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On Fekete-Szegö Problems for Certain Subclass of Analytic Functions

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Abstract

The purpose of the present paper is to derive the Fekete-Szegö inequality for the class $M_{\mu}^{\lambda,s}(\phi)$ ***of normalized analytic functions*** $f(z)$ ***defined on the open unit disk for which*** $z(\Theta_{\mu}^{\lambda,s}f(z))'/\Theta_{\mu}^{\lambda,s}f(z)$ ***lies in a region starlike with respect to 1 and symmetric with respect to the real axis***.

Keywords: *Analytic functions, starlike functions, subordination, Fekete Szegö inequality, linear operator.*

MSC Subj. Classifications: 30C45

1 Introduction

Let $U=\{z: z\in C \ |z|<1\}$ be the open unit disk and A denotes the class of functions f normalized by

$$f(z)=z+\sum_{k=2}^{\infty}a_k z^k, \tag{1.1}$$

which is analytic in the open unit disk U and satisfy the condition $f(0)=f'(0)-1=0$. We also denote by S the subclass of A consisting of functions which are also univalent in U. Let V denote the class of Schwarz functions, i.e. $w\in V$ if and only if w is analytic in

$U, w(0)=0$ and $|w(z)|<1$ on U. Further, let P denote the class of analytic functions in U such that $h(z)=1+p_1 z+\ldots$, $h(0)=1$ and $\Re h(z)>0,\ z\in U$,

$$h(z)=\frac{1+w(z)}{1-w(z)}$$

for some $w(z)\in V, z\in U$. For $f_i\in A$ given by

$$f_i(z)=z+\sum_{k=2}^{\infty}a_{k,i}z^k,\ (i=1,2),$$

the Hadamard product (or convolution) $f_1 * f_2$ of f_1 and f_2 is defined by

$$(f_1 * f_2)(z)=z+\sum_{k=2}^{\infty}a_{k,1}a_{k,2}z^k.$$

Let F and G be analytic functions in the unit disk U. The function F is subordinate to G, written $F\prec G$ if G is univalent, $F(0)=G(0)$ and $F(U)\subset G(U)$. In general, given two functions F and G which are analytic in U, the function F is said to be subordinate to G, if there exist a function w analytic in U with

$$w(0)=0 \text{ and } (\forall z\in U): |w(z)|<1,$$

such that

$$(\forall z\in U): F(z)=G(w(z)).$$

Let $\phi(z)$ be an analytic function with positive real part on A with $\phi(0)=1,\ \phi'(0)>0,$ which maps the unit disk U onto a region starlike with respect to 1 and symmetric with respect to the real axis. Let $S^*(\phi)$ be the class of functions $f\in S$ for which

$$\frac{zf'(z)}{f(z)}\prec\phi(z)\qquad z\in U,$$

and $C(\phi)$ be the class of functions $f\in S$ for which

$$1+\frac{zf''(z)}{f'(z)}\prec\phi(z)\qquad z\in U,$$

where $\prec$ denotes the subordination between analytic functions. These classes were introduced and studied by Ma and Minda [6]. They have obtained the Fekete-Szegö inequality for the functions in the class $C(\phi)$. For a brief history of Fekete-Szeg¨o problem for the class of starlike, convex, close-to convex functions and extended classes, see for example ([3], [7]-[12], [17]-[20]). Of course the main result shall refer back to Fekete and Szegö [13] in the year 1933. After 30 years or so, Keogh and Merkes [14] solved the problem for certain subclasses of univalent functions. Then Koepf [15, 16] gave excellent results for the class of close-to-convex functions. These articles ([13-16]) gave valuable results which inspired many to solve related problems for other extended classes.

For $\mu \in N, \lambda, s \in N \cup \{0\}$ the authors [1] introduced the operator $\Theta_\mu^{\lambda,s}$ defined by

$$\Theta_\mu^{\lambda,s} f(z) = z + \sum_{k=2}^{\infty} \frac{(k+\lambda-1)!(\mu-1)!}{\lambda!(k+\mu-2)!} k^s a_k z^k. \tag{1.2}$$

In the present paper, we obtain the Fekete- Szegö inequality for functions $f \in A$ in the class $M_\mu^{\lambda,s}(\phi)$ defined as follows:

Definition 1.1. *Let $\phi(z)$ be a univalent starlike function with respect to 1 which maps the unit disc U onto a region in the right half plane which is symmetric with respect to the real axis,* $\phi(0)=1$ and $\phi'(0)>1$. *A function $f \in A$ is in the class $M_\mu^{\lambda,s}(\phi)$ if*

$$\frac{z(\Theta_\mu^{\lambda,s} f(z))'}{\Theta_\mu^{\lambda,s} f(z)} \prec \phi(z), \qquad (\lambda, s \in N_0,\ \mu \in N).$$

We state the following lemmas that will be used in this paper.

Lemma 1.1 [4]. *If $p_1(z) = 1 + c_1 z + c_2 z^2 + ...$ is an analytic function with positive real part in U, then*

$$\left| c_2 - v c_1^2 \right| \le \begin{cases} -4v+2 & \text{if } v \le 0 \\ 2 & \text{if } 0 \le v \le 1, \\ 4v-2 & \text{if } v \ge 0 \end{cases}$$

when $v<0$ *or* $v>1$, the equality holds if and only if $p_1(z)=(1+z)/(1-z)$ or one of its rotations. If $0<v<1$, the equality holds if and only if $p_1(z)$ is $(1+z^2)/(1-z^2)$ or one of its rotations. If $v=0$, the equality holds if and only if

$$p_1(z)=\left(\frac{1}{2}+\frac{1}{2}\gamma\right)\frac{1+z}{1-z}+\left(\frac{1}{2}-\frac{1}{2}\gamma\right)\frac{1-z}{1+z}\qquad(0\le\gamma\le1),$$

or one of its rotations. If $v=1$, the equality holds if and only if p_1 is the reciprocal of one of the functions such that the equality holds in the case of $v=0$. Also the above upper bound is sharp, it can be improved as follows when $0<v<1$:

$$|c_2-vc_1^2|+v|c_1^2|\le2\qquad(0<v\le1/2)$$

and

$$|c_2-vc_1^2|+(1-v)|c_1^2|\le2\qquad(1/2<v\le1).$$

Lemma 1.2 ([5]). *If* $p(z)=1+c_1z+c_2z^2+\ldots$ *is a function with positive real part, then for any complex number* μ,

$$|c_2-\mu c_1^2|\le2\max\{1,|2\mu-1|\},$$

and the result is sharp for the functions given by

$$p_1(z)=\frac{1+z^2}{1-z^2}\ \ or\ \ p_1(z)=\frac{1+z}{1-z}.$$

Lemma 1.3 ([2]) *Let* K *be analytic in* U *with* $\Re K(z)>0$ *and be given by* $K(z)=1+c_1z+c_2z^2+\ldots$ *for* $z\in U$, *then*

$$\left|c_2-\frac{c_1^2}{2}\right|\le2-\frac{|c_1|^2}{2}.$$

2 Main Results

Our main result is the following:

Theorem. 2.1. *Let* $\phi(z)=1+B_1z+B_2z^2+...,$ *and Let*

$$\sigma_1=\frac{(\lambda+1)(\mu+1)2^{2s}}{2\mu(\lambda+2)3^s}\left[\frac{(B_2-B_1)+B_1^2}{B_1^2}\right]$$

$$\sigma_2=\frac{(\lambda+1)(\mu+1)2^{2s}}{2\mu(\lambda+2)3^s}\left[\frac{(B_2+B_1)+B_1^2}{B_1^2}\right],$$

$$\sigma_3=\frac{(\lambda+1)(\mu+1)2^{2s}}{2\mu(\lambda+2)3^s}\left[\frac{B_2+B_1^2}{B_1^2}\right],$$

$$\eta=\frac{\mu(\mu+1)B_2}{2(\lambda+1)(\lambda+2)3^s}-\frac{\tau\,\mu^2B_1^2}{(\lambda+1)^2 2^{2s}}+\frac{\mu(\mu+1)B_1^2}{2(\lambda+1)(\lambda+2)3^s},$$

$$\delta=\frac{\mu(\mu+1)B_1}{2(\lambda+1)(\lambda+2)3^s}.$$

If $f(z)$ *given by (1.1) belongs to* $M_{\mu}^{\lambda,s}(\phi)$, *then*

$$\left|a_3-\tau a_2{}^2\right|\le\begin{cases}\eta, & \text{if } \tau\le\sigma_1,\\ \delta, & \text{if } \sigma_1\le\tau\le\sigma_2;\\ -\eta, & \text{if } \tau\ge\sigma_2.\end{cases}$$

Further, If $\sigma_1\le\tau\le\sigma_3$, *then*

$$\left|a_3-\tau a_2{}^2\right|+\frac{(\lambda+1)(\mu+1)2^{2s-1}}{\mu(\lambda+2)3^s B_1^2}\Big[B_1-B_1^2+\frac{\tau\,\mu(\lambda+2)3^s-(\lambda+1)(\mu+1)2^{2s-1}}{(\lambda+1)(\mu+1)2^{2s-1}}B_1^2\Big]\left|a_2\right|^2\le\delta.$$

If $\sigma_3\le\tau\le\sigma_2$,

$$\left|a_3-\tau a_2^{\,2}\right|+\frac{(\lambda+1)(\mu+1)2^{2s-1}}{\mu(\lambda+2)3^s B_1^2}\Big[B_1+B_1^2$$
$$+\frac{\tau\,\mu(\lambda+2)3^s-(\lambda+1)(\mu+1)2^{2s-1}}{(\lambda+1)(\mu+1)2^{2s-1}}B_1^2\Big]\left|a_2\right|^{2}\leq\delta.$$

The result is sharp.

Proof. If $f(z)\in M_{\mu}^{\lambda,s}(\phi)$, then there is a Schwarz function $w(z)$, analytic in U with $w(0)=0$ and $|w(z)|<1$ in U such that

$$\frac{z(\Theta_{\mu}^{\lambda,s}f(z))'}{\Theta_{\mu}^{\lambda,s}f(z)}=\phi(w(z)).$$

Define a function $p_1(z)$ by

$$p_1(z)=\frac{1+w(z)}{1-w(z)}.$$

Since $w(z)$ is Schwarz function, we see that $\Re p_1(z)>0$ *and* $p_1(0)=1.$ Define a function $p(z)$ by

$$p(z)=\frac{z(\Theta_{\mu}^{\lambda,s}f(z))'}{\Theta_{\mu}^{\lambda,s}f(z)}=1+b_1z+b_2z^2+\dots. \qquad (2.1)$$

From (2.1), we obtain

$$\frac{(\lambda+1)}{\mu}2^s a_2=b_1 \ \ \textit{and}\ \ \frac{2(\lambda+1)(\lambda+2)}{\mu(\mu+1)}3^s a_3=b_2+\frac{(\lambda+1)^2}{\mu^2}2^{2s}a_2^{\,2} \qquad (2.2)$$

Since

$$p_1(z)=\frac{1+\phi^{-1}(p(z))}{1-\phi^{-1}(p(z))}$$

then

$$p(z)=\phi\left(\frac{p_1(z)-1}{p_1(z)+1}\right),$$

and

$$1+b_1 z+b_2 z^2+\ldots=\phi\left(\frac{c_1 z+c_2 z^2+\ldots}{2+c_1 z+c_2 z^2+\ldots}\right)$$

$$=\phi[\frac{1}{2}c_1 z+\frac{1}{2}(c_2-\frac{1}{2}c_1^2)z^2+\ldots] \tag{2.3}$$

$$=1+B_1\frac{1}{2}c_1 z+B_1\frac{1}{2}(c_2-\frac{1}{2}c_1^2)z^2+\ldots+B_2\frac{1}{4}c_1^2 z^2+\ldots$$

Then we obtain

$$b_1=\frac{1}{2}B_1 c_1 \quad \textit{and} \quad b_2=\frac{1}{2}B_1(c_2-\frac{1}{2}c_1^2)+\frac{1}{4}B_2 c_1^2. \tag{2.4}$$

From (2.2) and (2.4) we get

$$a_2=\frac{\mu B_1 c_1}{2(\lambda+1)2^s},$$

$$a_3=\frac{\mu(\mu+1)}{4(\lambda+1)(\lambda+2)3^s}\left[B_1(c_2-\frac{1}{2}c_1^2)+\frac{1}{2}B_2 c_1^2+\frac{B_1^2 c_1^2}{2}\right]$$

Therefore we have

$$a_3-\tau a_2^2=\frac{\mu(\mu+1)B_1}{4(\lambda+1)(\lambda+2)3^s}\left[c_2-c_1^2\left[\frac{1}{2}\left(1-\frac{B_2}{B_1}+\frac{\tau\,\mu(\lambda+2)3^s-(\lambda+1)(\mu+1)2^{2s-1}}{(\lambda+1)(\mu+1)2^{2s-1}}B_1\right)\right]\right]$$

$$=\frac{\mu(\mu+1)B_1}{4(\lambda+1)(\lambda+2)3^s}\left[c_2-vc_1^2\right],$$

where

$$v=\frac{1}{2}\left(1-\frac{B_2}{B_1}+\frac{\tau\,\mu(\lambda+2)3^s-(\lambda+1)(\mu+1)2^{2s-1}}{(\lambda+1)(\mu+1)2^{2s-1}}B_1\right).$$

If $\tau\le\sigma_1$, then by applying Lemma 1.1 and Lemma 1.3, we get

$$\left|a_3-\tau a_2^2\right|\le\frac{\mu(\mu+1)B_2}{2(\lambda+1)(\lambda+2)3^s}-\frac{\tau\,\mu^2 B_1^2}{(\lambda+1)^2 2^{2s}}+\frac{\mu(\mu+1)B_1^2}{2(\lambda+1)(\lambda+2)3^s}.$$

This is the first part of Theorem 1.1.

Similarly, if $\tau \geq \sigma_2$, we get

$$\left|a_3 - \tau a_2^{\ 2}\right| \leq -\frac{\mu(\mu+1)B_2}{2(\lambda+1)(\lambda+2)3^s} + \frac{\tau\,\mu^2 B_1^2}{(\lambda+1)^2 2^{2s}} - \frac{\mu(\mu+1)B_1^2}{2(\lambda+1)(\lambda+2)3^s}$$

If $\sigma_1 < \tau < \sigma_2$, we see that

$$\left|a_3 - \tau a_2^{\ 2}\right| = \frac{\mu(\mu+1)B_1}{4(\lambda+1)(\lambda+2)3^s}\left|c_2 - vc_1^2\right| \leq \frac{\mu(\mu+1)B_1}{2(\lambda+1)(\lambda+2)3^s}.$$

Further, If $\sigma_1 \leq \tau \leq \sigma_3$, then

$$\left|a_3 - \tau a_2^{\ 2}\right| + (\tau - \sigma_1)\left|a_2\right|^2$$
$$= \frac{\mu(\mu+1)B_1}{4(\lambda+1)(\lambda+2)3^s}\left|\, c_2 - v\, c_1^2\right| + \left(\tau - \frac{(\lambda+1)(\mu+1)2^{2s-1}}{\mu(\lambda+2)3^s}\left[\frac{(B_2 - B_1) + B_1^{\ 2}}{B_1^2}\right]\right)\frac{\mu^2 B_1^{\ 2}\left|c_1\right|^2}{4(\lambda+1)^2 2^{2s}}$$
$$= \frac{\mu(\mu+1)B_1}{2(\lambda+1)(\lambda+2)3^s}\left\{\frac{1}{2}\left[\left|\, c_2 - v\, c_1^2\right| + v\,\left|c_1\right|^2\right]\right\}$$
$$\leq \frac{\mu(\mu+1)B_1}{2(\lambda+1)(\lambda+2)3^s}.$$

Finally, we see that

$$\left|\, a_3 - \tau a_2^{\ 2}\right| + (\sigma_2 - \tau)\left|\, a_2\right|^2$$
$$= \frac{\mu(\mu+1)B_1}{4(\lambda+1)(\lambda+2)3^s}\left|\, c_2 - v\, c_1^2\right| + \left(\frac{(\lambda+1)(\mu+1)2^{2s-1}}{\mu(\lambda+2)3^s}\left[\frac{(B_2 + B_1) + B_1^{\ 2}}{B_1^2}\right] - \tau\right)\frac{\mu^2 B_1^{\ 2}\left|\, c_1\right|^{\ 2}}{4(\lambda+1)^2 2^{2s}}$$

$$= \frac{\mu(\mu+1)B_1}{2(\lambda+1)(\lambda+2)3^s}\left\{\frac{1}{2}\left[\left|\, c_2 - vc_1^2\right| + (1-v)\left|c_1\right|^2\right]\right\}$$
$$\leq \frac{\mu(\mu+1)B_1}{2(\lambda+1)(\lambda+2)3^s}.$$

To show that the bounds are sharp, we define functions $K_n^{\phi}(n = 2,3,...)$ by

$$\frac{z\,(\Theta_{\mu}^{\lambda,s}K_{n}^{\phi}(z))'}{\Theta_{\mu}^{\lambda,s}K_{n}^{\phi}(z)}=\phi(z^{n-1}),\quad K_{n}^{\phi}(0)=0=(K_{n}^{\phi}(0))'-1,$$

and the function F_{γ} and $G_{\gamma}(0\le\gamma\le 1)$ by

$$\frac{z\,(\Theta_{\mu}^{\lambda,s}F_{\gamma}(z))'}{\Theta_{\mu}^{\lambda,s}F_{\gamma}(z)}=\phi(\frac{z\,(z+\gamma)}{1+\gamma z}),\quad F_{\gamma}(0)=0=(F_{\gamma}(0))'-1,$$

and

$$\frac{z\,(\Theta_{\mu}^{\lambda,s}G_{\gamma}(z))'}{\Theta_{\mu}^{\lambda,s}G_{\gamma}(z)}=\phi(-\frac{z\,(z+\gamma)}{1+\gamma z}),\quad G_{\gamma}(0)=0=(G_{\gamma}(0))'-1.$$

Clearly, the functions $K_{n}^{\phi}, F_{\gamma}\, and\, G_{\gamma}\in M_{\mu}^{\lambda,s}(\phi)$. We write $K^{\phi}=K_{2}^{\phi}$ If $\tau<\sigma_{1}$ or $\tau>\sigma_{2}$, then the equality holds if and only if f is K^{ϕ} or one of its rotations. When $\sigma_{1}<\tau>\sigma_{2}$, then the equality holds if and only if f is K_{3}^{ϕ} or one of its rotations. If $\tau=\sigma_{1}$, then the equality holds if and only if f is F_{γ} or one of its rotations. If $\tau=\sigma_{2}$, then the equality holds if and only if f is G_{γ} or one of its rotations.

By making use of Lemma 1.2 we can easily obtain the next theorem.

Theorem. **2.2**. *Let* $\phi(z)=1+B_{1}z+B_{2}z^{2}+......$ *If* $f\ (z)$ *given by (1.1) belongs to* $M_{\mu}^{\lambda,s}(\phi)$, *then*

$$\left|a_{3}-\tau a_{2}^{2}\right|\le\frac{\mu(\mu+1)B_{1}}{2(\lambda+1)(\lambda+2)3^{s}}\max\left\{1,\left|\frac{\tau\,\mu(\lambda+2)3^{s}-(\lambda+1)(\mu+1)2^{2s-1}}{(\lambda+1)(\mu+1)2^{2s-1}}B_{1}-\frac{B_{2}}{B_{1}}\right|\right\}.$$

3 Open Problem

Considering the class $M_{\mu}^{\lambda,s}(\phi)$, can we obtain the sharp bounds for the functional $\left|a_{4}-\alpha a_{2}a_{3}\right|,\left|a_{4}-\alpha a_{2}a_{3}-\eta a_{2}^{3}\right|$ and $\left|a_{5}-\zeta a_{2}^{2}a_{3}\right|$, where the parameters α,η and ζ are all real numbers?

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