes with $*$. Using Theorem 2.10, we improve ([12], Theorem 2.3) as follows.

Theorem 2.11 *Let R be a 2-torsion free $*$-prime ring and I a nonzero $*$-ideal of R. If R admits a generalized derivation F associated with a nonzero derivation d, commuting with $*$, such that for each $x, y \in I$ either $F(x)F(y) - xy \in Z(R)$ or $F(x)F(y) + xy \in Z(R)$, then R is commutative.*

3 Open Problems

(i) Does Theorem 2.10 remain valid without the assumption that d commutes with $*$?
(ii) Does the condition $F(x)F(y) - yx \in Z(R)$ for all $x, y \in U$ imply that $U \subseteq Z(R)$?
(iii) Does the condition $F(x)F(y) + yx \in Z(R)$ for all $x, y \in U$ imply that $U \subseteq Z(R)$?

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