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The Structure of Some Classes of Sasakian Manifolds with respect to the Quarter-Symmetric Metric Connection

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Abstract

The object of the paper is to study conharmonic curvature tensor in Sasakian manifolds with respect to the quarter-symmetric metric connection. Some necessary and/or sufficient condition(s) for Sasakian manifolds to be quasi - conharmonically flat, ϕ - conharmonically flat and ξ - conharmonically flat with respect to the quarter-symmetric metric connection are obtained.

Keywords: *Sasakian manifold, conharmonic curvature tensor, quarter - symmetric metric connection, quasi-conharmonically flat, ϕ-conharmonically flat, ξ-conharmonically flat.*

1 Introduction

In 1924, A. Friedman and J.A. Schouten ([8, 19]) introduced the notion of a semi-symmetric linear connection on a differentiable manifold. In 1932, H.A. Hayden [10] introduced the idea of metric connection with torsion on a Riemannian manifold. In 1970, K. Yano [23] studied some curvature and derivational conditions for semi-symmetric connections in Riemannian manifolds. In 1975,

S. Golab [9] defined and studied quarter-symmetric linear connection on a differentiable manifold. A linear connection $\widetilde{\nabla}$ in an n-dimensional differentiable manifold is said to be a quarter-symmetric connection if its torsion tensor T is of the form

$$T(X,Y) = \widetilde{\nabla}_X Y - \widetilde{\nabla}_Y X - [X,Y], \tag{1}$$

$$= \eta(Y)\phi X - \eta(X)\phi Y, \tag{2}$$

where η is a 1-form and ϕ is a tensor of type $(1,1)$. In addition, a quarter-symmetric linear connection $\widetilde{\nabla}$ satisfies the condition $\widetilde{\nabla}_X g)(Y,Z) = 0$ for all $X,Y,Z \in \chi(M)$, where $\chi(M)$ is the Lie algebra of vector fields of the manifold M, then $\widetilde{\nabla}$ is said to be a quarter-symmetric metric connection. If we replace ϕX by X then the connection is called a semi-symmetric metric connection [23].

In 1980, R. S. Mishra and S. N. Pandey [13] studied quarter-symmetric metric connection and in particular, Ricci quarter-symmetric symmetric metric connection on Riemannian, Sasakian and Kaehlerian manifolds. Note that a quarter-symmetric metric connection is a Hayden connection with the torsion tensor of the form $(1,2)$. Studies of various types of quarter-symmetric metric connection and their properties include ([2, 3, 7, 13, 17, 18, 14]) and [24] among others.

Let M be an almost contact metric manifold equipped with an almost contact metric structure (ϕ, ξ, η, g). Since at each point $p \in M$ the tangent space T_pM can be decomposed into the direct sum $T_pM = \phi(T_pM) \oplus \{\xi_p\}$, where ξ_p is the one-dimensional linear subspace of T_pM generated by ξ_p, the conformal curvature tensor C is a map

$$C : T_pM \times T_pM \times T_pM \to \phi(T_pM) \oplus \{\xi_p\}, \quad p \in M. \tag{3}$$

It may be natural to consider the following cases: (1) the projection of the image of C in $\phi(T_pM)$ is zero; (2) the projection of the image of C in $\{\xi_p\}$ is zero; and (3) the projection of the image of $C|_{\phi(T_pM)\times\phi(T_pM)\times\phi(T_pM)}$ in $\phi(T_pM)$ is zero. An almost contact metric manifold satisfying the cases (1), (2) and (3) are said to be conformally symmetric [25], ξ-conformmally flat[26] and ϕ-conformally flat [6] repectively. In [25], it is proved that a conformally symmetric K-contact manifold is locally isometric to the unit sphere. In [26], it is proved that a K-contact manifold is ξ-conformally flat if and only if it is an η-Einstein Sasakian manifold. In [6], some necessary conditions for a K-contact manifold to be ϕ-conformally flat are proved. In [1], some results for ϕ-conformally flat, $\phi-$conharmonically flat and ϕ-concirculary flat on (k,μ)-contact manifolds were given. In [15], Weyl conformal curvature tensor, conharmonic curvature tensor and projective curvature tensor are discussed on Lorentzian para-Sasakian manifolds.

Apart from conformal curvature tensor, the conharmonic curvature tensor is another important tensor from the differential geometrical view point. The author of [16] considered some conditions on conharmonic curvature tensor K, which has many applications in Physics and Mathematics, on a hypersurface in the semi-Euclidean space E_S^{n+1}. He proved that every conharmonically Ricci-symmetric hypersurface M satisfying the condition $K \circ R = 0$ is pseudo symmetric. He also considered the condition $K \circ K = L_k Q(g, K)$ on hypersurfaces of the semi-Euclidean space E_S^{n+1}.

In a Riemannian manifold M of dimension $n \geq 3$, the conharmonic curvature tensor K is defined by [11]

$$K(X,Y)Z = R(X,Y)Z - \frac{1}{n-2}\{S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY\} \tag{4}$$

for $X, Y, Z \in TM$, where R is the curvature tensor and Q is the Ricci operator. Analogous to the consideration of conformal curvature tensor, we give the following definition:

Definition 1.1 *An almost contact metric manifold M is said to be quasi-conharmonically flat if*

$$g(K(X,Y)Z, \phi W) = 0, \qquad X, Y, Z, W \in TM, \tag{5}$$

ϕ-conharmonically flat if

$$g(K(\phi X, \phi Y)\phi Z, \phi W) = 0, \qquad X, Y, Z, W \in TM, \tag{6}$$

and ξ-conharmonically flat if

$$K(X,Y)\xi = 0, \qquad X, Y \in TM. \tag{7}$$

A Sasakian manifold is said to be an η-Einstein manifold if its Ricci tensor satisfies the condition

$$S(X,Y) = \alpha g(X,Y) + \beta \eta(X)\eta(Y) \tag{8}$$

where α and β are scalars.

The present paper is organized as follows: In section 2, the brief account of Sasakian manifolds were given. In the next section, we establish the relation between the Levi-Civita connection and the quarter-symmetric metric connection on a Sasakian manifold. In section 4, we consider three cases of conharmonic curvature tensor, analogous to those of conformal curvature tensor, and give definitions of quasi-conharmonically flat, ϕ-conharmonically flat and ξ-conharmonically flat almost contact metric manifolds with respect to the quarter-symmetric metric connection. It is proved that, if a Sasakian manifold is quasi-conharmonically flat with repsect to the quarter-symmetric metric connection $\widetilde{\nabla}$ then it is η-Einstein with respect to the connection ∇ and the scalar curvature with respect to the connection $\widetilde{\nabla}$ is zero. Necessary and sufficient conditions for a Sasakian manifold to be quasi-conharmonically flat, ϕ-conharmonically flat and ξ-conharmonically flat with respect to the connection $\widetilde{\nabla}$ are obtained.

2 Sasakian manifolds

Let M be an almost contact metric manifold of dimension $n(= 2m + 1)$ equipped with an almost contact metric structure (ϕ, ξ, η, g) consisting of a $(1, 1)$ tensor field ϕ, a vector field ξ, a 1-form η and a Riemannian metric g, satisfying

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta \circ \phi = 0, \tag{9}$$
$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad X, Y \in TM. \tag{10}$$

From (9) and (10) we easily get

$$g(\phi X, Y) = -g(X, \phi Y), \quad g(X, \xi) = \eta(X), \quad X, Y \in TM. \tag{11}$$

An almost contact metric manifold becomes a contact metric manifold if

$$g(\phi X, Y) = d\eta(X, Y), \quad X, Y \in TM \tag{12}$$

A contact metric structure is said to be K-contact if ξ is a killing with respect to g. If in such a manifold the relation

$$(\nabla_X \phi)Y = g(X, Y)\xi - \eta(Y)X \tag{13}$$

holds, where ∇ denotes the Levi-Civita connection of g, then M is called a Sasakian manifold. It is known that every Sasakian manifold is K-contact but converse is not true in general. However, a 3-dimensional K-contact manifold is Sasakian [12].

In a Sasakian manifold M equipped with the structure (ϕ, ξ, η, g), the following relations hold ([4, 22]):

$$\nabla_X \xi = -\phi X, \tag{14}$$
$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y, \tag{15}$$
$$R(\xi, X)Y = (\nabla_X \phi)Y, \tag{16}$$
$$S(X, \xi) = (n - 1)\eta(X), \tag{17}$$
$$S(\phi X, \phi Y) = S(X, Y) - (n - 1)\eta(X)\eta(Y) \tag{18}$$

for any vector fields $X, Y \in TM$. Where R and S denote the curvature tensor and the Ricci tensor of M, respectively.

In an n-dimensional Sasakian manifold M, if $\{e_1, ..., e_{n-1}, \xi\}$ is a local orthonormal basis of vector fields in M, then $\{\phi e_1, ..., \phi e_{n-1}, \xi\}$ is also a local orthonormal basis. Then it is easy to verify that

$$\sum_{i=1}^{n-1} g(e_i, e_i) = \sum_{i=1}^{n-1} g(\phi e_i, \phi e_i) = n - 1, \tag{19}$$

$$\sum_{i=1}^{n-1} g(e_i, Z)S(Y, e_i) = \sum_{i=1}^{n-1} g(\phi e_i, Z)S(Y, \phi e_i) = S(Y, Z) - (n-1)\eta(Z)(20)$$

$$\sum_{i=1}^{n-1} g(e_i, \phi Z)S(Y, e_i) = \sum_{i=1}^{n-1} g(\phi e_i, \phi Z)S(Y, \phi e_i) = S(Y, \phi Z), \tag{21}$$

$$\sum_{i=1}^{n-1} S(e_i, e_i) = \sum_{i=1}^{n-1} S(\phi e_i, \phi e_i) = r - (n-1), \tag{22}$$

$$\sum_{i=1}^{n-1} R(e_i, Y, Z, e_i) = \sum_{i=1}^{n-1} R(\phi e_i, Y, Z, \phi e_i) = S(Y, Z) - R(\xi, Y, Z, \xi), \tag{23}$$

$$\sum_{i=1}^{n-1} R(\phi e_i, \phi Y, \phi Z, \phi e_i) = S(\phi Y, \phi Z) - R(\xi, Y, Z, \xi) \tag{24}$$

for $Y, Z \in TM$. These results will be used in further sections. For more details we refer ([5, 21]).

3 Some properties of a Quarter-symmetric metric connection in a Sasakian manifolds

Let $\widetilde{\nabla}$ be a linear connection and ∇ be a Riemannian connection of an almost contact metric manifold M such that

$$\widetilde{\nabla}_X Y = \nabla_X Y + H(X, Y), \tag{25}$$

where H is a tensor of type $(1, 1)$. For $\widetilde{\nabla}$ to be a quarter-symmetric metric connection in M, we have [9]

$$H(X, Y) = \frac{1}{2}[T(X, Y) + T^{'}(X, Y) + T^{'}(Y, X)], \tag{26}$$

and

$$g(T^{'}(X, Y), Z) = g(T(Z, X), Y). \tag{27}$$

From (1) and (27) we get

$$T^{'}(X, Y) = g(\phi Y, X)\xi - \eta(X)\phi Y. \tag{28}$$

Using (1) and (28) in (26) we obtain

$$H(X,Y) = -\eta(X)\phi Y. \tag{29}$$

Hence a quarter-symmetric metric connection $\widetilde{\nabla}$ in a Sasakian manifold is given by

$$\widetilde{\nabla}_X Y = \nabla_X Y - \eta(X)\phi Y. \tag{30}$$

Therefore equation (30) is the relation between the Levi-Civita connection and the quarter-symmetric metric connection on a Sasakian manifold.

The curvature tensor $\widetilde{R}$ of the connection $\widetilde{\nabla}$ is given by

$$\widetilde{R}(X,Y)Z = \widetilde{\nabla}_X\widetilde{\nabla}_Y Z - \widetilde{\nabla}_Y\widetilde{\nabla}_X Z - \widetilde{\nabla}_{[X,Y]}Z. \tag{31}$$

From (30) it follows that

$$\widetilde{\nabla}_X\widetilde{\nabla}_Y Z = \widetilde{\nabla}_X\nabla_Y Z - \widetilde{\nabla}_X(\eta(Y)\phi Z). \tag{32}$$

In view of (30), (31) and (32), we obtain the formula for curvature $\widetilde{R}$ of the connection $\widetilde{\nabla}$ as

$$\widetilde{R}(X,Y)Z = R(X,Y)Z - 2d\eta(X,Y)\phi Z + \eta(X)(\nabla_Y\phi)Z - \eta(Y)(\nabla_X\phi)Z. \tag{33}$$

For a Sasakian manifold, in view of (13) we get

$$\begin{aligned}\widetilde{R}(X,Y)Z &= R(X,Y)Z - 2d\eta(X,Y)\phi Z + \eta(X)g(Y,Z)\xi \\ &\quad -\eta(Y)g(X,Z)\xi - \{\eta(X)Y - \eta(Y)X\}\eta(Z).\end{aligned} \tag{34}$$

where $R(X,Y)Z$ is the curvature tensor of the connection ∇. A relation between the curvature tensor of M with respect to the quarter-symmetric metric connection $\widetilde{\nabla}$ and the Levi-Civita connection ∇ is given by the equation (34). So from (34) and (15) we have

$$\widetilde{R}(X,Y)\xi = 2\{\eta(Y)X - \eta(X)Y\}, \tag{35}$$

$$\widetilde{R}(X,\xi)Y = 2\{\eta(Y)X - g(X,Y)\xi\} \tag{36}$$

and

$$\widetilde{R}(\xi,X)\xi = 2\{\eta(X)\xi - X\}. \tag{37}$$

Taking the innerproduct of (34) with W, we have

$$\begin{aligned}g(\widetilde{R}(X,Y)Z,W) &= g(R(X,Y)Z,W) - 2d\eta(X,Y)g(\phi Z,W) \\ &\quad +\{\eta(X)g(Y,Z) - \eta(Y)g(X,Z)\}\eta(W) \\ &\quad -\{\eta(X)g(Y,W) - \eta(Y)g(X,W)\}\eta(Z).\end{aligned} \tag{38}$$

Contracting (38) over X and W, we obtain

$$\widetilde{S}(Y,Z) = S(Y,Z) - 2d\eta(\phi Z, Y) + g(Y,Z) + (n-2)\eta(Y)\eta(Z), \tag{39}$$

where $\widetilde{S}$ and S are the Ricci tensors of the connections $\widetilde{\nabla}$ and ∇, respectively. So, in a Sasakian manifold, the Ricci tensor of the quarter-symmetric metric connection is symmetric. It follows from (17) and (39) that

$$\widetilde{S}(\phi Y, \phi Z) = \widetilde{S}(Y,Z) - 2(n-1)\eta(Y)\eta(Z), \tag{40}$$
$$\widetilde{S}(Y,\xi) = 2(n-1)\eta(Y)\eta(Z). \tag{41}$$

Again, contracting (39) over Y and Z, we get

$$\widetilde{r} = r + 2(n-1), \tag{42}$$

where $\widetilde{r}$ and r are the scalar curvatures of the connection $\widetilde{\nabla}$ and ∇, respectively. So we have the following theorem:

Theorem 3.1 *For a Sasakian manifold M with respect to the quarter-symmetric metric connection $\widetilde{\nabla}$*

1. *The curvature tensor $\widetilde{R}$ is given by (34),*
2. $\widetilde{R}(X,Y,Z,W) + \widetilde{R}(Y,X,Z,W) = 0,$
3. $\widetilde{R}(X,Y,Z,W) + \widetilde{R}(X,Y,W,Z) = 0,$
4. $\widetilde{R}(X,Y,Z,W) - \widetilde{R}(Z,W,X,Y) = 0,$
5. *The Ricci tensor $\widetilde{S}$ is given b (39),*
6. $\widetilde{S}(\phi X, \phi Y) = \widetilde{S}(Y,Z) - 2(n-1)\eta(Y)\eta(Z),$
7. *The Ricci tensor $\widetilde{S}$ is symmetric,*
8. *The scalar curvature $\widetilde{r}$ is given by (42).*

In an n-dimensional Sasakian manifold M with respect to the quarter-symmetric metric connection $\widetilde{\nabla}$, if $\{e_1, ..., e_{n-1}, \xi\}$ is a local orthonormal basis of vector fields in M, then $\{\phi e_1, ..., \phi e_{n-1}, \xi\}$ is also a local orthonormal basis. Then using (20) we have

$$\sum_{i=1}^{n-1} g(e_i, Z)\widetilde{S}(Y, e_i) = \sum_{i=1}^{n-1} g(\phi e_i, Z)\widetilde{S}(Y, \phi e_i) = \widetilde{S}(Y,Z) - 2(n-1)\eta(Y)\eta(Z) \tag{43}$$

for all $Y, Z \in TM$. In a Sasakian manifold with respect to the connection $\widetilde{\nabla}$ we have

$$\widetilde{R}(\xi, Y, Z, \xi) = -2d\eta(\phi Z, Y), \tag{44}$$
$$\widetilde{S}(\xi, \xi) = 2(n-1) \tag{45}$$

and

$$\widetilde{Q}\xi = 2(n-1)\xi. \tag{46}$$

From (44), (23) and (24) it follows that

$$\sum_{i=1}^{n-1} \widetilde{R}(e_i, Y, Z, e_i) = \sum_{i=1}^{n-1} \widetilde{R}(\phi e_i, Y, Z, \phi e_i) = \widetilde{S}(Y, Z) + 2d\eta(\phi Z, Y), \tag{47}$$
$$\begin{aligned}\sum_{i=1}^{n-1} \widetilde{R}(e_i, \phi Y, \phi Z, e_i) &= \sum_{i=1}^{n-1} \widetilde{R}(\phi e_i, \phi Y, \phi Z, \phi e_i) \\ &= \widetilde{S}(Y, Z) - 2g(Y, Z) - 2(n-2)\eta(Y)\eta(Z),\end{aligned} \tag{48}$$

Also from (45) and (22) we get

$$\sum_{i=1}^{n-1} \widetilde{S}(e_i, e_i) = \sum_{i=1}^{n-1} \widetilde{S}(\phi e_i, \phi e_i) = \tilde{r} - 2(n-1), \tag{49}$$
$$\sum_{i=1}^{n-1} \widetilde{S}(\phi e_i, \phi Z) g(\phi Y, \phi e_i) = \widetilde{S}(\phi Y, \phi Z). \tag{50}$$

These results will be useful in the next section.

3.1 Some structure theorems on Sasakian manifolds with respect to the quarter-symmetric metric connection

The conharmonic curvature tensor $\widetilde{K}$ of an n-dimensional almost contact metric manifold with respect to quarter-symmetric metric connection $\widetilde{\nabla}$ is given by

$$\begin{aligned}\widetilde{K}(X, Y)Z &= \widetilde{R}(X, Y)Z \\ &-\frac{1}{n-2}\{\widetilde{S}(Y, Z)X - \widetilde{S}(X, Z)Y + g(Y, Z)\widetilde{Q}X - g(X, Z)\widetilde{Q}Y\} \quad X, Y, Z \in TM.\end{aligned} \tag{51}$$

Analogous to the Definition(1.1), we give the following Definition with respect to the connection $\widetilde{\nabla}$.

Definition 3.2 *An almost contact metric manifold M is said to be quasi-conharmonically flat with respect to the quarter-symmetric metric connection if*

$$g(\widetilde{K}(X, Y)Z, \phi W) = 0, \quad X, Y, Z, W \in TM, \tag{52}$$

ϕ-conharmonically flat with respect to the quarter-symmetric metric connection if

$$g(\widetilde{K}(\phi X, \phi Y)\phi Z, \phi W) = 0, \quad X, Y, Z, W \in TM, \tag{53}$$

and ξ-conharmonically flat with respect to the quarter-symmetric metric connection if

$$\widetilde{K}(X, Y)\xi = 0, \quad X, Y \in TM, \tag{54}$$

We begin with the following:

Theorem 3.3 *Let M be an n-dimensional $(n > 3)$ Sasakian manifold with respect to quarter-symmetric metric connection $\widetilde{\nabla}$. If M satisfies*

$$g(\widetilde{K}(\phi X, Y)Z, \phi W) = 0, \quad X, Y, Z, W \in TM, \tag{55}$$

then the scalar curvature with respect to the connection $\widetilde{\nabla}$ is zero and M is an η-Einstein manifold with respect to the connection $\widetilde{\nabla}$.

Proof. Let M be an n-dimensional $(n > 3)$ Sasakian manifold with respect to the connection $\widetilde{\nabla}$. From (51) we have

$$\begin{aligned} g(\widetilde{K}(\phi X, Y)Z, \phi W) &= g(\widetilde{R}(\phi X, Y)Z, \phi W) \\ &- \frac{1}{n-2}\{\widetilde{S}(Y, Z)g(\phi X, \phi W) - \widetilde{S}(\phi X, Z)g(Y, \phi W) \\ &+ g(Y, Z)\widetilde{S}(\phi X, \phi W) - g(\phi X, Z)\widetilde{S}(Y, \phi W)\} \end{aligned} \tag{56}$$

for $X, Y, Z, W \in TM$. For a local orthonormal basis of vector fields $\{e_1, ..., e_{n-1}, \xi\}$ in M, then (56) gives

$$\begin{aligned} \sum_{i=1}^{n-1} g(\widetilde{K}(\phi e_i, Y)Z, \phi e_i) &= \sum_{i=1}^{n-1} g(\widetilde{R}(\phi e_i, Y)Z, \phi e_i) \\ &- \frac{1}{n-2}\sum_{i=1}^{n-1}\{\widetilde{S}(Y, Z)g(\phi e_i, \phi e_i) - \widetilde{S}(\phi e_i, Z)g(Y, \phi e_i) \\ &+ g(Y, Z)\widetilde{S}(\phi e_i, \phi e_i) - g(\phi e_i, Z)\widetilde{S}(Y, \phi e_i)\}. \end{aligned} \tag{57}$$

for $Y, Z \in TM$. Using (47), (19), (43) and (49) in above equation, we get

$$\begin{aligned} \sum_{i=1}^{n-1} g(\widetilde{K}(\phi e_i, Y)Z, \phi e_i) &= \widetilde{S}(Y, Z) + 2d\eta(\phi Z, Y) \\ &- \frac{1}{n-2}\{(n-3)\widetilde{S}(Y, Z) - 4d\eta(\phi Z, Y) \\ &(\widetilde{r} - 2(n-1))g(Y, Z) + 4n\eta(Y)\eta(Z)\} \end{aligned} \tag{58}$$

for $Y, Z \in TM$. If M satisfies (55), then from (58) we have

$$\begin{aligned}\widetilde{S}(Y,Z) + 2d\eta(\phi Z, Y) &= \frac{1}{n-2}\{(n-3)\widetilde{S}(Y,Z) - 4d\eta(\phi Z, Y) \\ &\quad +(\tilde{r} - 2(n-1))g(Y,Z) + 4n\eta(Y)\eta(Z)\}\end{aligned}$$

for $Y, Z \in TM$. This is equivalent to

$$\widetilde{S}(Y,Z) = (\tilde{r} - 2)g(Y,Z) + 2n\eta(Y)\eta(Z)\} \tag{59}$$

for $Y, Z \in TM$, where (12) is used. Putting $Z = \xi$ in (59) and using (41) and $\eta(\xi) = 1$, we get $\tilde{r} = 0$ (i.e., the scalar curvature with respect to the connection $\widetilde{\nabla}$ is zero) and consequently (59) reduces to

$$\widetilde{S}(Y,Z) = -2g(Y,Z) + 2n\eta(Y)\eta(Z). \tag{60}$$

This means that the manifold is an η-Einstein with respect to the connection $\widetilde{\nabla}$.

Theorem 3.4 *An n-dimensional $(n > 3)$ Sasakian manifold M is quasi-conharmonically flat with respect to the quarter-symmetric metric connection $\widetilde{\nabla}$ if and only if*

$$\begin{aligned}\widetilde{R}(X,Y,Z,\phi W) &= -\frac{4}{(n-2)}\{g(Y,Z)g(X,\phi W) - g(X,Z)g(Y,\phi W)\} \quad (61)\\ &\quad +\frac{2n}{(n-2)}\{\eta(Y)g(X,\phi W) - \eta(X)g(Y,\phi W)\}\eta(Z),\end{aligned}$$

for all $X, Y, Z, W \in TM$.

Proof. Let M is quasi-conharmonically flat with respect to the connection $\widetilde{\nabla}$, using (60) in

$$\begin{aligned}g(\widetilde{K}(X,Y)Z,\phi W) &= \widetilde{R}(X,Y,Z,\phi W) - \frac{1}{(n-2)}\{\widetilde{S}(Y,Z)g(X,\phi W) - \widetilde{S}(X,Z)g(Y,\phi W) \\ &\quad +g(Y,Z)\widetilde{S}(X,\phi W) - g(X,Z)\widetilde{S}(Y,\phi W)\}\end{aligned}$$

we obtain (61). The converse is straightforward.

Theorem 3.5 *An n-dimensional $(n > 3)$ Sasakian manifold is ϕ-conharmonically flat with respect to the quarter-symmetric metric connection $\widetilde{\nabla}$ if and only if M satisfies*

$$\widetilde{R}(\phi X,\phi Y,\phi Z,\phi W) = -\frac{4}{(n-2)}\{g(\phi Y,\phi Z)g(\phi X,\phi W) - g(\phi X,\phi Z)g(\phi Y,\phi W)\} \tag{62}$$

for all $X, Y, Z, W \in TM$

Proof. Let M be an n-dimensional Sasakian manifold. From (51) we have

$$g(\widetilde{K}(\phi X,\phi Y)\phi Z,\phi W) = g(\widetilde{R}(\phi X,\phi Y)\phi Z,\phi W) - \frac{1}{(n-2)}\{\widetilde{S}(\phi Y,\phi Z)g(\phi X,\phi W) - \widetilde{S}(\phi X,\phi Z)g(\phi Y,\phi W) + \widetilde{S}(\phi X,\phi W)g(\phi Y,\phi Z) - \widetilde{S}(\phi Y,\phi W)g(\phi X,\phi Z)\} \tag{63}$$

for all $X, Y, Z, W \in TM$. For an orthonormal basis of vector fileds $\{e_1, ..., e_{n-1}, \xi\}$ in M, from (63) it follows that

$$\sum_{i=1}^{n-1} g(\widetilde{K}(\phi e_i,\phi Y)\phi Z,\phi e_i) = \sum_{i=1}^{n-1} g(\widetilde{R}(\phi e_i,\phi Y)\phi Z,\phi e_i) - \frac{1}{(n-2)}\sum_{i=1}^{n-1}\{\widetilde{S}(\phi Y,\phi Z)g(\phi e_i,\phi e_i) - \widetilde{S}(\phi e_i,\phi Z)g(\phi Y,\phi e_i) + \widetilde{S}(\phi e_i,\phi e_i)g(\phi Y,\phi Z) - \widetilde{S}(\phi Y,\phi e_i)g(\phi e_i,\phi Z)\} \tag{64}$$

for all $Y, Z \in TM$. Which in view of (19), (48), (49) and (50) becomes

$$\sum_{i=1}^{n-1} g(\widetilde{K}(\phi e_i,\phi Y)\phi Z,\phi e_i) = \widetilde{S}(Y,Z) - 2g(Y,Z) - 2(n-2)\eta(Y)\eta(Z) - \frac{1}{(n-2)}\{(n-3)\widetilde{S}(\phi Y,\phi Z) + (\tilde{r} - 2(n-1))g(\phi Y,\phi Z)\} \tag{65}$$

for all $Y, Z \in TM$. If M is ϕ-conharmonically flat with respect to the connection $\widetilde{\nabla}$, using (54), (40) and (10) in (65) we get

$$\widetilde{S}(Y,Z) = \{\tilde{r} - 2\}g(Y,Z) + \{2n - \tilde{r}\}\eta(Y)\eta(Z) \quad Y, Z \in TM \tag{66}$$

Putting $Z = \xi$ in (66) and using (41) and $\eta(\xi) = 1$, we get $\tilde{r} = 0$ and consequently (66) reduces to (60). By replacing X by ϕX and Y by ϕY in (60) one can get $\widetilde{S}(\phi Y,\phi Z) = -2g(\phi Y,\phi Z)$ for $Y, Z \in TM$. Now using this value in (63) with (53) we obtain (62). The converse is obvious.

Theorem 3.6 *Let M^n be an n-dimensional $(n > 3)$ Sasakian manifold. Then the following statements are equivalent:*

1. *M is conharmonically flat with respect to the connection $\widetilde{\nabla}$.*
2. *M is ϕ-conharmonically flat with respect to the connection $\widetilde{\nabla}$.*
3. *The curvature tensor with respect to the connection $\widetilde{\nabla}$ of M is given by*

$$\widetilde{R}(X,Y)Z = -\frac{4}{n-2}\{g(Y,Z)X - g(X,Z)Y\} - \frac{2n}{n-2}\{g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi - \eta(Y)\eta(Z)X + \eta(X)\eta(Z)Y\} \quad for \ \ all \ \ X, Y, Z \in TM. \tag{67}$$

Proof. Let M be an n-dimensional ($n > 3$) Sasakian manifold. From (52) and (54) it is obvious that (1) implies (2). Now, assume that (2) is true. In a Sasakian manifold, in view of (35) and (36) we can verify

$$\begin{aligned}\widetilde{R}(\phi^2X, \phi^2Y, \phi^2Z, \phi^2W) \;=\; & \widetilde{R}(X, Y, Z, W) + 2\{g(X, Z)\eta(Y)\eta(W) \\ & -g(Y, Z)\eta(X)\eta(W) + g(Y, W)\eta(X)\eta(Z) - g(X, W)\eta(Y)\eta(Z)\}\end{aligned} \tag{68}$$

for all $X, Y, Z, W \in TM$. By changing X, Y, Z, W to $\phi X, \phi Y, \phi Z, \phi W$, respectively in (62) and using (68) we get (67). Hence, the statement (2) implies the statement (3). Next, we assume the statement (3) is true. On contracting (67) it follows (60). Using (60) and (67) in (51) we get the statement (1). This completes the proof.

Theorem 3.7 *Let M be an n-dimensional ($n > 3$) η-Einstein manifold with respect to the quarter-symmetric metric connection $\widetilde{\nabla}$. Then M is ξ-conharmonically flat with repsect to the quarter-symmetric metric connection.*

Proof. Suppose M be an n-dimensional η-Einstein manifold with respect to the quarter-symmetric metric connection. then there exists functions α and β such that

$$\widetilde{S}(X, Y) = g(\widetilde{Q}X, Y) = \alpha g(X, Y) + \beta\eta(X)\eta(Y). \tag{69}$$

But, from (45) we also have

$$\alpha + \beta = 2(n - 1). \tag{70}$$

on the other hand, the scalar curvature with respect to the connection $\widetilde{\nabla}$ satisfies:

$$\widetilde{r} = \sum_{i=1}^{n} \widetilde{S}(X, Y) = n\alpha + \beta. \tag{71}$$

By virtue of (70) and (71), we have from (69) that

$$\widetilde{S}(X, Y) = \left(\frac{\widetilde{r}}{n-1} - 2\right) g(X, Y) + \left(2n - \frac{\widetilde{r}}{n-1}\right) \eta(X)\eta(Y). \tag{72}$$

For a local orthonormal basis of vector fields $\{e_1, ..., e_{n-1}, \xi\}$ in M, then from (72) we get $\widetilde{r} = 0$ and consequently (72) reduces to (60). By taking account of (35) and (72) in formula (51) we get the required result.

Theorem 3.8 *A Sasakian manifold M is ξ-conharmonically flat with respect to the quarter-symmetric metric connection $\widetilde{\nabla}$ if and only if it is an η-Einstein manifold with respect to the connection $\widetilde{\nabla}$.*

Proof. We only have to prove that a ξ-conharmonically flat Sasakian manifold with respect to the connection $\widetilde{\nabla}$ is an η-Einstein manifold with respect to the connection $\widetilde{\nabla}$. The converse follows from the Theorem(3.7).

For a ξ-conharmonically flat Sasakian manifold with respect to the connection $\widetilde{\nabla}$ and by (51) and $g(X,\xi)=\eta(X)$, we have

$$\begin{aligned} g(\widetilde{K}(X,Y)\xi,W) &= g(\tilde{R}(X,Y)\xi,W) \\ &-\frac{1}{(n-2)}\{\widetilde{S}(Y,\xi)g(X,W)-\widetilde{S}(X,\xi)g(Y,W) \\ &+\eta(Y)\widetilde{S}(X,W)-\eta(X)\widetilde{S}(Y,W)\}. \end{aligned} \tag{73}$$

for all $X,Y,W\in TM$. For a local orthonormal basis $\{e_1,...,e_{n-1},\xi\}$ of vector fields in M, from (73) we get

$$\begin{aligned} g(\widetilde{K}(e_i,Y)\xi,e_i) &= g(\tilde{R}(e_i,Y)\xi,e_i) \\ &-\frac{1}{(n-2)}\{\widetilde{S}(Y,\xi)g(e_i,e_i)-\widetilde{S}(e_i,\xi)g(Y,e_i) \\ &+\eta(Y)\widetilde{S}(e_i,e_i)-\eta(e_i)\widetilde{S}(Y,e_i)\}. \end{aligned} \tag{74}$$

for all $Y\in TM$. If M is ξ-conharmonically flat with respect to the connection $\widetilde{\nabla}$ and using (54), (19), (43) and (49) in above equation, we get $\widetilde{r}=0$. Now putting $Y=\xi$ in (74) and using (9) and (41) we have

$$\begin{aligned} g(\widetilde{K}(X,\xi)\xi,W) &= g(\tilde{R}(X,\xi)\xi,W) \\ &-\frac{1}{(n-2)}\{\widetilde{S}(X,W)-4(n-1)\eta(X)\eta(W)\}. \end{aligned} \tag{75}$$

for all $Y,Z\in TM$. Using (53) and (37) in above equation, we get (60).

Corollary 3.9 *Let M be a ξ-conharmonically flat Sasakian manifold with respect to the quarter-symmetric metric connection. If there exists functions K_1 and K_2 such that*

$$(\nabla_X Q)Y-(\nabla_Y Q)X=K_1X+K_2Y, \tag{76}$$

then,

$$QX=\alpha_1 X. \tag{77}$$

proof. By equating (39) and (60), we have

$$S(X,W)=-5g(X,W)+(n+4)\eta(X)\eta(W). \tag{78}$$

This implies $QX=\alpha_1X+\beta_1\eta(X)\xi$, where $\alpha_1=-5$ and $\beta_1=(n+4)$. Thus we have:

$$\begin{aligned} (\nabla_X Q)Y-(\nabla_Y Q)X &= (X\alpha_1)Y-(Y\alpha_1)X+(X\beta_1)\eta(Y)\xi \\ &-(Y\beta_1)\eta(X)\xi-\beta_1\{2g(\phi X,Y)\xi+\eta(Y)\phi X-\eta(X)\phi Y\}. \end{aligned} \tag{79}$$

Replacing X by ϕX and Y by ϕY in (79) we get:

$$(\nabla_{\phi X}Q)\phi Y - (\nabla_{\phi Y}Q)\phi X = (\phi X\alpha_1)\phi Y - (\phi Y\alpha_1)\phi X - 2\beta_1 g(\phi^2 X, \phi Y)\xi. \tag{80}$$

from (76) and (80) we obtain $(K_1+(\phi Y\alpha_1))\phi Y+(K_2-(\phi X\alpha_1))\phi Y = 2\beta_1 g(\phi^2 X, \phi Y)\xi$, which shows that $-2\beta_1 g(\phi^2 X, \phi Y) = 0$. Here by replacing X by ϕY, we obtain $\beta_1 g(\phi X, \phi Y) = 0$ and hence $\beta_1 = 0$.

Finally, it is easy to prove the following Corollary.

Corollary 3.10 *On a Sasakian manifold M, the following conditions are equivalent.*

1. *M is ξ-conharmonically flat with respect to the connection $\widetilde{\nabla}$;*
2. *$\eta(\widetilde{K}(X,Y)Z) = 0$;*
3. *$\phi^2\widetilde{K}(X,Y)Z = -\widetilde{K}(X,Y)Z$; and*
4. *$\widetilde{K}(X,Y)\xi = 0$,*

for all $X, Y, Z \in TM$.

4 Open Problems

1. Efforts can be made to study the structure of some classes of trans-Sasakin manifolds with respect to semi-symmetric and Quarter-symmetric metric connections.

2. In this paper, we have discussed some classes of Kenmotsu manifolds with respect to quarter-symmetric metric connection on Conhamonic curvature tensor. One can work on different types of curvature tensors like conformal, quasi-conformal, concircular, Projective, Pseudo-projective, T-curvature tensor, etc.,

3. One can study to give some non-trivial examples of conharmonic flat metric Kenmotsu manifolds with respect to quarter-symmetric metric connection.

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