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Study of a Perturbed Transport Problem in a Bounded Domain With a Convex Angle

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Abstract

We consider a perturbed transport problem in a domain having an isolated corner point. We prove the existence and the uniqueness of the solution, for small data in weighted Sobolev spaces $\mathcal{V}_{\xi}^{k,2}$, where the index ξ characterizes the power growth of the solution near the angular point.

Keywords: *singular domains, transport equation, viscoelastic fluids, weakly compressible, weighted Sobolev space*

1 Introduction

In the 20th century and particularly in its second half, a great number of mathematical articles and books on elliptic boundary problems in domains with a singular boundary were published. The first works, which develop a general theory for elliptic problems in domains having angular or conical points, are those of G.I. Eskin [4, 5], Ya.B. Lopatinskii [20] and V.A. Kondrat'ev [18, 19]. Let us also cite the books by P. Grisvard [9, 10], and by V.A. Kozlov, V.G. Maz'ya and J. Rossman [16, 17]. There are many interesting papers and it is difficult to make a complete list of them. Let us recall in particular the papers of M. Dauge (for example [3]), M.G. Garroni, V.A. Solonnikov and M.A. Vivaldi [11], P. Grisvard and G. Iooss [6], and G. Sweers [27].

The mathematical analysis of several problems intervening in fluid engineering, and posed in domains with a singular boundary, is still open. Recently, the problems of Stokes, Neumann and Navier-Stokes in domains having corners were studied in several papers. We can recall the works of R.B. Kellogg and J.E. Osborn [12], S.A. Nazarov, A. Novotný and K. Pileckas [23], S.A. Nazarov, M. Specovius-Neugebauer and J.H. Videman [24], J.R. Kweon and R.B. Kellogg [13, 14, 15], A. Kokotov and B. Plamenevskii (see for example [21]). In parallel to these theoretical works, several numerical studies have been published (see for example [1, 2]). In the case of viscoelastic fluids, many attempts have been made to study the problem of re-entrant corner (*i.e* having an angle $> \pi$): see for instance [26, 8].

Our objective here is to study the existence and uniqueness of the solution to a perturbed transport problem, in a domain having corners. This work[1] is organized as follows. After giving the modeling of the problem under study and describing the flow domain in Section 2, we present the notation and the functional setting in Section 3. Section 4 consists the main Theorem. We give some auxiliary results and preliminary lemmas in Section 5. In Section 6, we prove the main theorem. Lastly, some perspectives are posed.

2 Modeling and Description of The Domain

2.1 Modeling

We study the existence and uniqueness of the solution to a perturbed transport problem in a singular domain Ω with boundary $\Gamma = \partial\Omega$. Let us consider the problem: find p such that

$$L[\mathbf{z}]^{-1}p + \mathrm{a}(\mathbf{z}\cdot\nabla)p = q, \qquad \text{in } \Omega, \tag{1}$$

where q and $\mathbf{z}$ are given, a assumed to be a positive constant and

$$L[\mathbf{z}] = (1-\mathrm{b})I + \mathrm{b}(I + \mathrm{c}(\mathbf{z}\cdot\nabla)),$$

with b and c are two real positive number such that $0 < \mathrm{b} < 1$. A simpler transport equation has been resolved in [23]. We adapt their proof to our case and derive estimates of the solution.

2.2 Description of The Domain

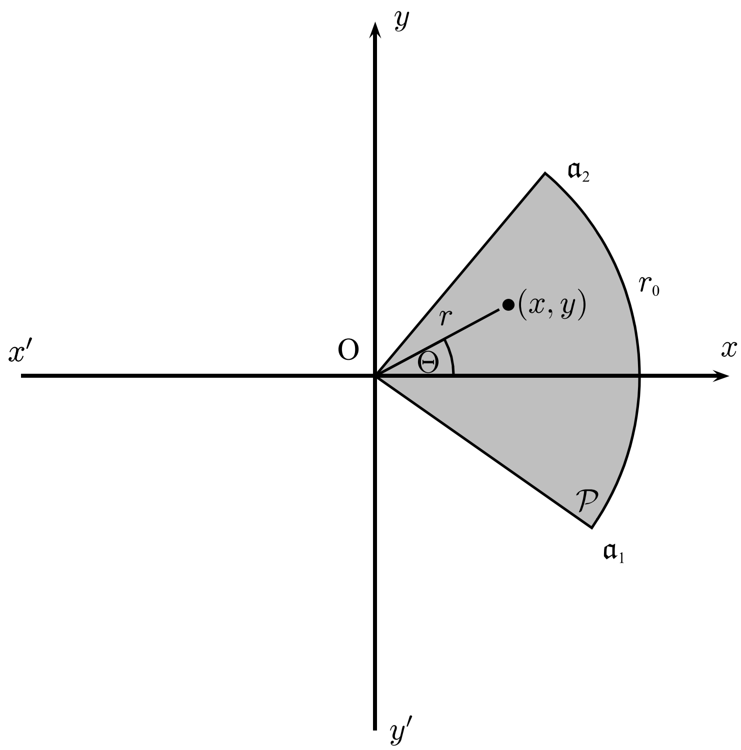
For the sake of simplicity, we consider a bounded domain $\Omega \subset \mathbb{R}^2$ with one single corner point placed at the origin O. Near its angular point, Ω is, locally, a convex angular sector $\mathcal{P}$ of angle measured by $\theta_0 \in (0,\pi)$; more precisely, $\theta_0 = |\mathfrak{a}_2 - \mathfrak{a}_1|$ where the angle measures $\mathfrak{a}_1 \in (-\pi/2, 0)$ and $\mathfrak{a}_2 \in (0, \pi/2)$ are given with respect to the x-axis. In particular, Γ coincides near O with the

[1]based on a study completed in collaboration with C. Guillopé and R. Talhouk [7].

half lines $\{\theta = \mathfrak{a}_1, r > 0\}$ and $\{\theta = \mathfrak{a}_2, r > 0\}$. A point $\mathbf{x} = (x, y)$ of $\mathbb{R}^2$ is given by its polar coordinates (r, θ), where $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$. The angular sector $\mathcal{P}$ is then, in polar coordinates, the set $(0, r_0) \times (\mathfrak{a}_1, \mathfrak{a}_2)$,

$$\mathcal{P} \;=\; \{(x, y) \in \mathbb{R}^2; r = \sqrt{x^2 + y^2} \text{ and } \Theta = \tan^{-1}\left(\frac{y}{x}\right) \leq \theta_0 \text{ such that } 0 < r \leq r_0\},$$

where $r_0 > 0$ is a given real number. We also define $\epsilon_0 = \frac{\pi}{\theta_0} > 1$.



We suppose the following conditions:

1. $B(O, r_0) \cap \Omega$ coincides with $\mathcal{P}$, where $B(O, r_0)$ is the ball centered at O, with radius r_0;

2. Γ is regular outside O, say C^3 at least.

3 Notation, Spaces and Norms

In this paper, C will denote a generic constant depending on certain data, but not depending on the unknown functions; we shall make explicit this dependence on the data. The letters C_i, $i = 1, 2, \cdots$, will denote some particular such constants. In particular, C_*, C_1 and C_2 will be constants taking different values, depending only on the specified data: $C_* = C(\Omega, k, \xi)$, $C_1 = C(\Omega, k, \xi, \mathrm{c})$

and $C_2 = C(\Omega, k, \xi, \mathrm{c}, \mathrm{b})$. Spaces of scalar functions and of vector-functions are not distinguished in our notation.

If u is a function or a vector defined almost everywhere on a bounded domain Ω in $\mathbb{R}^2$, one denotes by $|u|$ its absolute value in $\mathbb{R}$ or its Euclidean norm in $\mathbb{R}^2$. The norm of Lebesgue spaces $\mathrm{L}^2(\Omega)$ or $\mathrm{L}^\infty(\Omega)$ are denoted by $||\cdot||_0$ or $||\cdot||_\infty$. $||\cdot||_k$, $k \in \mathbb{N}^*$, denote the norm of Sobolev spaces $W^{k,2}(\Omega)$. $W^{k-\frac{1}{2},2}(\Gamma)$ is the associated trace function space on Γ. We denote by $W^{k,2}_{\mathrm{loc}}(\Omega)$ the usual local space associated to $W^{k,2}(\Omega)$. We will use also the homogeneous Sobolev spaces $W^{k,2}_0(\Omega)$ of functions of $W^{k,2}(\Omega)$ vanishing on Γ, the Hölder spaces $C^{m+\alpha}(\Omega)$, $m \in \mathbb{N}$, $0 < \alpha < 1$, and the space $C_0^\infty(\mathbb{R}^2)$ or $C_0^\infty(\Omega)$ of C^∞-functions vanishing at infinity or having compact support in Ω.

3.1 Hölder and Sobolev Weighted Spaces

Let ξ be a real number. For all $k \in \mathbb{N}$, we define the weighted Sobolev space $\mathcal{V}^{k,2}_\xi(\Omega)$ as the space of functions $u = u(\mathbf{x})$ of $\mathrm{L}^2(\Omega)$, for which the norm $||\cdot||_{k,2,\xi}$ is finite, where

$$||u||^2_{k,2,\xi} = \sum_{i=0}^{k} ||r^{\xi-k+i}\, u||^2_i, \tag{2}$$

with $r = |\mathbf{x}|$. $\mathcal{V}^{k,2}_\xi(\Omega)$ is a Banach space for its norm $||\cdot||_{k,2,\xi}$. For simplicity, we denote by $[\![\cdot]\!]_{k,\xi}$ the $\mathcal{V}^{k,2}_\xi(\Omega)$-norm. If necessary, we may also make explicit the dependence on the domain Ω, by denoting $[\![\cdot]\!]_{k,\xi;\Omega}$ the $\mathcal{V}^{k,2}_\xi(\Omega)$-norm. Note that, in the case of regular domain Ω (C^k for example), $\mathcal{V}^{k,2}_\xi(\Omega)$ coincide with $W^{k,2}(\Omega)$.

For $k \in \mathbb{N}^*$, $\mathcal{V}^{k-\frac{1}{2},2}_\xi(\Gamma)$ is the space of traces on Γ of functions of $\mathcal{V}^{k,2}_\xi(\Omega)$; it is equipped with the norm

$$[u]_{k-\frac{1}{2},\xi} = \inf\{[\![v]\!]_{k,\xi}; v \in \mathcal{V}^{k,2}_\xi(\Omega) \text{ such that } v_{|\Gamma} = u\}.$$

For $m \in \mathbb{N}$ and $0 < \delta < 1$, $\Lambda^{m+\delta}_\xi(\Omega)$ is the weighted Hölder space equipped with the norm

$$||u||_{\Lambda^{m+\delta}_\xi} = \left(\sum_{k=0}^{m} ||r^{\xi-m-\delta+k} u||_{C^k(\Omega)} + ||r^\xi u||_{C^{m+\delta}(\Omega)}\right).$$

4 The Main Result

Recall that $\epsilon_0 = \pi/\theta_0$, defined in Section 2.2, characterizes the angular point of the singular domain.

Theorem 4.1. *Let Ω be a bounded domain in $\mathbb{R}^2$, as defined in Section 2.2. Let $k \in \mathbb{N}^*$ and $\xi \in \mathbb{R}$ be such that $\xi \leq k$. Suppose that*

$$\mathbf{z} \in \mathcal{V}_{\xi}^{k+2,2}(\Omega), \quad q \in \mathcal{V}_{\xi}^{k+1,2}(\Omega).$$

Then there exist two positive constants $\widehat{\alpha} = \widehat{\alpha}(\Omega, \xi, k, \mathrm{b})$ and $\widehat{\mathrm{a}} = \widehat{\mathrm{a}}(\Omega, k, \xi, \mathrm{b})$ such that if

$$\mathrm{a} < \widehat{\mathrm{a}}, \tag{3}$$

and

$$\widehat{\alpha}\mathrm{c}[\![\mathbf{z}]\!]_{k+2,\xi} < 1, \tag{4}$$

then Problem (1) admits a unique solution $p \in \mathcal{V}_{\xi}^{k+1,2}(\Omega)$.

Moreover, the following estimates hold

$$[\![p]\!]_{k+1,\xi} \leq C_2[\![q]\!]_{k+1,\xi}, \tag{5}$$

$$[\![\Delta p]\!]_{k-1,\xi} \leq C_2\left([\![\Delta q]\!]_{k-1,\xi} + \left([\![\mathbf{z}]\!]_{k+2,\xi} + \mathrm{a}\,\mathrm{b}\right)[\![p]\!]_{k+1,\xi}\right). \tag{6}$$

5 Auxiliary and Preliminary Results

Lemma 5.1. *Let $k \in \mathbb{N}^*$ and $\xi \in \mathbb{R}$ such that $\xi < k$. There exists a constant $C_* = C(\Omega, k, \xi)$ such that for all $f \in \mathcal{V}_{\xi}^{k,2}(\Omega)$, $g \in \mathcal{V}_{\xi}^{k+1,2}(\Omega)$, $l \in \mathcal{V}_{\xi}^{k+2,2}(\Omega)$, and $h \in \mathcal{V}_{\xi}^{k-1,2}(\Omega)$.*

$$[\![fg]\!]_{k,\xi} \leq C_*[\![f]\!]_{k,\xi}[\![g]\!]_{k+1,\xi}\,, \tag{7}$$

$$[\![(l\cdot\nabla)l]\!]_{k,\xi} \leq C_*[\![l]\!]_{k+2,\xi}^2\,, \tag{8}$$

$$[\![gh]\!]_{k-1,\xi} \leq C_*[\![g]\!]_{k+1,\xi}[\![h]\!]_{k-1,\xi}\,, \tag{9}$$

$$[\![lf]\!]_{k-1,\xi} \leq C_*[\![l]\!]_{k+2,\xi}[\![f]\!]_{k,\xi}\,. \tag{10}$$

Proof.

The estimates (7) and (8) are already shown in [23]. We will show (9).

Let $k \in \mathbb{N}^*$ and $\ell = 0, 1, \cdots, k-1$. By the Leibniz formula, and the definition of the norm (2) in $\mathcal{V}_{\xi}^{k+2,2}(\Omega)$, we have

$$[\![\nabla^{\ell}(gh)]\!]_{0,\xi-k+1+\ell}^2 \leq \sum_{\substack{i+j=\ell\\0\leq j\leq\ell}} \int_{\Omega} |\nabla^i g|^2 |\nabla^j h|^2 |\mathbf{x}|^{2(\xi-k+1+\ell)}\,d\mathbf{x}.$$

For $\xi < k$ and $i + j = \ell$, we calculate

$$\begin{aligned}\int_{\Omega} |\nabla^i g|^2 |\nabla^j h|^2 |\mathbf{x}|^{2(\xi-k+1+\ell)}\,d\mathbf{x} &= \int_{\Omega} |\nabla^i g|^2 |\mathbf{x}|^{2(\xi-k+i)} |\nabla^j h|^2 |\mathbf{x}|^{2(\xi-k+1+j)} |\mathbf{x}|^{2(k-\xi)}\,d\mathbf{x}\\ &\leq C_*[\![\nabla^i g]\!]_{\Lambda^0_{\xi-k+i}}^2 [\![\nabla^j h]\!]_{0,\xi-k+1+j}^2.\end{aligned}$$

For $i \in \{0, \cdots, \ell\}$, using the inclusions $\mathcal{V}_\xi^{k+1-i,2}(\Omega) \subset \Lambda^0_{\xi-k+i}(\Omega)$, we obtain

$$\begin{aligned}\int_\Omega |\nabla^i g|^2 |\nabla^j h|^2 |\mathbf{x}|^{2(\xi-k+1+\ell)}\, d\mathbf{x} &\leq [\![\nabla^i g]\!]^2_{k+1-i,\xi} [\![\nabla^j h]\!]^2_{0,\xi-k+1+j} \\ &\leq C_* [\![g]\!]^2_{k+1,\xi} [\![h]\!]^2_{k-1,\xi}.\end{aligned}$$

To finish the proof of estimate (9), we sum for ℓ between 0 and $k-1$, and obtain

$$[\![gh]\!]^2_{k-1,\xi} = \sum_{\ell=0}^{k-1} [\![\nabla^\ell (gh)]\!]^2_{0,\xi-k+1+\ell} \leq C_* [\![g]\!]^2_{k+1,\xi} [\![h]\!]^2_{k-1,\xi}.$$

We use the same method to show (10). □

Define the operator

$$Z[\mathbf{z}] \equiv I + \mathrm{c}(\mathbf{z} \cdot \nabla) : \mathcal{V}_\xi^{k+1,2}(\Omega) \longrightarrow \mathcal{V}_\xi^{k+1,2}(\Omega).$$

The following proposition shows that $Z[\mathbf{z}]$ is invertible in certain Sobolev weighted spaces, which will be appropriate for our problem.

Proposition 5.2 ([23], Theorem 5.6)**.** *Let $k \in \mathbb{N}^*$ and $\xi \leq k$. Suppose that $g \in \mathcal{V}_\xi^{k+1,2}(\Omega)$ and $\mathbf{z} \in \mathcal{V}_\xi^{k+2,2}(\Omega)$, with $\mathbf{z} \cdot n_{|\Gamma} = 0$. There exists a constant $\alpha_1 = \alpha_1(\xi, k)$ such that if*

$$\alpha_1 \mathrm{c} [\![\mathbf{z}]\!]_{k+2,\xi} < 1, \tag{11}$$

then the problem

$$u + \mathrm{c}(\mathbf{z} \cdot \nabla) u = g,$$

admits a unique solution $u \in \mathcal{V}_\xi^{k+1,2}(\Omega)$.

Moreover, there holds the following estimate

$$[\![u]\!]_{k+1,\xi} \leq C_1 [\![g]\!]_{k+1,\xi}. \tag{12}$$

with $C_1 = C(\Omega, k, \xi, \mathrm{c})$.

The properties of $L[\mathbf{z}]$ are given in the following proposition.

Proposition 5.3. *Let $k \in \mathbb{N}^*$ and $\xi \leq k$. We suppose that $\mathbf{z} \in \mathcal{V}_\xi^{k+2,2}(\Omega)$, with $\mathbf{z} \cdot n_{|\Gamma} = 0$.*
There exists a positive constant $\widetilde{\alpha} = \widetilde{\alpha}(\xi, k, \mathrm{b})$ such that

$$\widetilde{\alpha} \mathrm{c} [\![\mathbf{z}]\!]_{k+2,\xi} < 1,$$

the operator $L[\mathbf{z}] : \mathcal{V}_\xi^{k+1,2}(\Omega) \longrightarrow \mathcal{V}_\xi^{k+1,2}(\Omega)$ is an isomorphism.

6 Proof of Theorem 4.1

We will show this theorem in several steps.

Step 1. Approximation of the domain and of the transport equation. Let $B(O,\mathrm{r})$ be the ball of the plane having center O and radius $\mathrm{r} > 0$. Let χ be a function of $\mathcal{C}^\infty(\mathbb{R}^2)$ such that

$$\chi \equiv 0 \text{ in } B(O,1), \quad \chi \equiv 1 \text{ in } \mathbb{R}^2 \setminus B(O,2).$$

We define the cut-off function χ_s, for $s > 0$, by $\chi_s(\mathbf{x}) = \chi\left(\frac{\mathbf{x}}{2s}\right)$, for all $\mathbf{x} \in \mathbb{R}^2$. Note that, for all $s > 0$, there exists a constant c such that for all $\alpha \in \mathbb{N}^2$,

$$|\nabla^\alpha \chi_s(\mathbf{x})| \ \le \ c\,|\mathbf{x}|^{-|\alpha|}, \quad \forall \mathbf{x} \in \mathbb{R}^2.$$

Define $\{\Omega_s\}_{s>0}$ be a family of sub domains of Ω such that

$$\Omega_s \subset \Omega \backslash B(O,s), \quad \Omega_{s_1} \subset \Omega_{s_2} \quad \text{for } s_2 < s_1,$$

$$\partial\Omega_s \in C^{k+2}, \quad \partial\Omega_s \subset \overline{\Omega}, \quad \partial\Omega_s \backslash B(O,2s) = \partial\Omega \backslash B(O,2s).$$

Define $\mathbf{z}_s = \mathbf{z}\chi_s$. We shall first solve the following equation, which approximates (1):

$$L[\mathbf{z}_s]^{-1} p_s + \mathrm{a}(\mathbf{z}_s \cdot \nabla) p_s = q, \quad \text{in } \Omega_s. \tag{13}$$

Step 2. Existence of solutions to the transport equation (13). For fixed $s > 0$, if $\mathbf{z} \in \mathcal{V}_\xi^{k+2,2}(\Omega)$ and $q \in \mathcal{V}_\xi^{k+1,2}(\Omega)$, then $\mathbf{z}_s|_{\Omega_s} \in W^{k+2,2}(\Omega_s)$ and $q|_{\Omega_s} \in W^{k+1,2}(\Omega_s)$. Problem (13) is solved in the following lemma, whose demonstration is given in [28].

Lemma 6.1. [28] *Let $\Omega_s \in C^{k+2}$ be a bounded domain with $k \in \mathbb{N}^*$. Suppose that*

$$\mathbf{z}_s \in W^{k+2,2}(\Omega_s) \cap W_0^{1,2}(\Omega_s), \quad q \in W^{k+1,2}(\Omega_s).$$

Then there exist two constants $\alpha_3 = \alpha_3(\Omega, k, \omega)$ and $\mathrm{a}_1 = \mathrm{a}_1(\Omega, k, \mathrm{b})$ such that if

$$\alpha_3 \mathrm{c}||\mathbf{z}_s||_{k+2} < 1, \quad \mathrm{a} < \mathrm{a}_1,$$

Problem (13) admits a unique solution in $W^{k+1,2}(\Omega_s)$. Moreover, the estimates

$$\begin{aligned} ||p_s||_{k+1} &\le C_2||q||_{k+1}, \\ ||\Delta p_s||_{k-1} &\le C_2\left(||\Delta q||_{k-1} + (||\mathbf{z}||_{k+2} + \mathrm{a}\,\mathrm{b})\,||p_s||_{k+1}\right) \end{aligned}$$

hold.

Step 3. Estimates of p_s in $\mathcal{V}_\xi^{k+1,2}(\Omega_s)$.

Lemma 6.2. *Let $s > 0$, $k \in \mathbb{N}^*$ and $\xi < k$. Let $\mathbf{z}_s \in W^{k+2,2}(\Omega_s) \cap W_0^{1,2}(\Omega_s)$, and $q \in \mathcal{V}_\xi^{k+1,2}(\Omega_s)$. There exist two constants $\overline{\alpha} = \overline{\alpha}(\Omega, k, \xi, \mathrm{b})$ and $\overline{\mathrm{a}} = \overline{\mathrm{a}}(\Omega, k, \xi, \mathrm{b})$, such that if*

$$\mathrm{a} < \overline{\mathrm{a}}, \tag{14}$$

and

$$\overline{\alpha}\, \mathrm{c} [\![\mathbf{z}_s]\!]_{k+2,\xi;\Omega_s} < 1, \tag{15}$$

the following estimates

$$[\![p_s]\!]_{k+1,\xi;\Omega_s} \leq C_2 [\![q]\!]_{k+1,\xi}, \tag{16}$$
$$[\![\Delta p_s]\!]_{k-1,\xi;\Omega_s} \leq C_2 \left[[\![\Delta q]\!]_{k-1,\xi;\Omega_s} + \left([\![\mathbf{z}_s]\!]_{k+2,\xi;\Omega_s} + \mathrm{a}\, \mathrm{b} \right) [\![p_s]\!]_{k+1,\xi} \right] \tag{17}$$

hold for any solution $p_s \in W^{k+1,2}(\Omega_s)$ to Problem (13), with a constant C_2 independent of s.

Proof.
Assume

$$\widetilde{\alpha}\, \mathrm{c} [\![\mathbf{z}_s]\!]_{k+2,\xi;\Omega_s} < 1, \tag{18}$$

where $\widetilde{\alpha}$ has been defined in Proposition 5.3; thus $L[\mathbf{z}_s] = (1-\mathrm{b})I + \mathrm{b} Z[\mathbf{z}_s]^{-1}$ is an isomorphism in $\mathcal{V}_\xi^{k+2,2}(\Omega_s)$. The transport problem (13) is then equivalent to

$$p_s + \mathrm{a}\, L[\mathbf{z}_s](\mathbf{z}_s \cdot \nabla) p_s = L[\mathbf{z}_s] q \equiv \widetilde{q}^s \quad \text{in } \Omega_s,$$

which is written in the form

$$L_1[\mathbf{z}_s] p_s + \mathrm{a}\, \mathrm{b}\, L_2[\mathbf{z}_s] p_s = \widetilde{q}^s \quad \text{in } \Omega_s,$$

with $L_1[\mathbf{z}_s] = I + \mathrm{a}(1-\mathrm{b})(\mathbf{z}_s \cdot \nabla)$, and $L_2[\mathbf{z}_s] = Z[\mathbf{z}_s]^{-1}(\mathbf{z}_s \cdot \nabla)$.

We now study the operators $L_1[\mathbf{z}_s]$ and $L_2[\mathbf{z}_s]$. For $\mathbf{z}_s \in \mathcal{V}_\xi^{k+2,2}(\Omega_s)$, we apply Proposition 5.2: there exists a constant $\alpha_4 = \alpha_4(\xi, k)$ such that if

$$\alpha_4 \mathrm{a}(1-\mathrm{b}) [\![\mathbf{z}_s]\!]_{k+2,\xi;\Omega_s} < 1, \tag{19}$$

the operator $L_1[\mathbf{z}_s] : \mathcal{V}_\xi^{k+1,2}(\Omega_s) \longrightarrow \mathcal{V}_\xi^{k+1,2}(\Omega_s)$ is an isomorphism. In addition, for all $u \in \mathcal{V}_\xi^{k+1,2}(\Omega_s)$ such that $L_1[\mathbf{z}_s] u = u + \mathrm{a}(1-\mathrm{b})(\mathbf{z}_s \cdot \nabla) u$, we have

$$[\![L_1[\mathbf{z}_s] u]\!]_{k+1,\xi;\Omega_s} \leq C_2 [\![u]\!]_{k+1,\xi;\Omega_s},$$
$$[\![L_1[\mathbf{z}_s]^{-1} u]\!]_{k+1,\xi;\Omega_s} \leq C_2 [\![u]\!]_{k+1,\xi;\Omega_s}.$$

In the other hand, for all $u \in \mathcal{V}_\xi^{k+1,2}(\Omega_s)$, we have $L_2[\mathbf{z}_s] u = \frac{1}{\mathrm{c}}(I - Z[\mathbf{z}_s]^{-1}) u$. Due to Proposition 5.2, $L_2[\mathbf{z}_s]$ is a continuous mapping from $\mathcal{V}_\xi^{k+1,2}(\Omega_s)$ into itself if

$$\alpha_1 \mathrm{c} [\![\mathbf{z}_s]\!]_{k+2,\xi;\Omega_s} < 1, \tag{20}$$

and we have

$$[\![L_2[\mathbf{z}_s]z]\!]_{k+1,\xi;\Omega_s} \leq C_2[\![z]\!]_{k+1,\xi;\Omega_s}.$$

Under conditions (18), (19) and (20), we can deduce

$$[\![p_s]\!]_{k+1,\xi;\Omega_s} \leq C_2\Big([\![\widetilde{q^s}]\!]_{k+1,\xi;\Omega_s} + \mathrm{a}\,\mathrm{b}[\![p_s]\!]_{k+1,\xi;\Omega_s}\Big).$$

For $\mathrm{a} < \min(2\mathrm{b}C_2, 1)$, we have

$$[\![p_s]\!]_{k+1,\xi;\Omega_s} \leq C_2[\![\widetilde{q^s}]\!]_{k+1,\xi;\Omega_s}.$$

By Proposition 5.3, $\widetilde{q^s} = L[\mathbf{z}_s]q$ satisfies

$$[\![q_s]\!]_{k+1,\xi;\Omega_s} \leq 2C_2[\![q]\!]_{k+1,\xi;\Omega_s};$$

we then deduce inequality (16).

To show inequality (17), we apply the operator $\Delta = \mathrm{Tr}\nabla^2$ to equation (13) and obtain

$$L[\mathbf{z}_s]^{-1}p_s + \mathrm{a}(\mathbf{z}_s\cdot\nabla)p_s = q.$$

According to the calculations done in [28], we have $\nabla^2(L[\mathbf{z}_s]^{-1}p_s) = L[\mathbf{z}_s]^{-1}\nabla^2 p_s + A + B + \mathrm{b}(1-\mathrm{b})\mathrm{c}^2 C$ with

$$\begin{aligned} A &= 2\Big((D_{\mathbf{z}_s}L[\mathbf{z}_s]^{-1})\nabla\mathbf{z}_s\Big)\nabla p_s,\\ B &= \Big((D_{\mathbf{z}_s}L[\mathbf{z}_s]^{-1})\nabla^2\mathbf{z}_s\Big)p_s,\\ C &= -L[\mathbf{z}_s]^{-1}Z[\mathbf{z}_s]^{-1}(\nabla\mathbf{z}_s)^T\nabla\Big(L[\mathbf{z}_s]^{-1}Z[\mathbf{z}_s]^{-1}(\nabla\mathbf{z}_s)^T\nabla(Z[\mathbf{z}_s]^{-1}L[\mathbf{z}_s]^{-1}p_s)\Big), \end{aligned}$$

where $D_{\mathbf{z}_s}L[\mathbf{z}_s]^{-1}$ is the differential operator of $L[\mathbf{z}_s]^{-1}$ with respect to $\mathbf{z}_s$. The operators $L[\mathbf{z}_s]^{-1}$ and Tr commute, so that

$$L[\mathbf{z}_s]^{-1}\Delta p_s + \mathrm{a}(\mathbf{z}_s\cdot\nabla)\Delta p_s = Q, \tag{21}$$

where

$$Q \equiv \Delta q - \mathrm{a}\nabla\mathbf{z}_s\cdot\nabla^2 p_s - \mathrm{a}(\mathbf{z}_s\cdot\nabla)p_s - \mathrm{Tr}A - \mathrm{Tr}B - \mathrm{b}(1-\mathrm{b})\mathrm{c}^2\mathrm{Tr}\,C$$

is in $\mathcal{V}_\xi^{k-1,2}(\Omega_s)$. In the transport equation (21), we can estimate Δp_s by using inequality (16); the above calculations thus imply that there is a constant, still denoted C_2 independent of s, such that

$$[\![\Delta p_s]\!]_{k-1,\xi;\Omega_s} \leq C_2[\![Q]\!]_{k-1,\xi;\Omega_s}.$$

To estimate Q, we will use Lemma 5.1:

$$\begin{aligned} [\![Q]\!]_{k-1,\xi;\Omega_s} \leq\ & [\![\Delta q]\!]_{k-1,\xi;\Omega_s} + [\![\mathbf{z}_s]\!]_{k+2,\xi;\Omega_s}[\![p_s]\!]_{k+1,\xi;\Omega_s} + \mathrm{b}(1-\mathrm{b})\mathrm{c}^2C_*[\![\mathbf{z}_s]\!]^2_{k+2,\xi;\Omega_s}[\![p_s]\!]_{k+1,\xi;\Omega_s}\\ & + \mathrm{a}\Big([\![\mathbf{z}_s]\!]_{k+2,\xi;\Omega_s}[\![\nabla^2 p_s]\!]_{k-1,\xi;\Omega_s} + [\![\mathbf{z}_s]\!]_{k+2,\xi;\Omega_s}[\![\nabla p_s]\!]_{k,\xi;\Omega_s}\Big). \end{aligned}$$

Therefore,

$$[\![\Delta p_s]\!]_{k-1,\xi;\Omega_s} \leq C_2 \left([\![\Delta q]\!]_{k-1,\xi;\Omega_s} + ([\![\mathbf{z}_s]\!]_{k+2,\xi;\Omega_s} + \mathrm{a}\,\mathrm{b})\, [\![p_s]\!]_{k+1,\xi;\Omega_s}\right),$$

and this under the assumption (15). To finish, let us note that we may choose $\overline{\mathrm{a}} < \min(2\mathrm{b}C_2, 1)$ and $\overline{\alpha} = \max(\alpha_1, \alpha_3, 2\alpha_4(1-\mathrm{b})C_2/\mathrm{b})$. □

Step 4. The limit process. First of all, we remark that

$$[\![\mathbf{z}_s]\!]_{k+2,\xi;\Omega_s} \leq C_3 [\![\mathbf{z}]\!]_{k+2,\xi};$$

where $C_3 > 0$ is a constant independent of s.

Let $\widehat{\alpha} = C_3\overline{\alpha}$ and $\overline{\mathrm{a}} = \widehat{\mathrm{a}}$; then the conditions (3) and (4) immediately give the conditions (14) and (15). Therefore, according to Lemma 6.1, there exists a unique solution p_s of Problem (13), which satisfies the estimates (16) and (17).

Let $\{s_i\}_{i=1}^{\infty}$ be a sequence of real positive numbers decreasing to 0.

We define $\Omega_i = \Omega_{s_i}$ and $p_i = p_{s_i}$, and note that

$$\Omega_{i_1} \subset \Omega_{i_2} \text{ for } i_1 > i_2, \quad \bigcup_{i=1}^{\infty} \Omega_i = \Omega.$$

For all $i \in \mathbb{N}^*$, the function p_i is the unique solution of Problem (13) in Ω_i, and satisfies (16). For any fixed m, there exists a subsequence $(p_{i_j}^m)_j$ of $(p_i)_i$ converging weakly in $W^{k+1,2}(\Omega_m)$, and strongly in $W^{1,2}(\Omega_m)$, to $p^m \in W^{k+1,2}(\Omega_m)$. Let $n \in \mathbb{N}^*$ be fixed. For $n_1 > n$, we can extract from $(p_{i_j}^n)_j$ a subsequence denoted $(p_{i_j}^{n_1})_j$, which converges weakly in $W^{k+1,2}(\Omega_{n_1})$, and strongly in $W^{1,2}(\Omega_{n_1})$, to p^{n_1}. Thus

$$p^n = p^{n_1}, \quad \text{a.e.} \quad \mathbf{x} \in \Omega_n.$$

We define the function p in Ω as follows: for all $\mathbf{x} \in \Omega$, $p(\mathbf{x}) = p^n(\mathbf{x})$ where n is chosen large enough for $\mathbf{x}$ to be in Ω_n.

Then $p \in W^{k+1,2}_{\mathrm{loc}}(\Omega)$ and satisfies the transport equation

$$L[\mathbf{z}]^{-1}p + \mathrm{a}\Big(\mathbf{z}\cdot\nabla\Big)p = q, \quad \text{in } \Omega, \tag{22}$$

in the sense that

$$\int_\Omega \Big[L[\mathbf{z}]^{-1}p + \mathrm{a}\Big(\mathbf{z}\cdot\nabla\Big)p\Big]\,\phi\, d\mathbf{x} = \int_\Omega q\phi\, d\mathbf{x}, \quad \forall\phi \in C_0^\infty(\Omega).$$

This shows that p satisfies (22) almost everywhere in Ω, and since $p \in W^{k+1,2}_{\mathrm{loc}}(\Omega)$ with $k \geq 1$, equation (22) is valid for all $\mathbf{x} \in \Omega$.

7 Perspectives

A question can be posed about the possibility of adapting the result of single convex angle in case of convex polygon (*i.e.* case of bounded domain with finite number of convex corners). It seems possible to extend this result by constructing a partition of unit and find the convenable space. Another problem is how to improve the existence result in case of a bounded domain with both convex and concave corners. This latter, which is very important because of its real occurrence, needs a particular study of the solution's behavior in the concave corner.

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