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Minimal Immersion and Harmonic Maps in Heisenberg Group $\mathbb{H}^{2n+1}$

Essin TURHAN and Talat KÖRPINAR

Firat University, Department of Mathematics, 23119, Elazig, TURKEY
e-mail: essin.turhan@gmail.com

Abstract

In this work, we prove a Weierstrass representation formula for simply connected immersed minimal surfaces in Heisenberg group $\mathbb{H}^{2n+1}$***. Using the Weierstrass representation we also give a simple proof of the fact that minimal immersions is harmonic maps on the domain.***

Keywords: *harmonic map, Heisenberg group, immersion, minimal surface, Weierstrass representation.*

1 Introduction

For the first time Weierstrass representation for conformal immersion of surface into $\mathbb{R}^3$ appeared in the result of variational problem on search of minimal surface restricted by the some curve [19]. Generalization of Weierstrass formulae for surfaces with mean curvature $H \neq 0$ was proposed by Eisenhart in 1909 [7].

It has been shown [12] that Weierstrass representations are very useful and suitable tools for the systematic study of minimal surfaces immersed in n-dimensional spaces. This subject has a long and rich history. It has been extensively investigated since the initial works of Weierstrass [19]. In the literature there exists a great number of applications of the Weierstrass representation to various domains of Mathematics, Physics, Chemistry and Biology. In particular in such areas as quantum field theory and statistical physics [8], chemical physics, fluid dynamics and membranes [16], minimal surfaces play an essential role. More recently it is worth mentioning that works by Kenmotsu [10], Hoffmann [9], Osserman [15], Budinich [5], Konopelchenko [6,11] and Bobenko [3, 4] have made very significant contributions to constructing minimal surfaces in a systematic way and to understanding their intrinsic geometric properties as well as their integrable dynamics. The type of extension of the Weierstrass representation which has been useful in three-dimensional applications to multidimensional spaces will continue to generate many additional applications to physics and mathematics. According to [13] integrable deformations of surfaces are generated by the Davey–Stewartson hierarchy of 2+1 dimensional soliton equations. These deformations of surfaces inherit all the remarkable properties of soliton equations. Geometrically such deformations are characterised by the invariance of an infinite set of functionals over surfaces, the simplest being the Willmore functional.

Surface theory has been intensively studied in mathematics and physics. The application of the theory to solitary wave phenomena in physics yields socalled “soliton geometry”. An important branch is the Weierstrass representation of the surface in constant curvature space. The representation makes us study surfaces and their properties by means of analysis methods. A classical example of such an approach is given by the Weierstrass representation for the minimal surface in $\mathbb{R}^3$.

D. A. Berdinski and I. A. Taimanov gave a representation formula for minimal surfaces in 3-dimensional Lie groups in terms of spinors and Dirac operators (see [1]).

In this work, we prove a Weierstrass representation formula for simply connected immersed minimal surfaces in Heisenberg group $\mathbb{H}^{2n+1}$. Using the Weierstrass representation we also give a simple proof of the fact that minimal immersions is harmonic maps on the domain.

2 Heisenberg Group $\mathbb{H}^{2n+1}$

We begin with a well-known description of the Heisenberg group of dimension $2n+1$. Let $\mathbb{R}^n\times \mathbb{R}^n\times\mathbb{R}$ be the Euclidean space with coordinates (x, y, t) where $x = (x_1, ..., x_n) \in \mathbb{R}^n$, $y = (y_1, ..., y_n) \in \mathbb{R}^n$, $t \in \mathbb{R}$. Then the Heisenberg group $\mathbb{H}^{2n+1}$ is this space with the following multiplication rule:

$$(x, y, t)\cdot(x', y', t') = (x + x', y + y', t + t' + (\langle x, y'\rangle - \langle x', y\rangle)),$$

where $\langle , \rangle$ is a scalar product in $\mathbb{R}^n$. The element zero, 0 = (0,..., 0), is the unit of this group structure.

Let $\mathbb{H}^{2n+1} = (\mathbb{R}^{2n+1}, g)$ be the Heisenberg group endowed with the Riemannian metric g which is defined by

$$g = \sum_{i=1}^{n}(dx_i^2 + dy_i^2) + \left[dt + \sum_{i=1}^{n}(y_i dx_i - x_i dy_i)\right]^2.$$

Note that the metric g is left invariant.

The vector fields

$$X_i = \frac{\partial}{\partial x_i} - y_i \frac{\partial}{\partial t}, \quad Y_i = \frac{\partial}{\partial y_i} + x_i \frac{\partial}{\partial t}, \quad T = \frac{\partial}{\partial t} \tag{2.1}$$

are then left-invariant vector fields. We define the left-invariant metric on $\mathbb{H}^{2n+1}$ by taking X_i, Y_i, T as the orthonormal frame.

The bracket relations for our left-invariant fields

$$[X_i, Y_i] = 2T, \quad [X_i, Y_j] = [X_i, T] = [Y_i, T] = 0, \quad i \neq j. \tag{2.2}$$

3 Harmonic Map and Minimal Surface in $\mathbb{H}^{2n+1}$

In this section, we obtain an integral representation formula for minimal surfaces in the Heisenberg group $\mathbb{H}^{2n+1}$.

We will denote with $\Omega \subseteq \mathbb{C} \cong \mathbb{R}^2$ a simply connected domain with a complex coordinate $z = u + iv$, $u, v \in \mathbb{R}$. Also we will use the standard notations for complex derivatives:

$$\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}\right), \quad \frac{\partial}{\partial \overline{z}} := \frac{1}{2}\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right). \tag{3.1}$$

For $X \in h^{2n+1}$, denote by $\mathsf{ad}(X)^*$ the adjoint operator of $\mathsf{ad}(X)$, i.e., it satisfies the equation

$$g\left([X, Y], Z\right) = g\left(Y, \mathsf{ad}(X)^* (Z)\right), \tag{3.2}$$

for any $Y, Z \in h^{2n+1}$. Let U be the symmetric bilinear operator on h^{2n+1} defined by

$$U(X, Y) := \frac{1}{2}\left\{\mathsf{ad}(X)^*(Y) + \mathsf{ad}(Y)^*(X)\right\}. \tag{3.3}$$

We have

$$U(X_i, T) = Y_i \ , \ U(Y_i, T) = X_i \ , \ U(X_i, Y_i) = U(T, T) = 0. \tag{3.4}$$

Lemma 3.1 *(see [18]) Let D be a simply connected domain. A smooth map $\varphi : D \longrightarrow \mathbb{H}^{2n+1}$ is harmonic if and only if*

$$\left(\varphi^{-1}\varphi_u\right)_u + \left(\varphi^{-1}\varphi_v\right)_v - \mathsf{ad}\left(\varphi^{-1}\varphi_u\right)^*\left(\varphi^{-1}\varphi_u\right) - \mathsf{ad}\left(\varphi^{-1}\varphi_v\right)^*\left(\varphi^{-1}\varphi_v\right) = 0 \tag{3.5}$$

holds.

Let $z = u + iv$. Then in terms of complex coordinates z, $\bar{z}$, the harmonic map equation (3.4) can be written as

$$\frac{\partial}{\partial \bar{z}}\left(\varphi^{-1}\frac{\partial \varphi}{\partial z}\right) + \frac{\partial}{\partial z}\left(\varphi^{-1}\frac{\partial \varphi}{\partial \bar{z}}\right) - 2U\left(\varphi^{-1}\frac{\partial \varphi}{\partial z}, \varphi^{-1}\frac{\partial \varphi}{\partial \bar{z}}\right) = 0. \tag{3.6}$$

Let $\varphi^{-1}d\varphi = Adz + \bar{A}d\bar{z}$. Then, (3.6) is equivalent to

$$A_{\bar{z}} + \bar{A}_z = 2U\left(A, \bar{A}\right). \tag{3.7}$$

The Maurer–Cartan equation is given by

$$A_{\bar{z}} - \bar{A}_z = \left[A, \bar{A}\right]. \tag{3.8}$$

(3.7) and (3.8) can be combined to a single equation

$$A_{\bar{z}} = U\left(A, \bar{A}\right) + \frac{1}{2}\left[A, \bar{A}\right]. \tag{3.9}$$

(3.8) is both the integrability condition for the differential equation $\varphi^{-1}d\varphi = Adz + \bar{A}d\bar{z}$ and the condition for φ to be a harmonic map.

For a smooth map $\varphi : \Sigma \longrightarrow \mathbb{H}^{2n+1}$, we write $\varphi(z) = (x^i(z), y^i(z), t(z))$. The following is obtained by straightforward calculation.

Lemma 3.2

$$A = \left(t_z + y^i x^i_z - x^i y^i_z\right) T + \sum_{i=1}^{n}\left(x^i_z X_i + y^i_z Y_i\right). \tag{3.10}$$

Lemma 3.3

$$\overline{A} = \left(t_{\bar{z}} + y^i x^i_{\bar{z}} - x^i y^i_{\bar{z}}\right) T + \sum_{i=1}^{n}\left(x^i_{\bar{z}} X_i + y^i_{\bar{z}} Y_i\right). \tag{3.11}$$

Theorem 3.4 *$\varphi : \Omega \longrightarrow \mathbb{H}^{2n+1}$ is harmonic if and only if the following equations hold:*

$$t_{z\bar{z}} + \sum_{i=1}^{n}\left(y^i x^i_{z\bar{z}} - x^i y^i_{z\bar{z}}\right) = 0. \tag{3.12}$$

Proof. Using (3.5) and (3.7), we have (3.12).

We choose

$$\xi^i = x^i_z dz,\ \eta^i = y^i_z dz,\ w = \left(t_z + y^i x^i_z - x^i y^i_z\right) dz. \tag{3.13}$$

Theorem 3.5

$$\overline{\partial} w = \sum_{i=1}^{n} \left(\xi^i \wedge \overline{\eta}^i - \overline{\xi}^i \wedge \eta^i \right). \tag{3.14}$$

Proof. From $w = (t_z + y^i x^i_z - x^i y^i_z)\, dz$, we have

$$\overline{\partial} w = \left[t_{z\overline{z}} + \sum_{i=1}^{n} \left(y^i x^i_{z\overline{z}} + x^i_z y^i_{\overline{z}} - x^i_{\overline{z}} y^i_z - x^i y^i_{z\overline{z}} \right) \right] dz \wedge d\overline{z}.$$

Using Theorem 3.4, we get

$$\overline{\partial} w = \sum_{i=1}^{n} \left(x^i_z y^i_{\overline{z}} - x^i_{\overline{z}} y^i_z \right) dz \wedge d\overline{z}.$$

Making necessary calculations, we have (3.14).

Theorem 3.6 *Let Σ be a simply connected Riemann surface. If an 2n+1-tuple of 1-forms (ξ^i, η^i, w) on Σ satisfy (3.5), then*

$$\varphi(z) = \left(2\int_{z_0}^{z} \operatorname{Re} \xi^i, 2\int_{z_0}^{z} \operatorname{Re} \eta^i, 2\int_{z_0}^{z} \operatorname{Re} \left(w + x^i \eta^i - y^i \xi^i \right) \right) \tag{3.15}$$

gives a harmonic map $\Sigma \longrightarrow \mathbb{H}^{2n+1}$.

Proof. By Theorem 3.4, we see that φ is a harmonic map if and only if φ satisfy (3.15).

From (3.8) we have

$$x^i(z) = 2\int_{z_0}^{z} \operatorname{Re} \xi^i, \quad y^i(z) = 2\int_{z_0}^{z} \operatorname{Re} \eta^i, \quad t(z) = 2\int_{z_0}^{z} \operatorname{Re} \left(w + x^i \eta^i - y^i \xi^i \right). \tag{3.16}$$

The theorem is proved.

By using Theorem 3.5 and (3.5), we have the following theorem:

Theorem 3.7

$$t_{z\overline{z}} = \sum_{i=1}^{n} \left(\xi^i \wedge \overline{\partial} \overline{\eta}^i - \overline{\partial} \xi^i \wedge \overline{\eta}^i \right). \tag{3.17}$$

4 Open Problem

In this work, we obtain relationship between harmonic maps and representation formula. We have given some explicit characterizations of these maps. Additionally, problems such as; investigation of relationship between biharmonic maps and representation formula.

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