Int. J. Open Problems Compt. Math., Vol. 3, No. 4, December 2010
ISSN 1998-6262; Copyright ©ICSRS Publication, 2010
www.i-csrs.org

Boundary Value Problem For Fractional Integro-Differential Equations With Nonlocal Conditions

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Abstract

In this paper, the existence problems of the solution of boundary value fractional integrodifferential equations with nonlocal conditions are investigated. The results are obtained using Banach and Krasnoselkii fixed point theorems.

Keywords: *Fractional calculus, existence and uniqueness; integrodifferential equations; fixed point theorems; nonlocal conditions.*

1 Introduction

Fractional differential equations are emerged as a new branch of applied mathematics by which many physical and engineering approaches can be modelled. The fact that fractional differential equations are considered as alternative models to nonlinear differential equations which induced extensive researches in various fields including the theoretical part (see [1]-[11] and references therein). The existence and uniqueness problems of fractional nonlinear differential and integrodifferential equations as a basic theoretical part of some applications are investigated by many authors (see for examples [3], [6], and [7]). In [1], the authors obtained sufficient conditions for the existence of solutions for a class of boundary value problem of fractional differential equations (in the case of $1 < \alpha \leq 2$) involving the Caputo fractional derivative and nonlocal conditions using the Banach and Schaefer's fixed points theorems. The Cauchy problems for some fractional abstract differential equations (in the case of $0 < \alpha \leq 1$) with nonlocal conditions are investigated by the authors in [2] and [10] using the Banach and Krasnoselkii fixed point theorems. The Banach fixed point theorem is used in [4] and [8] to investigate the existence problems of fractional integrodifferential equations (in the case of $0 < \alpha \leq 1$) in Banach spaces. Motivated by these works we study in this paper the existence of solution of boundary value problem for fractional integrodifferential equations (in the case of $1 < \alpha \leq 2$) in Banach spaces by using Banach and Krasnoselkii fixed point theorems.

2 Preliminaries

We need some basic definitions and properties of fractional calculus which will be used in this paper.

Definition 2.1 *A real function $f(t)$ is said to be in the space C_μ, $\mu \in \boldsymbol{R}$ if there exists a real number $p > \mu$, such that $f(t) = t^p f_1(t)$, where $f_1 \in C[0,\infty)$; and it is said to be in the space C_μ^n if and only if $f^{(n)} \in C_\mu, n \in \boldsymbol{N}$.*

Definition 2.2 *A function $f \in C_\mu$, $\mu \geq -1$ is said to be fractional integrable of order $\alpha > 0$ if*

$$I^\alpha f(t) = (I^\alpha f)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds < \infty,$$

and if $\alpha = 0$, then $I^0 f(t) = f(t)$.

Next, we introduce the Caputo fractional derivative.

Definition 2.3 *The fractional derivative in the Caputo sense is defined as*

$$D^{(\alpha)} f(t) = \left(D^{(\alpha)} f\right)(t) = I^{n-\alpha}\left(\frac{d^n f}{dt^n}\right)(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} \left(\frac{d^n f(s)}{ds^n}\right) ds$$

for $n-1 < \alpha \leq n$, $n \in \boldsymbol{N}$, $t > 0$, $f \in C_{-1}^n$. In particular, if $1 < \alpha \leq 2$, then $D^{(\alpha)} f(t) = \frac{1}{\Gamma(2-\alpha)} \int_0^t (t-s)^{1-\alpha} f''(s) ds$, where $f''(s) = \frac{d^2 f(s)}{ds^2}$, and $f \in C_{-1}^2$ is a function with values in abstract space X.

The identity

$$\left(I^\alpha D^{(\alpha)} f\right)(t) = f(t) + a + bt, \tag{2.1}$$

where $t \in J$, a, b are constants and other properties of the fractional operators used in the general theory of fractional differential equations can be found in [5], [9], and [11].

Let $Y = C^2(J, X)$ be a Banach space of all functions $x(t)$ having at most continuous second derivatives from a compact interval $J = [0, T]$ into a Banach space X. Let $D = \{(t,s) : 0 \leq s \leq t \leq T\}$ be subset of $\mathbf{R}^2$.

Consider the fractional nonlinear integrodifferential equation

$$\begin{cases} D^{(\alpha)}x(t) = f(t, x(t)) + \int\limits_0^t g(t, s, x(s))ds, \\ x(0) = h(x), \ x(T) = k(x), \end{cases} \tag{2.2}$$

where $1 < \alpha \leq 2$, and the nonlinear functions f, g, h, and k satisfy the following hypotheses:

(H1) $f : J \times Y \to Y$, $g : D \times Y \to Y$ are continuous functions and there exists a positive constant C such that

$$\begin{cases} \|f(t, x) - f(t, y)\| \leq C \|x - y\| \\ \|g(t, s, x) - g(t, s, y)\| \leq C \|x - y\| \end{cases}$$

for any $t \in J$, $(t, s) \in D$, $x, y \in Y$. Moreover, let $A = \sup_{t\in J} \|f(t, 0)\|$, $B = \sup_{(t,s)\in D} \|g(t, s, 0)\|$, and $L = \max\{A, B, C\}$.

(H2) $h : Y \to Y$, and $k : Y \to Y$ are continuous functions such that

$$\begin{cases} \|h(x) - h(y)\| \leq C \|x - y\| \\ \|k(x) - k(y)\| \leq C \|x - y\| \end{cases}$$

for any $x, y \in Y$.

Lemma 2.4 *The equation*

$$\begin{cases} D^{(\alpha)}x(t) = f(t) + \int\limits_0^t g(t, s)ds, \\ x(0) = h(x), \ x(T) = k(x), \end{cases} \tag{2.3}$$

is equivalent to the integral equation

$$x(t) = \left(\frac{T - t}{T}\right) h(x) + \frac{t}{T}k(x) - \frac{t}{T}I^\alpha y(T) + I^\alpha y(t) \tag{2.4}$$

where $y(t) = f(t) + \int\limits_0^t g(t, s)ds$ is a fractional integrable (of order α) function.

Proof. Applying the fractional integral operator I^α to both sides of equation (2.3), and using the identity (2.1), we get

$$\begin{aligned} I^\alpha D^{(\alpha)}x(t) &= I^\alpha f(t) + I^\alpha \int\limits_0^t g(t, s)ds \\ x(t) + a + bt &= \frac{1}{\Gamma(\alpha)} \int\limits_0^t (t - s)^{\alpha-1} f(s)ds + \frac{1}{\Gamma(\alpha)} \int\limits_0^t (t - s)^{\alpha-1} \left(\int\limits_0^s g(s, r)dr\right) ds. \end{aligned}$$

Now, if $t=0$, we have $a=-h(x)$, and if $t=T$, we have

$$k(x)-h(x)+bT=\frac{1}{\Gamma(\alpha)}\int_0^T(T-s)^{\alpha-1}f(s)ds+\frac{1}{\Gamma(\alpha)}\int_0^T(T-s)^{\alpha-1}\left(\int_0^s g(s,r)dr\right)ds$$

which implies that

$$\begin{aligned} b &= \frac{-k(x)}{T}+\frac{h(x)}{T}+\frac{1}{T\Gamma(\alpha)}\int_0^T(T-s)^{\alpha-1}f(s)ds \\ &\quad +\frac{1}{T\Gamma(\alpha)}\int_0^T(T-s)^{\alpha-1}\left(\int_0^s g(s,r)dr\right)ds. \end{aligned}$$

Therefore

$$\begin{aligned} x(t) &= h(x)+\frac{tk(x)}{T}-\frac{th(x)}{T}-\frac{t}{T\Gamma(\alpha)}\int_0^T(T-s)^{\alpha-1}f(s)ds \\ &\quad -\frac{t}{T\Gamma(\alpha)}\int_0^T(T-s)^{\alpha-1}\left(\int_0^s g(s,r)dr\right)ds \\ &\quad +\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s)ds+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\left(\int_0^s g(s,r)dr\right)ds \\ &= \left(\frac{T-t}{T}\right)h(x)+\frac{t}{T}k(x) \\ &\quad -\frac{t}{T\Gamma(\alpha)}\int_0^T(T-s)^{\alpha-1}\left(f(s)+\left(\int_0^s g(s,r)dr\right)\right)ds \\ &\quad +\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\left(f(s)+\left(\int_0^s g(s,r)dr\right)\right)ds \end{aligned}$$

which is equation (2.4). On the other hand, applying the fractional differential operator $D^{(\alpha)}$ to both sides of equation (2.4), it is easily to get equation (2.3).

In view of Lemma 2.4, equation (2.2) is equivalent to the integral equation

$$x(t)=\left(\frac{T-t}{T}\right)h(x)+\frac{t}{T}k(x)-\frac{t}{T}\left(I^\alpha F(x)\right)(T)+\left(I^\alpha F(x)\right)(t)$$

where $F(x)=f(s,x(s))+\left(\int_0^s g(s,r,x(r))dr\right)$ is a fractional integrable (of order α) nonlinear operator. The operator F satisfies the following estimates

$$\left\|\left(I^\alpha F(x)\right)(t)\right\| \leq \left(I^\alpha\left\|F(x)-F(0)\right\|\right)(t)+\left(I^\alpha\left\|F(0)\right\|\right)(t)$$

$$\leq \frac{Lt^\alpha}{\Gamma(\alpha+1)}\left(1+\frac{t}{\alpha+1}\right)(1+\|x\|)$$

and

$$\|I^\alpha\left(F(x)-F(y)\right)(t)\| \leq \frac{Lt^\alpha}{\Gamma(\alpha+1)}\left(1+\frac{t}{\alpha+1}\right)\|x-y\|$$

for every $x, y \in Y$, $t \in J$.

3 Existence problems

We prove the existence of the fractional nonlinear integrodifferential equation (2.2) by using the well-known Banach fixed point theorem. The following condition is essential to get the contraction property.

(H3) Let $0 < q < 1$, and r be a positive finite real number such that

$$\begin{cases} q \geq L\left(1+\frac{2T^\alpha}{\Gamma(\alpha+1)}\left(1+\frac{T}{\alpha+1}\right)\right) \\ r \geq (1-q)^{-1}\left(\|h(0)\|+\|k(0)\|+\frac{2LT^\alpha}{\Gamma(\alpha+1)}\left(1+\frac{T}{\alpha+1}\right)\right). \end{cases}$$

Moreover, let $B_r = \{y \in Y : \|y\| \leq r\}$.

Theorem 3.1 *If the hypotheses (H1)-(H3) are satisfied, then the fractional integrodifferential equation (2.2) has a unique solution on J.*

Proof. Define the operator $\Psi : Y \to Y$ by

$$\Psi x(t) = \left(\frac{T-t}{T}\right)h(x) + \frac{t}{T}k(x) - \frac{t}{T}\left(I^\alpha F(x)\right)(T) + \left(I^\alpha F(x)\right)(t).$$

We show that Ψ has a fixed point on B_r. This fixed point is then a solution of equation (2.2). Firstly, we show that $\Psi B_r \subset B_r$. Let $x \in B_r$, then

$$\begin{aligned}
\|\Psi x(t)\| &\leq \left\|\left(\frac{T-t}{T}\right)h(x)\right\| + \left\|\frac{t}{T}k(x)\right\| + \left\|\frac{t}{T}\left(I^\alpha F(x)\right)(T)\right\| + \|\left(I^\alpha F(x)\right)(t)\| \\
&\leq \frac{T-t}{T}\|h(x)\| + \frac{t}{T}\|k(x)\| + \frac{t}{T}\|\left(I^\alpha F(x)\right)(T)\| + \|\left(I^\alpha F(x)\right)(t)\| \\
&\leq \left(\frac{T-t}{T}\right)\|h(0)\| + C\left(\frac{T-t}{T}\right)\|x\| + C\frac{t}{T}\|x\| + \frac{t}{T}\|k(0)\| \\
&\quad + \frac{Lt^\alpha}{\Gamma(\alpha+1)}\left(1+\frac{t}{\alpha+1}\right)(1+\|x\|) \\
&\quad + \frac{tLT^\alpha}{T\Gamma(\alpha+1)}\left(1+\frac{T}{\alpha+1}\right)(1+\|x\|) \\
&\leq \|h(0)\| + \|k(0)\| + \frac{2LT^\alpha}{\Gamma(\alpha+1)}\left(1+\frac{T}{\alpha+1}\right) \\
&\quad + L\left(1+\frac{2T^\alpha}{\Gamma(\alpha+1)}\left(1+\frac{T}{\alpha+1}\right)\right)\|x\| \\
&\leq (1-q)\,r + qr = r.
\end{aligned}$$

Hence, the operator Ψ maps B_r into itself. Next, we prove that Ψ is a contraction mapping on B_r. Let $x, y \in B_r$, then

$$\begin{aligned}
\|\Psi x(t) - \Psi y(t)\| &= \Big\| \left(\frac{T-t}{T}\right) h(x) + \frac{t}{T} k(x) - \frac{t}{T} \left(I^\alpha F(x)\right)(T) + \left(I^\alpha F(x)\right)(t) \\
&\quad - \left(\frac{T-t}{T}\right) h(y) - \frac{t}{T} k(y) + \frac{t}{T} \left(I^\alpha F(y)\right)(T) - \left(I^\alpha F(y)\right)(t) \Big\| \\
&\le \left(\frac{T-t}{T}\right) \|h(x) - h(y)\| + \frac{t}{T} \|k(x) - k(y)\| \\
&\quad + \frac{t}{T} \left(I^\alpha \|F(y) - F(x)\|\right)(T) + \left(I^\alpha \|F(x) - F(y)\|\right)(t) \\
&\le L \left(\frac{T-t}{T}\right) \|x - y\| + L \frac{t}{T} \|x - y\| \\
&\quad + \frac{LT^\alpha}{\Gamma(\alpha+1)} \left(1 + \frac{T}{\alpha+1}\right) \|x - y\| \\
&\quad + \frac{Lt^\alpha}{\Gamma(\alpha+1)} \left(1 + \frac{t}{\alpha+1}\right) \|x - y\| \\
&\le L \left(1 + \frac{2T^\alpha}{\Gamma(\alpha+1)} \left(1 + \frac{T}{\alpha+1}\right)\right) \|x - y\| \le q \|x - y\| .
\end{aligned}$$

Hence, the operator Ψ has a unique fixed point which is a solution to equation (2.2).

The next result is based on the following well-known fixed point theorem.

Theorem 3.2 *(Krasnoselkii) Let S be a closed convex and nonempty subset of a Banach space X. Let P and Q be two operators such that*
(i) $Px + Qy \in S$ whenever $x, y \in S$;
(ii) P is a contraction mapping;
(iii) Q is compact and continuous.
Then there exists $z \in S$ such that $z = Pz + Qz$.

To apply the above theorem we need the following condition instead of the condition (H1).

(H4) The functions $f : J \times Y \to Y$, and $g : D \times Y \to Y$ are jointly continuous and there exists a positive constant L such that

$$\|f(t, y)\| + t \sup_{s \in J} \|g(t, s, y)\| \le L$$

for all $(t, y) \in J \times Y$.

Theorem 3.3 *If the hypotheses (H2) and (H4) are satisfied, and if $C < 1$, then the fractional integrodifferential equation (2.2) has a solution on J.*

Proof. Let $r \geq (1-C)^{-1}\left(\|h(0)\| + \|k(0)\| + \frac{2LT^{\alpha}}{\Gamma(\alpha+1)}\right)$. Define the operators P, and Q on the compact set $B_r = \{y \in Y : \|y\| \leq r\} \subset Y$ by

$$\begin{cases} Px(t) = \left(\frac{T-t}{T}\right) h(x) + \frac{t}{T} k(x) \\ Qy(t) = \left(I^{\alpha} F(y)\right)(t) - \frac{t}{T}\left(I^{\alpha} F(y)\right)(T). \end{cases}$$

We observe that

$$\begin{aligned} \left(I^{\alpha}\|F(y)\|\right)(t) &\leq I^{\alpha}\|f(t, y(t))\| + I^{\alpha}\left(\int_0^t \|g(t,s,y(s))\|\, ds\right) \\ &\leq \frac{LT^{\alpha}}{\Gamma(\alpha+1)}, \end{aligned}$$

hence

$$\|Qy(t)\| \leq \frac{2LT^{\alpha}}{\Gamma(\alpha+1)} \tag{3.1}$$

and

$$\begin{aligned} \|Px(t) + Qy(t)\| &\leq \|Px(t)\| + \|Qy(t)\| \\ &= \left\|\left(\frac{T-t}{T}\right) h(x) + \frac{t}{T} k(x)\right\| \\ &\quad + \left\|\left(I^{\alpha} F(y)\right)(t) - \frac{t}{T}\left(I^{\alpha} F(y)\right)(T)\right\| \\ &\leq \frac{(T-t)}{T}\|h(0)\| + + \frac{t}{T}\|k(0)\| + \frac{2LT^{\alpha}}{\Gamma(\alpha+1)} \\ &\quad + \frac{Ct}{T}\|x\| + \frac{C(T-t)}{T}\|x\|. \end{aligned}$$

Therefore, if $x, y \in B_r$, then $Px + Qy \in B_r$. On the other hand, it is easily to show that the operator P is a contraction. Indeed, since

$$\begin{aligned} \|Px(t) - Py(t)\| &\leq \left(\frac{T-t}{T}\right)\|h(x) - h(y)\| + \frac{t}{T}\|k(x) - k(y)\| \\ &\leq C\left(\frac{T-t}{T}\right)\|x - y\| + C\frac{t}{T}\|x - y\| \\ &= C\|x - y\|. \end{aligned}$$

By the hypothesis (H4), the operator Q is continuous and by the inequality (3.1), it is uniformly bounded on B_r. For the equicontinuity of $Qy(t)$, let $t_1, t_2 \in J$, and $y \in B_r$, we have

$$\begin{aligned} &\left\|\left(I^{\alpha} F(y)\right)(t_1) - \left(I^{\alpha} F(y)\right)(t_2)\right\| \\ = &\frac{1}{\Gamma(\alpha)}\left\|\int_0^{t_1}(t_1 - s)^{\alpha-1} f(s, y(s))ds + \int_0^{t_1}(t_1 - s)^{\alpha-1}\left(\int_0^s g(s,r,y(r))dr\right) ds\right. \end{aligned}$$

$$
\begin{aligned}
& -\frac{1}{\Gamma(\alpha)}\int_0^{t_2}(t_2-s)^{\alpha-1}f(s,y(s))ds-\int_0^{t_2}(t_2-s)^{\alpha-1}\left(\int_0^{s}g(s,r,y(r))dr\right)ds\Bigg\| \\
\leq\ & \frac{L}{\Gamma(\alpha)}\int_0^{t_1}\left|(t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}\right|ds+\frac{L}{\Gamma(\alpha)}\int_{t_1}^{t_2}\left|(t_2-s)^{\alpha-1}\right|ds \\
\leq\ & \frac{L}{\Gamma(\alpha+1)}\left(2\left|t_2-t_1\right|^{\alpha}+\left|t_2^{\alpha}-t_1^{\alpha}\right|\right)
\end{aligned}
$$

hence by (H4) one can get

$$
\begin{aligned}
\|Qy(t_1)-Qy(t_2)\| &\leq \|\left(I^{\alpha}F(y)\right)(t_1)-\left(I^{\alpha}F(y)\right)(t_2)\| \\
&\quad +\left\|\frac{t_2}{T}\left(I^{\alpha}F(y)\right)(T)-\frac{t_1}{T}\left(I^{\alpha}F(y)\right)(T)\right\| \\
&\leq \frac{L}{\Gamma(\alpha+1)}\left(2\left|t_2-t_1\right|^{\alpha}+\left|t_2^{\alpha}-t_1^{\alpha}\right|\right)+\left|t_2-t_1\right|\frac{LT^{\alpha}}{\Gamma(\alpha+2)} \\
&= \frac{L}{\Gamma(\alpha+1)}\left(2\left|t_2-t_1\right|^{\alpha}+\left|t_2^{\alpha}-t_1^{\alpha}\right|+T^{\alpha-1}\left|t_2-t_1\right|\right).
\end{aligned}
$$

As $t_2 \to t_1,$ the right-hand side of the above inequality tends to zero which gives the equicontinuity of $Qy(t)$. So $Q(B_r)$ is relatively compact. By the Arzela Ascoli Theorem, Q is compact. Hence by the Krasnoselkii theorem there exists a solution to equation (2.2).

4 Open Problem

Let $Y = C^2(J,X)$ be a Banach space of all functions $x(t)$ having at most continuous second derivatives from a compact interval $J = [0,T]$ into a Banach space X. Consider the fractional semilinear differential equation

$$
\begin{cases}
D^{(\alpha)}x(t) = Ax(t) + f(t,x(t)), \\
x(0) = h(x),\ x'(0) = k(x),
\end{cases}
$$

where $1 < \alpha \leq 2$, A is the infinitesimal generator of a strongly continuous cosine family $C(\cdot)$ of bounded linear operators on X. We denote by $S(\cdot)$ the associated sine function which is defined by $S(t)x = \int_0^t C(s)xds$ and the non-linear functions f, h and k satisfy appropriate conditions. The problem may be solved if one can establish a mild integral solution of the above equation and then a fixed point theorem can be applied to prove the existence problem.

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