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Triple Positive Solutions for Singular Integral Boundary Value Problems[1]

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Abstract

In this paper, we consider the following singular Sturm-Liouville integral boundary value problems for second-order ordinary differential equations

$$\begin{cases} u''(t) + h(t)f(t, u(t), u'(t)) = 0, \quad t \in (0,1), \\ au(0) - bu'(0) = \int_0^1 u(t)\mathrm{d}\xi(t), cu(1) + du'(1) = \int_0^1 u(t)\mathrm{d}\eta(t), \end{cases}$$

where h is allowed to be singular at $t = 0$ and (or) $t = 1$. By using the Avery-Peterson fixed point theorem, we will establish a result on the existence of positive solutions for the above system. Finally, we give an example to illustrate our result.

Keywords: *Integral boundary value problem; Positive solution; Avery-Peterson fixed point theorem; A priori estimate.*

1 Introduction

In this paper, we consider the existence of at least three positive solutions for the following singular Sturm-Liouville integral boundary value problems for

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second-order ordinary differential equations

$$\begin{cases} u''(t) + h(t)f(t, u(t), u'(t)) = 0, \\ au(0) - bu'(0) = \int_0^1 u(t)\mathrm{d}\xi(t), cu(1) + du'(1) = \int_0^1 u(t)\mathrm{d}\eta(t), \end{cases} \tag{1.1}$$

where $a, b, c, d \geq 0$ with $ac + ad + bc > 0$; $h \in C((0,1), [0, +\infty))$ may be singular at $t = 0$ and (or) $t = 1$, and $\int_0^1 h(s)\mathrm{d}s < \infty$; $f \in C([0,1] \times [0, +\infty) \times (-\infty, +\infty), [0, +\infty))$; $\xi(t)$ and $\eta(t)$ are increasing on $[0,1]$ and right continuous on $\in [0,1)$, left continuous at $t = 1$, with $\xi(0) = \eta(0) = 0$; $\int_0^1 u(t)\mathrm{d}\xi(t)$ and $\int_0^1 u(t)\mathrm{d}\eta(t)$ denote the Riemann-Stieltjes integrals of u with respect to ξ and η, respectively.

We are here interested in the case where f depends explicitly on u'. There are many papers dealing with second order multi-point boundary value problems when f is independent of u', see for example [4–12] and references cited therein. For abstract spaces, Guo et al. [4], Liu [7], Zhao et al. [12], by using fixed point theorems of strict-set-contractions, the authors obtained some sufficient conditions for the existence of at least one or two positive solutions to two, three, multi-point boundary value problems.

When f involves u' explicitly, the existence of at least two or three solutions for second order boundary value problems in scalar space has been studied in a number of papers, see for example [2, 3] and references cited therein. In [3], Chandra et al. discussed the existence of at least one solution for the following boundary value problems in a Banach space

$$\begin{cases} u''(t) + f(t, u(t), u'(t)) = \theta, \quad 0 < t < 1, \\ au(0) + bu'(0) = u_0, \quad cu(1) + du'(1) = u_1. \end{cases} \tag{1.2}$$

In [2], Bernfeld et al. obtained some results on the existence of solution for the problems (1.2) using the method of lower and upper solutions.

Motivated by the works mentioned above, in this paper, by using the Avery-Peterson fixed point theorem, we will establish a result on the existence of at least three positive solutions to the problem (1.1). It is interesting to note that, unlike most of the authors who discuss multipoint or integral boundary value problems, we do not endeavor to construct a new Green's function associated with (1.1).

This paper is organized as follows. Section 2 gives some preliminary results, the focus being on deriving some a priori estimates, required in the proof of our main result. The main results are stated and proved in Section 3, then followed by an example to illustrate the usefulness of our main results.

2 A priori estimates and lemmas

Let

$$\rho := ac + ad + bc, \quad \mu(t) := b + at, \quad \nu(t) := c(1-t) + d.$$

Now we list our hypotheses:

(H1) $\rho > 0$, $a > \xi(1) = \int_0^1 \mathrm{d}\xi(t)$, and $c > \eta(1) = \int_0^1 \mathrm{d}\eta(t)$.

(H2) $h \in C((0,1),[0,+\infty))$ and $0 < \int_0^1 h(s)\mathrm{d}s < \infty$.

(H3) $f \in C([0,1] \times [0,+\infty) \times (-\infty,+\infty),[0,+\infty))$.

(H4) $\kappa := \kappa_1\kappa_4 - \kappa_2\kappa_3 > 0$, $\kappa_1 > 0$, $\kappa_4 > 0$, where

$$\kappa_1 := 1 - \frac{1}{\rho}\int_0^1 \nu(t)\mathrm{d}\xi(t), \kappa_2 := \frac{1}{\rho}\int_0^1 \mu(t)\mathrm{d}\xi(t), \kappa_3 := \frac{1}{\rho}\int_0^1 \nu(t)\mathrm{d}\eta(t),$$
$$\kappa_4 := 1 - \frac{1}{\rho}\int_0^1 \mu(t)\mathrm{d}\eta(t).$$

Lemma 2.1 *Assume that $f \in L[0,1]$ is nonnegative, then the following boundary value problem*

$$\begin{cases} -u''(t) = f(t), \\ au(0) - bu'(0) = \int_0^1 u(t)\mathrm{d}\xi(t), cu(1) + du'(1) = \int_0^1 u(t)\mathrm{d}\eta(t) \end{cases} \tag{2.1}$$

is equivalent to

$$u(t) = \int_0^1 k(t,s)f(s)\mathrm{d}s + \frac{\nu(t)}{\rho}\int_0^1 u(t)\mathrm{d}\xi(t) + \frac{\mu(t)}{\rho}\int_0^1 u(t)\mathrm{d}\eta(t), \tag{2.2}$$

where

$$k(t,s) := \frac{1}{\rho}\begin{cases} \mu(s)\nu(t), & 0 \le s \le t \le 1, \\ \nu(s)\mu(t), & 0 \le t \le s \le 1. \end{cases} \tag{2.3}$$

Simple computations show the following result.

Lemma 2.2 *Suppose that (H1) and (H4) hold. Let u uniquely solve Problem* (2.1), *then*

$$\int_0^1 u(t)\mathrm{d}\xi(t) = \frac{1}{\kappa}\left[\kappa_4\int_0^1 \mathrm{d}\xi(t)\int_0^1 k(t,s)f(s)\mathrm{d}s + \kappa_2\int_0^1 \mathrm{d}\eta(t)\int_0^1 k(t,s)f(s)\mathrm{d}s\right], \tag{2.4}$$

and

$$\int_0^1 u(t)\mathrm{d}\eta(t) = \frac{1}{\kappa}\left[\kappa_3\int_0^1 \mathrm{d}\xi(t)\int_0^1 k(t,s)f(s)\mathrm{d}s + \kappa_1\int_0^1 \mathrm{d}\eta(t)\int_0^1 k(t,s)f(s)\mathrm{d}s\right]. \tag{2.5}$$

The following a priori estimates will be required in the proof of our main result.

Lemma 2.3 *Suppose that (H1)-(H4) hold. Let u uniquely solve Problem* (2.1) *and $\sigma \in (0,1/2)$ be fixed. Then*

$$u(t) \geq 0, \forall t \in [0,1], \text{ and } \min_{\sigma \leq t \leq 1-\sigma} u(t) \geq \tau_1 \max_{0 \leq t \leq 1} |u(t)|,$$

where $\tau_1 := \min\left\{\frac{\nu(1-\sigma)}{\nu(0)}, \frac{\mu(\sigma)}{\mu(1)}\right\}$. Moreover, the following inequality holds:

$$\max_{0 \leq t \leq 1} |u(t)| \leq \tau_2 \max_{0 \leq t \leq 1} |u'(t)|, \tag{2.6}$$

where

$$\tau_2 = \max\left\{1 + \frac{b + \int_0^1 t \mathrm{d}\xi(t)}{a - \int_0^1 \mathrm{d}\xi(t)}, \quad 1 + \frac{d + \int_0^1 (1-t) \mathrm{d}\eta(t)}{c - \int_0^1 \mathrm{d}\eta(t)}\right\}.$$

Proof. We first prove the nonnegativity of u. By relation (2.2) and the nonnegativity of f, it suffices to verify $\int_0^1 u(t)\mathrm{d}\xi(t) \geq 0$ and $\int_0^1 u(t)\mathrm{d}\eta(t) \geq 0$, as are evident from (2.4) and (2.5).

By definition(see (2.3)), we have $k(t,s) = k(s,t)$. Note μ is increasing on $[0,1]$ and ν is decreasing on $[0,1]$. This implies

$$k(t,s) \leq k(s,s), \forall t, s \in [0,1].$$

If $t \in [\sigma, 1-\sigma]$ and $s \in [0,1]$, then it follows from the definition of $k(t,s)$ that

$$\frac{k(t,s)}{k(s,s)} = \begin{cases} \frac{\nu(t)}{\nu(s)}, & 0 \leq s \leq t \leq 1 \\ \frac{\mu(t)}{\mu(s)}, & 0 \leq t \leq s \leq 1 \end{cases} \geq \begin{cases} \frac{\nu(1-\sigma)}{\nu(0)} \\ \frac{\mu(\sigma)}{\mu(1)} \end{cases} \geq \tau_1,$$

so that $k(t,s) \geq \tau_1 k(s,s)$. Consequently, we have by Lemma 2.1

$$\begin{aligned} u(t) &\geq \tau_1 \int_0^1 k(s,s) f(s)\mathrm{d}s + \frac{\nu(1-\sigma)}{\rho} \int_0^1 u(t)\mathrm{d}\xi(t) + \frac{\mu(\sigma)}{\rho} \int_0^1 u(t)\mathrm{d}\eta(t) \\ &\geq \tau_1 \left(\int_0^1 k(s,s) f(s)\mathrm{d}s + \frac{\nu(0)}{\rho} \int_0^1 u(t)\mathrm{d}\xi(t) + \frac{\mu(1)}{\rho} \int_0^1 u(t)\mathrm{d}\eta(t)\right) \end{aligned}$$

for all $\sigma \leq t \leq 1-\sigma$. From (2.2), we obtain

$$\begin{aligned} &\max_{0 \leq t \leq 1} u(t) \\ &= \max_{0 \leq t \leq 1} \left(\int_0^1 k(t,s) f(s)\mathrm{d}s + \frac{\nu(t)}{\rho} \int_0^1 u(t)\mathrm{d}\xi(t) + \frac{\mu(t)}{\rho} \int_0^1 u(t)\mathrm{d}\eta(t)\right) \\ &\leq \int_0^1 k(s,s) f(s)\mathrm{d}s + \frac{\nu(0)}{\rho} \int_0^1 u(t)\mathrm{d}\xi(t) + \frac{\mu(1)}{\rho} \int_0^1 u(t)\mathrm{d}\eta(t). \end{aligned}$$

Combining the preceding inequalities we conclude

$$\min_{\sigma\le t\le 1-\sigma} u(t) \ge \tau_1 \max_{0\le t\le 1} |u(t)|.$$

From (2.1), we have $u''(t) = -f(t) \le 0$, then $u'(t)$ is decreasing, that is $u'(1) \le u'(t) \le u'(0), \forall t \in [0,1]$. So we arrive at

$$\max_{t\in[0,1]} |u'(t)| = \max\{|u'(0)|, |u'(1)|\}.$$

Now we distinguish between two cases.

Case 1. $\max_{t\in[0,1]} |u'(t)| = |u'(0)|$. From the concavity of $u(t)$, we get $u'(0) \ge \frac{u(t)-u(0)}{t}$. Combining this with the first boundary condition of (1.1), we obtain

$$\int_0^1 u'(0)t\mathrm{d}\xi(t) \ge \int_0^1 u(t)\mathrm{d}\xi(t) - \int_0^1 u(0)\mathrm{d}\xi(t) = au(0) - bu'(0) - \int_0^1 u(0)\mathrm{d}\xi(t)$$

and thus

$$u(0) \le \frac{b+\int_0^1 t\mathrm{d}\xi(t)}{a-\int_0^1 \mathrm{d}\xi(t)} u'(0).$$

Therefore,

$$u(t) \le u'(0)t + u(0) \le \left(1 + \frac{b+\int_0^1 t\mathrm{d}\xi(t)}{a-\int_0^1 \mathrm{d}\xi(t)}\right) u'(0).$$

That is,

$$\max_{t\in[0,1]} |u(t)| \le \left(1 + \frac{b+\int_0^1 t\mathrm{d}\xi(t)}{a-\int_0^1 \mathrm{d}\xi(t)}\right) \max_{t\in[0,1]} |u'(t)|. \tag{2.7}$$

Case 2. Let $\max_{t\in[0,1]} |u'(t)| = |u'(1)|$. By the concavity of $u(t)$, we have $u'(1) \le \frac{u(t)-u(1)}{t-1}$. The second boundary condition of (1.1) leads to

$$\begin{aligned}(-u'(1)) \int_0^1 (1-t)\mathrm{d}\eta(t) &\ge \int_0^1 u(t)\mathrm{d}\eta(t) - \int_0^1 u(1)\mathrm{d}\eta(t)\\ &= cu(1) + du'(1) - u(1)\int_0^1 \mathrm{d}\eta(t)\end{aligned}$$

and thus

$$u(1) \le \frac{d+\int_0^1 (1-t)\mathrm{d}\eta(t)}{c-\int_0^1 \mathrm{d}\eta(t)} (-u'(1)).$$

Consequently,

$$u(t) \le (-u'(1))(1-t) + u(1) \le \left(1 + \frac{d+\int_0^1 (1-t)\mathrm{d}\eta(t)}{c-\int_0^1 \mathrm{d}\eta(t)}\right) (-u'(1)).$$

That is,

$$\max_{0\le t\le 1}|u(t)| \le \left(1+\frac{d+\int_0^1(1-t)\mathrm{d}\eta(t)}{c-\int_0^1\mathrm{d}\eta(t)}\right)\max_{t\in[0,1]}|u'(t)|. \tag{2.8}$$

Now (2.6) follows from (2.7) and (2.8) immediately. This completes the proof.

Let

$$\overline{A} := \frac{\kappa_4\xi(1)+\kappa_2\eta(1)}{\kappa}, \quad \overline{B} := \frac{\kappa_3\xi(1)+\kappa_1\eta(1)}{\kappa},$$

then

$$\int_0^1 u(s)d\xi(s) \le \overline{A}\int_0^1 k(s,s)f(s)\mathrm{d}s, \quad \int_0^1 u(s)d\eta(s) \le \overline{B}\int_0^1 k(s,s)f(s)\mathrm{d}s.$$

Note we have proved that $k(t,s) \ge \tau_1 k(s,s)$, $\forall t \in [\sigma, 1-\sigma]$ and $s \in [0,1]$. From Lemma 2.2, we have

$$\int_0^1 u(s)\mathrm{d}\xi(s) \ge \overline{A}\tau_1\int_0^1 k(s,s)f(s)\mathrm{d}s, \quad \int_0^1 u(s)\mathrm{d}\eta(s) \ge \overline{B}\tau_1\int_0^1 k(s,s)f(s)\mathrm{d}s.$$

Let

$E := C^1[0,1]$ with the norm $\|u\| = \max\{\|u\|_0, \|u'\|_0\}$, where $\|u\|_0 = \max_{0\le t\le 1}|u(t)|$.

Define our work cone K by

$$K := \{u \in E:\ u \text{ is nonnegative and concave on } [0,1], \text{ and } \min_{\sigma\le t\le 1-\sigma} u(t) \ge \tau_1 \max_{0\le t\le 1}|u(t)|\}.$$

Suppose that u is a solution of (1.1), then for any $t \in [0,1]$

$$u(t) = \int_0^1 k(t,s)h(s)f(s,u(s),u'(s))\mathrm{d}s + \frac{\nu(t)}{\rho}\int_0^1 u(t)\mathrm{d}\xi(t) + \frac{\mu(t)}{\rho}\int_0^1 u(t)\mathrm{d}\eta(t).$$

Define the operator T by

$$(Tu)(t) := \int_0^1 k(t,s)h(s)f(s,u(s),u'(s))\mathrm{d}s + \frac{\nu(t)}{\rho}\int_0^1 u(t)\mathrm{d}\xi(t) + \frac{\mu(t)}{\rho}\int_0^1 u(t)\mathrm{d}\eta(t), 0 \le t \le 1.$$

Now $T : K \to K$ is a completely continuous operator. Clearly $u \in K$ is a positive solution of (1.1) if and only if $u \in K \setminus \{0\}$ is a fixed point of T.

Let γ and θ be nonnegative continuous convex functionals on a cone K, α be a nonnegative continuous concave functional on K, ψ be a nonnegative continuous functional on K, and m_1, m_2, m_3 and m_4 be positive numbers. Define the convex sets

$$P(\gamma, m_4) := \{u \in K : \gamma(u) < m_4\},$$

$$P(\gamma, \alpha, m_2, m_4) := \{u \in K : m_2 \le \alpha(u), \gamma(u) \le m_4\},$$

$$P(\gamma, \theta, \alpha, m_2, m_3, m_4) := \{u \in K : m_2 \le \alpha(u), \theta(u) \le m_3, \gamma(u) \le m_4\},$$

and a closed set

$$Q(\gamma, \psi, m_1, m_4) := \{u \in K : m_1 \le \psi(u), \gamma(u) \le m_4\}.$$

Now we state a fixed point theorem due to Avery and Peterson [1].

Lemma 2.4 *Let K be a cone in a real Banach space E. Let γ and θ be nonnegative continuous convex functionals on K, α be a nonnegative continuous concave functional on K, and ψ be a nonnegative continuous functional on K satisfying $\psi(\lambda u) \le \lambda\psi(u)$ for $0 \le \lambda \le 1$, such that for some positive numbers ε and m_4,*

$$\alpha(u) \le \psi(u) \text{ and } \|u\| \le \varepsilon\gamma(u),$$

for all $u \in \overline{P(\gamma, m_4)}$. Suppose $T : \overline{P(\gamma, m_4)} \to \overline{P(\gamma, m_4)}$ is completely continuous and there exist positive numbers m_1, m_2 and m_3 with $m_1 < m_2$ such that
(C1) $\{u \in P(\gamma, \theta, \alpha, m_2, m_3, m_4) : \alpha(u) > m_2\} \ne \emptyset$ and $\alpha(Tu) > m_2$ for $u \in P(\gamma, \theta, \alpha, m_2, m_3, m_4)$;
(C2) $\alpha(Tu) > m_2$ for $u \in P(\gamma, \alpha, m_2, m_4)$ with $\theta(Tu) > m_3$;
(C3) $0 \notin Q(\gamma, \psi, m_1, m_4)$ and $\psi(Tu) < m_1$ for $u \in Q(\gamma, \psi, m_1, m_4)$ with $\psi(u) = m_1$.
Then T has at least three fixed points $u_1, u_2, u_3 \in \overline{P(\gamma, m_4)}$ such that $\gamma(u_i) \le m_4$ for $i = 1, 2, 3$; $m_2 < \alpha(u_1)$; $m_1 < \psi(u_2)$ with $\alpha(u_2) < m_2$; $\psi(u_3) < m_1$.

3 Main result

Let

$$\lambda_1 := \frac{(c - \eta(1)) \int_0^1 t^2 \mathrm{d}\xi(t) + (a - \xi(1))(c + 2d - \int_0^1 t^2 \mathrm{d}\eta(t))}{(a - \xi(1))(c + d - \int_0^1 t \mathrm{d}\eta(t)) + (c - \eta(1))(b + \int_0^1 t \mathrm{d}\xi(t))}$$

and

$$\lambda_2 := \frac{(b + \int_0^1 t \mathrm{d}\xi(t))(c + 2d - \int_0^1 t^2 \mathrm{d}\eta(t)) - (c + d - \int_0^1 t \mathrm{d}\eta(t)) \int_0^1 t^2 \mathrm{d}\xi(t)}{(a - \xi(1))(c + d - \int_0^1 t \mathrm{d}\eta(t)) + (c - \eta(1))(b + \int_0^1 t \mathrm{d}\xi(t))}.$$

Let

$$u_0(t) := -mt^2 + \lambda_1 mt + m\lambda_2. \tag{3.1}$$

where $m := \frac{4m_2}{\tau_1^2(\lambda_1^2+4\lambda_2)}$. Then $u_0(t)$ is concave. By direct computation we have that u_0 satisfies boundary value conditions of (1.1).

Let the nonnegative continuous convex functionals γ and θ, the nonnegative continuous concave functional α, and the nonnegative continuous functional ψ be defined on the cone K by

$$\alpha(u) = \min_{\sigma\le t\le 1-\sigma} u(t), \quad \gamma(u) = \max_{0\le t\le 1} |u'(t)|, \quad \theta(u) = \psi(u) = \max_{0\le t\le 1} u(t). \tag{3.2}$$

For convenience, we set

$$N := \left[1 + \frac{\nu(0)\overline{A}}{\rho} + \frac{\mu(1)\overline{B}}{\rho}\right]\int_0^1 k(s,s)h(s)\mathrm{d}s,$$

$$\begin{aligned} S := \max\Big\{ &|\frac{a}{\rho}\int_0^1 \nu(s)h(s)\mathrm{d}s| + \frac{c\overline{A}}{\rho}\int_0^1 k(s,s)h(s)\mathrm{d}s + \frac{a\overline{B}}{\rho}\int_0^1 k(s,s)h(s)\mathrm{d}s,\\ &|\frac{c}{\rho}\int_0^1 \mu(s)h(s)\mathrm{d}s| + \frac{c\overline{A}}{\rho}\int_0^1 k(s,s)h(s)\mathrm{d}s + \frac{a\overline{B}}{\rho}\int_0^1 k(s,s)h(s)\mathrm{d}s\Big\}, \end{aligned}$$

$$\begin{aligned} M := \min\Big\{ &\int_0^1 k(\sigma,s)h(s)\mathrm{d}s + \frac{\nu(\sigma)\overline{A}\tau_1}{\rho}\int_0^1 k(s,s)h(s)\mathrm{d}s\\ &+ \frac{\mu(\sigma)\overline{B}\tau_1}{\rho}\int_0^1 k(s,s)h(s)\mathrm{d}s,\\ &\int_0^1 k(1-\sigma,s)h(s)\mathrm{d}s + \frac{\nu(1-\sigma)\overline{A}\tau_1}{\rho}\int_0^1 k(s,s)h(s)\mathrm{d}s\\ &+ \frac{\mu(1-\sigma)\overline{B}\tau_1}{\rho}\int_0^1 k(s,s)h(s)\mathrm{d}s\Big\}. \end{aligned}$$

Theorem 3.1 *Suppose that (H1)-(H4) hold, and there exist m_1, m_2 and m_4 such that $0 < m_1 < m_2 \le \frac{\tau_1^2(\lambda_1^2+4\lambda_2)}{4\max\{|\lambda_1|,|\lambda_1-2|\}} m_4$ and*

(B1) $f(t,u,v) \le \frac{m_4}{S}$ for $(t,u,v) \in [0,1]\times[0,\tau_2 m_4]\times[-m_4,m_4]$.

(B2) $f(t,u,v) > \frac{m_2}{M}$ for $(t,u,v) \in [\sigma,1-\sigma]\times[m_2,m_2\tau_1^{-2}]\times[-m_4,m_4]$.

(B3) $f(t,u,v) \le \frac{m_1}{N}$ for $(t,u,v) \in [0,1]\times[0,m_1]\times[-m_4,m_4]$.

Then (1.1) has at least three positive solutions u_1, u_2, u_3 such that

$\max_{0\le t\le 1}|u_i'(t)| \le m_4$ for $i = 1,2,3$; $m_2 < \min_{\sigma\le t\le 1-\sigma} u_1(t)$; $m_1 < \max_{0\le t\le 1} u_2(t)$ with $\min_{\sigma\le t\le 1-\sigma} u_2(t) < m_2$; $\max_{0\le t\le 1} u_3(t) < m_1$.

Proof. By (3.2), we know that α, γ, θ and ψ are continuous nonnegative functionals on K satisfying $\psi(\lambda u) \le \lambda\psi(u)$ for $0 \le \lambda \le 1$, so that for some positive numbers $\varepsilon = \tau_2 > 1$ and m_4,

$$\alpha(u) \le \psi(u) \text{ and } \|u\| \le \varepsilon\gamma(u),$$

for all $u \in \overline{P(\gamma, m_4)}$. If $u \in \overline{P(\gamma, m_4)}$, then $\gamma(u) = \max_{0\le t\le 1} |u'(t)| < m_4$. By Lemma 2.3, we find

$$\max_{0\le t\le 1} |u(t)| \le \tau_2 \max_{0\le t\le 1} |u'(t)| < \tau_2 m_4.$$

It follows from the assumption (B1) and Lemma 2.3 that

$$\begin{aligned}
\gamma(Tu) &= \max_{0\le t\le 1} |(Tu)'(t)| = \max\{|(Tu)'(0)|, |(Tu)'(1)|\} \\
&= \max\Big\{|\frac{a}{\rho}\int_0^1 \nu(s)h(s)f(s,u(s),u'(s))\mathrm{d}s| + |\frac{c}{\rho}\int_0^1 u(t)\mathrm{d}\xi(t)| \\
&+ |\frac{a}{\rho}\int_0^1 u(t)\mathrm{d}\eta(t)|, |\frac{c}{\rho}\int_0^1 \mu(s)h(s)f(s,u(s),u'(s))\mathrm{d}s| + |\frac{c}{\rho}\int_0^1 u(t)\mathrm{d}\xi(t)| \\
&+ |\frac{a}{\rho}\int_0^1 u(t)\mathrm{d}\eta(t)|\Big\} \le \frac{m_4}{S}\max\Big\{|\frac{a}{\rho}\int_0^1 \nu(s)h(s)\mathrm{d}s| + \frac{c\overline{A}}{\rho}\int_0^1 k(s,s)h(s)\mathrm{d}s \\
&+ \frac{a\overline{B}}{\rho}\int_0^1 k(s,s)h(s)\mathrm{d}s, |\frac{c}{\rho}\int_0^1 \mu(s)h(s)\mathrm{d}s| + \frac{c\overline{A}}{\rho}\int_0^1 k(s,s)h(s)\mathrm{d}s \\
&+ \frac{a\overline{B}}{\rho}\int_0^1 k(s,s)h(s)\mathrm{d}s\Big\} = m_4.
\end{aligned}$$

Therefore, $T : \overline{P(\gamma, m_4)} \to \overline{P(\gamma, m_4)}$. Recall $u_0(t)$ is concave and satisfies the boundary value conditions of (1.1), that is,

$$au_0(0) - bu_0'(0) = \int_0^1 u_0(t)\mathrm{d}\xi(t) \text{ and } cu_0(1) + du_0'(1) = \int_0^1 u_0(t)\mathrm{d}\eta(t).$$

Thus $u_0 \in K$, we have

$$\alpha(u_0) = \min_{\sigma\le t\le 1-\sigma} u_0(t) \ge \tau_1 \frac{\lambda_1^2 + 4\lambda_2}{4} m = \frac{m_2}{\tau_1} > m_2, \tag{3.3}$$

$$\theta(u_0) = \max_{0\le t\le 1} u_0(t) = u_0(\frac{\lambda_1}{2}) = \frac{\lambda_1^2 + 4\lambda_2}{4} m = \frac{m_2}{\tau_1^2} > m_2, \tag{3.4}$$

$$\gamma(u_0) = \max_{0\le t\le 1} |u_0'(t)| = \max\{|u_0'(0)|, |u_0'(1)|\} = m\max\{|\lambda_1|, |\lambda_1 - 2|\} \le m_4. \tag{3.5}$$

(3.3) and (3.4) lead to $\alpha(u_0) > \tau_1\theta(u_0)$ and then $u_0 \in P(\gamma, \theta, \alpha, m_2, m_2\tau_1^{-2}, m_4)$ and $\{u \in P(\gamma, \theta, \alpha, m_2, m_2\tau_1^{-2}, m_4) : \alpha(u) > m_2\} \ne \emptyset$. Consequently, if $u \in P(\gamma, \theta, \alpha, m_2, m_2\tau_1^{-2}, m_4)$, then $m_2 < u(t) \le m_2\tau_1^{-2}, |u'(t)| \le m_4$ for $\sigma \le t \le$

$1-\sigma$. In view of (B2), we get

$$\begin{aligned}\alpha(Tu) &= \min_{\sigma\le t\le 1-\sigma}(Tu)(t) = \min\{(Tu)(\sigma),(Tu)(1-\sigma)\}\\ &= \min\Big\{\int_0^1 k(\sigma,s)h(s)f(s,u(s),u'(s))\mathrm{d}s + \frac{\nu(\sigma)}{\rho}\int_0^1 u(t)\mathrm{d}\xi(t)\\ &+ \frac{\mu(\sigma)}{\rho}\int_0^1 u(t)\mathrm{d}\eta(t), \int_0^1 k(1-\sigma,s)h(s)f(s,u(s),u'(s))\mathrm{d}s\\ &+ \frac{\nu(1-\sigma)}{\rho}\int_0^1 u(t)\mathrm{d}\xi(t) + \frac{\mu(1-\sigma)}{\rho}\int_0^1 u(t)\mathrm{d}\eta(t)\Big\}\\ &\ge \frac{m_2}{M}\min\Big\{\int_0^1 k(\sigma,s)h(s)\mathrm{d}s + \frac{\nu(\sigma)\overline{A}\tau_1}{\rho}\int_0^1 k(s,s)h(s)\mathrm{d}s\\ &+ \frac{\mu(\sigma)\overline{B}\tau_1}{\rho}\int_0^1 k(s,s)h(s)\mathrm{d}s, \int_0^1 k(1-\sigma,s)h(s)\mathrm{d}s\\ &+ \frac{\nu(1-\sigma)\overline{A}\tau_1}{\rho}\int_0^1 k(s,s)h(s)\mathrm{d}s + \frac{\mu(1-\sigma)\overline{B}\tau_1}{\rho}\int_0^1 k(s,s)h(s)\mathrm{d}s\Big\}\\ &= m_2.\end{aligned}$$

That is, $\alpha(Tu)\ge m_2$ for $u\in P(\gamma,\theta,\alpha,m_2,m_2\tau_1^{-2},m_4)$. So (C1) of Lemma 2.4 is satisfied. Next we prove that (C2) of Lemma 2.4 is satisfied. Indeed, if $u\in P(\gamma,\alpha,m_2,m_4)$ with $\theta(Tu)>m_2\tau_1^{-2}$, then $\alpha(Tu)=\min_{\sigma\le t\le 1-\sigma}(Tu)(t)\ge \tau_1\max_{0\le t\le 1}(Tu)(t) = \tau_1\theta(Tu)\ge \tau_1 m_2\tau_1^{-2} > m_2$. Finally, we assert (C3) of Lemma 2.4 is satisfied. It is obvious that $0\notin Q(\gamma,\psi,m_1,m_4)$. Suppose that $u\in Q(\gamma,\psi,m_1,m_4)$ with $\psi(u)=m_1$. By (B3), we have

$$\psi(Tu)=\max_{0\le t\le 1}(Tu)(t)\le \frac{m_1}{N}\left[1+\frac{\nu(0)\overline{A}}{\rho}+\frac{\mu(1)\overline{B}}{\rho}\right]\int_0^1 k(s,s)h(s)\mathrm{d}s = m_1.$$

Therefore, the (C1)-(C3) of Lemma 2.4 are satisfied, then (1.1) has at least three positive solutions u_1,u_2,u_3 such that

$$\max_{0\le t\le 1}|u_i'(t)|\le m_4 \ \text{ for } i=1,2,3;\quad m_2<\min_{\sigma\le t\le 1-\sigma}u_1(t);\quad m_1<\max_{0\le t\le 1}u_2(t)\ \text{ with }$$
$$\min_{\sigma\le t\le 1-\sigma}u_2(t)<m_2;\quad \max_{0\le t\le 1}u_3(t)<m_1.$$

An example Consider the problem

$$\begin{cases} u''(t)+h(t)f(t,u(t),u'(t))=0, t\in(0,1),\\ u(0)=\frac{1}{4}u(\frac{1}{4})+\frac{1}{2}u(\frac{1}{2}),\quad u(1)=\frac{1}{3}u(\frac{1}{4})+\frac{1}{2}u(\frac{1}{2}).\end{cases}\tag{3.6}$$

where $h(t)=\frac{1}{\sqrt{t(1-t)}}$, $f(t,u,v)=\begin{cases}10^{-2}t+\frac{11}{5}u^{\frac{3}{2}}+10^{-6}v, & u<16,\\ 10^{-2}t+\frac{704}{5}+10^{-6}v, & u\ge 16.\end{cases}$

We define $\xi(t)$ and $\eta(t)$ by

$$\xi(t)=\begin{cases}0, & [0,\frac{1}{4}),\\ \frac{1}{4}, & [\frac{1}{4},\frac{1}{2}),\\ \frac{3}{4}, & [\frac{1}{2},1],\end{cases}\qquad \eta(t)=\begin{cases}0, & [0,\frac{1}{4}),\\ \frac{1}{3}, & [\frac{1}{4},\frac{1}{2}),\\ \frac{5}{6}, & [\frac{1}{2},1].\end{cases}$$

Then the problem (3.6) is equivalent to the following system

$$\begin{cases}u''(t)+h(t)f(t,u(t),u'(t))=0, t\in(0,1),\\ u(0)=\int_0^1 u(t)\mathrm{d}\xi(t),\quad u(1)=\int_0^1 u(t)\mathrm{d}\eta(t).\end{cases}$$

It is easy to see that the problem (3.6) is equivalent to the following nonlinear integral equation:

$$u(t)=\int_0^1 k(t,s)h(s)f(s,u(s),u'(s))\mathrm{d}s+(1-t)\int_0^1 u(t)\mathrm{d}\xi(t)+t\int_0^1 u(t)\mathrm{d}\eta(t),$$

where $k(t,s):=\begin{cases}(1-t)s, & 0\le s\le t\le 1,\\ (1-s)t, & 0\le t\le s\le 1.\end{cases}$ It is easy to check that hypotheses (H1)-(H4) hold. Let $\sigma:=\frac{1}{4}$. By direct calculation, we obtain

$$\tau_1=\frac{1}{4},\quad \tau_2=4,\quad \int_0^1 k(s,s)h(s)\mathrm{d}s=\frac{\pi}{8},\quad \int_0^1 k(\frac{1}{4},s)h(s)\mathrm{d}s=\frac{5\pi}{24}-\frac{\sqrt{3}}{4},$$

$$\int_0^1 k(\frac{3}{4},s)h(s)\mathrm{d}s=\frac{\pi}{6}-\frac{\sqrt{3}}{4},\quad \int_0^1(1-s)h(s)\mathrm{d}s=\int_0^1 sh(s)\mathrm{d}s=\frac{\pi}{2},$$

$$\overline{A}=\frac{73}{21},\quad \overline{B}=\frac{81}{21}, \lambda_1=\frac{91}{84},\ \lambda_2=\frac{133}{168},\ \ N\approx 3.2725,\ \ S\approx 4.4506,\ \ M\approx 0.4599.$$

If we choose $m_1=\frac{1}{8}$, $m_2=1$ and $m_4=10^4$, then $m\approx 14.7456$ and $f(t,u,v)$ satisfies

$$f(t,u,v)\le\frac{m_4}{S}\approx 2246.888,\qquad (t,u,v)\in[0,1]\times[0,40000]\times[-10000,10000],$$

$$f(t,u,v)>\frac{m_2}{M}\approx 2.1744,\qquad (t,u,v)\in[\frac{1}{4},\frac{3}{4}]\times[1,16]\times[-10000,10000],$$

$$f(t,u,v)\le\frac{m_1}{N}\approx 0.1528,\qquad (t,u,v)\in[0,1]\times[0,\frac{1}{8}]\times[-10000,10000].$$

Then all hypotheses of Theorem 3.1 hold. Hence, the problem (3.6) has at least three positive solutions u_1, u_2, u_3 such that

$$\max_{0\le t\le 1} |u_i'(t)| \le 10^4 \text{ for } i = 1,2,3; \quad 1 < \min_{\sigma\le t\le 1-\sigma} u_1(t); \quad \frac{1}{8} < \max_{0\le t\le 1} u_2(t) \text{ with } \min_{\sigma\le t\le 1-\sigma} u_2(t) < 1; \quad \max_{0\le t\le 1} u_3(t) < \frac{1}{8}.$$

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