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On a Concept of Nonuniform Exponential Instability of Evolution Operators in Banach Spaces

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Abstract

In this paper we give necessary and sufficient conditions for a concept of nonuniform exponential instability (in the sense of Barreira-Valls) of evolution operators. We motivate our approach with illustrative examples.

Keywords: *evolution operators; nonuniform exponential instability.*

1 Introduction

The importance of the role played by the notions of exponential instability and expansivity in the theory of exponential dichotomy is illustrated by the appearance in the last few years of fundamental papers (for example, see [6] and [7]). We note that all these works consider only the case of uniform exponential instability.

The aim of this note is to generalize in a natural way the notion of uniform exponential instability of evolution operators in Banach spaces. Our approach is from nonuniform point of view (see [1] and [5]). We prove variants for the case of nonuniform exponential instability of well-known results due to Datko [3] and Rolewicz [9]. We emphasize that in our proof we do not need to assume the invertibility of the evolution operators.

2 Notions and preliminaries

Let X be a Banach space and $\mathcal{B}(X)$ be the Banach algebra of all bounded linear operators acting on X. The norms on X and on $\mathcal{B}(X)$ are both denoted by $\| \cdot \|$ and we put

$$\Delta = \{(t,s) \in \mathbb{R}_+^2 : t \geq s\} \text{ and } T = \{(t,s,t_0) \in \mathbb{R}_+^3 : t \geq s \geq t_0\}.$$

Let us first recall the classical notion of evolution operator:

Definition 2.1 An operator-valued function $U : \Delta \longrightarrow \mathcal{B}(X)$ is said to be *an evolution operator* if the following conditions hold:
e_1) $U(t,t) = I$ (the identity on X), for all $t \geq 0$;
e_2) $U(t,s)U(s,t_0) = U(t,t_0)$, for all $(t,s,t_0) \in T$;
e_3) for each $x \in X$, the function $(t,s) \longmapsto U(t,s)x$ is continuous.

Definition 2.2 The evolution operator $U : \Delta \longrightarrow \mathcal{B}(X)$ is said to be with *uniform exponential decay* if there are $M \geq 1$ and $\omega > 0$ such that

$$M \| U(t,s)x \| \geq e^{-\omega(t-s)} \| x \|, \text{ for all } (t,s,x) \in \Delta \times X.$$

An important class of evolution operators is given by:

Definition 2.3 The evolution operator $U : \Delta \longrightarrow \mathcal{B}(X)$ is called *uniformly exponentially unstable* if there are $N \geq 1$ and $\nu > 0$ such that

$$N \| U(t,s)x \| \geq e^{\nu(t-s)} \| x \|, \text{ for all } (t,s,x) \in \Delta \times X. \tag{1}$$

Remark 2.4 The evolution operator U is uniformly exponentially unstable if and only if there are $N \geq 1$ and $\nu > 0$ such that

$$\| U(s,t_0)x_0 \| \leq Ne^{-\nu(t-s)} \| U(t,t_0)x_0 \|, \text{ for all } (t,s,t_0,x_0) \in T \times X.$$

Remark 2.5 *If $U : \Delta \longrightarrow \mathcal{B}(X)$ is an uniformly exponentially unstable evolution operator then it has uniform exponential decay. The converse is not necessarily valid.*

Example 2.6 Let $X = \mathbb{R}^2$ endowed with the Euclidean norm. The evolution operator

$$U(t,s)(x_1,x_2) = (\xi_1, \xi_2),$$

where

$$\xi_1 = e^{t-s} \cos t \, (x_1 \cos s + x_2 \sin s) + e^{-(t-s)} \sin t \, (x_1 \sin s - x_2 \cos s)$$

$$\xi_2 = e^{t-s} \sin t \, (x_1 \cos s + x_2 \sin s) - e^{-(t-s)} \cos t \, (x_1 \sin s - x_2 \cos s)$$

has uniform exponential decay, but it is not uniformly exponentially unstable.

Proof. We have

$$\begin{aligned}\| U(t,s)(x_1,x_2) \|^2 &= e^{2(t-s)}(x_1\cos s + x_2 \sin s)^2 + e^{-2(t-s)}(x_1 \sin s - x_2\cos s)^2 \\ &\geq e^{-2(t-s)}(x_1^2+x_2^2).\end{aligned}$$

This yields

$$\| U(t,s)x \| \geq e^{-(t-s)} \| x \|, \text{ for all } (t,s,x) \in \Delta \times X$$

and hence U has uniform exponential decay. If we assume that U is uniformly exponentially unstable then there are constants $N, \nu > 0$ such that

$$N \| U(t,s)x \| \geq e^{\nu(t-s)} \| x \|, \text{ for all } (t,s,x) \in \Delta \times X.$$

In particular, for $s = 0$ and $x = (0,1)$, we obtain $Ne^{-t} \geq e^{\nu t}$, for all $t \geq 0$, which is obviously false. Therefore, U is not uniformly exponentially unstable.

It can be easily seen that if $U : \Delta \longrightarrow \mathcal{B}(X)$ is an uniformly exponentially unstable evolution operator then the following relation holds:

$$\lim_{t\to\infty} \| U(t,t_0)x_0 \| = \infty, \text{ for all } (t_0,x_0) \in \mathbb{R}_+ \times X,\ x_0 \neq 0. \tag{2}$$

The above relation defines the notion of instability. Our objective is to consider a nonuniform version of exponential instability such that this relation still holds true. We note that there are some concepts of nonuniform exponential instability, but for all of them the property (2) fails (see for instance [4] and the references therein). In this paper we propose the following concept:

Definition 2.7 The evolution operator $U : \Delta \longrightarrow \mathcal{B}(X)$ is said to be *exponentially unstable* (in the sense of Barreira-Valls [1]) if there are $N \geq 1$, $\alpha \geq 0$ and $\nu > 0$ with $\alpha < \nu$ such that

$$Ne^{\alpha t} \| U(t,s)x \| \geq e^{\nu(t-s)} \| x \|, \text{ for all } (t,s,x) \in \Delta \times X. \tag{3}$$

Remark 2.8 The evolution operator $U : \Delta \longrightarrow \mathcal{B}(X)$ is exponentially unstable if and only if there are $N \geq 1$, $\alpha \geq 0$ and $\nu > 0$ with $\alpha < \nu$ such that

$$\| U(s,t_0)x_0 \| \leq Ne^{\alpha t}e^{-\nu(t-s)} \| U(t,t_0)x_0 \|, \text{ for all } (t,s,t_0,x_0) \in T \times X.$$

Example 2.9 *The evolution operator*

$$U(t,s)x = e^{f(t)-f(s)}x, \text{ where } f(t) = \frac{t}{2+\cos t},\ t \geq 0$$

is exponentially unstable, but it is not uniformly exponentially unstable.

Proof. We have

$$\begin{aligned} f(t)-f(s) &= \frac{t}{2+\cos t}-\frac{s}{2+\cos s} \\ &= -t\left(\frac{1}{3}-\frac{1}{2+\cos t}\right)+s\left(\frac{1}{3}-\frac{1}{2+\cos s}\right)+\frac{1}{3}(t-s) \\ &= -\frac{t(\cos t-1)}{3(2+\cos t)}+\frac{s(\cos s-1)}{3(2+\cos s)}+\frac{1}{3}(t-s) \\ &= \frac{2t\sin^2\frac{t}{2}}{3(2+\cos t)}-\frac{2s\sin^2\frac{s}{2}}{3(2+\cos s)}+\frac{1}{3}(t-s) \\ &\geq -\frac{2}{3}s+\frac{1}{3}(t-s),\ t\geq s\geq 0, \end{aligned}$$

which implies

$$e^{\frac{2}{3}t}\parallel U(t,s)x\parallel\geq e^{t-s}\parallel x\parallel,\ \text{for all } (t,s,x)\in\Delta\times X.$$

Therefore, U is exponentially unstable. If we assume that U is uniformly exponentially unstable, then there are $N,\nu>0$ such that

$$N\parallel U(t,s)x\parallel\geq e^{\nu(t-s)}\parallel x\parallel,\ \text{for all } (t,s,x)\in\Delta\times X.$$

In particular, for $t=2n\pi+2\pi$, $s=2n\pi+\pi$, $n\in\mathbb{N}$ and $x\in X$ with $\parallel x\parallel=1$, we have

$$Ne^{\frac{\pi}{3}}e^{-\frac{2}{3}(2n\pi+\pi)}\geq e^{\nu\pi},\ \text{for every } n\in\mathbb{N},$$

which is false.

An equivalent definition of exponential instability is given by:

Proposition 2.10 *The evolution operator $U:\Delta\longrightarrow\mathcal{B}(X)$ is exponentially unstable if and only if there are $N\geq 1$, $\beta\geq 0$ and $\gamma>0$ such that*

$$\parallel U(s,t_0)x_0\parallel\leq Ne^{\beta s}e^{-\gamma(t-s)}\parallel U(t,t_0)x_0\parallel,\ \textit{for all } (t,s,t_0,x_0)\in T\times X.$$

Proof. Necessity. It follows from relation

$$\parallel U(s,t_0)x_0\parallel\leq Ne^{\alpha s}e^{-(\nu-\alpha)(t-s)}\parallel U(t,t_0)x_0\parallel,\ \text{for } (t,s,t_0,x_0)\in T\times X,$$

where N,α and ν are given by Definition 2.7.

Sufficiency. We have

$$\parallel U(s,t_0)x_0\parallel\leq Ne^{\beta t}e^{-(\gamma+\beta)(t-s)}\parallel U(t,t_0)x_0\parallel,$$

for all $(t,s,t_0,x_0)\in T\times X$, which implies that U is exponentially unstable with constants $\alpha=\beta$ and $\nu=\beta+\gamma$.

Definition 2.11 The evolution operator $U : \Delta \longrightarrow \mathcal{B}(X)$ is said to be with *exponential decay* if there are $M \geq 1$, $\varepsilon \geq 0$ and $\omega > 0$ such that

$$Me^{\varepsilon t_0} \parallel U(t,t_0)x_0 \parallel \geq e^{-\omega(t-t_0)} \parallel x_0 \parallel, \text{ for all } (t,t_0,x_0) \in \Delta \times X. \quad (4)$$

Remark 2.12 The evolution operator $U : \Delta \longrightarrow \mathcal{B}(X)$ has exponential decay if and only if there are $M \geq 1$, $\varepsilon \geq 0$ and $\omega > 0$ such that

$$\parallel U(s,t_0)x_0 \parallel \leq Me^{\varepsilon s} e^{\omega(t-s)} \parallel U(t,t_0)x_0 \parallel, \text{ for all } (t,s,t_0,x_0) \in T \times X.$$

Remark 2.13 If the evolution operator U has uniform exponential decay, then it has exponential decay. The following example shows that the converse is not valid.

Example 2.14 *The evolution operator*

$$U(t,s) = e^{f(t)-f(s)}I, \text{ where } f(t) = -2t + t\cos t,$$

has exponential decay, but it has not uniform exponential decay.

Proof. Successively, we obtain

$$\begin{aligned} f(t) - f(s) &= -(t-s) + t(\cos t - 1) - s(\cos s - 1) \\ &= -(t-s) - 2t\sin^2\frac{t}{2} + 2s\sin^2\frac{s}{2} \\ &\geq -(t-s) - 2t, \ \forall\, t \geq s \geq 0. \end{aligned}$$

Thus,

$$e^{2t} \parallel U(t,s)x \parallel \geq e^{-(t-s)} \parallel x \parallel, \text{ for all } (t,s,x) \in \Delta \times X,$$

which is equivalent with

$$e^{2s} \parallel U(t,s)x \parallel \geq e^{-3(t-s)} \parallel x \parallel, \text{ for all } (t,s,x) \in \Delta \times X.$$

We assume for a contradiction that U has uniform exponential decay. Letting $t = 2n\pi + \pi$, $s = 2n\pi$, $n \in \mathbb{N}$ and $x \in X$ with $\parallel x \parallel = 1$ in Definition 2.2, we deduce that

$$e^{4n\pi} \leq Me^{-3\pi+\omega\pi}, \text{ for all } n \in \mathbb{N},$$

which is not true. Therefore, U has not uniform exponential decay.

3 The main results

Let $\mathcal{R}$ be the set of all non-decreasing functions $R : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ with the property $R(t) > 0$, for every $t > 0$. First we give a necessarily condition for exponentially unstable evolution operators.

Proposition 3.1 *If the evolution operator $U : \Delta \longrightarrow \mathcal{B}(X)$ is exponentially unstable then there are $R \in \mathcal{R}$ and constants $K \geq 1$, $\beta \geq 0$ and $\gamma > \beta$ such that*

$$\sum_{n=0}^{[t-t_0]} R\left(e^{\gamma n} \parallel U(t-n,t_0)x_0 \parallel\right) \leq R\left(Ke^{\beta t} \parallel U(t,t_0)x_0 \parallel\right),$$

for all $(t, t_0, x_0) \in \Delta \times X$.

Proof. It follows by a simple computation for $R(t) = t,\ t \geq 0$, $\beta = \alpha$, $\gamma \in (\alpha, \nu)$ and $K = \max\left\{\frac{N}{1-e^{\gamma-\nu}}, 1\right\}$, where constants $N \geq 1$, $\alpha \geq 0$ and $\nu > \alpha$ are given by Definition 2.7. Indeed, we have

$$\begin{aligned}\sum_{n=0}^{[t-t_0]} e^{\gamma n} \parallel U(t-n,t_0)x_0 \parallel &\leq Ne^{\alpha t}\left(\sum_{n=0}^{[t-t_0]} e^{-(\nu-\gamma)n}\right) \parallel U(t,t_0)x_0 \parallel \\ &\leq Ke^{\beta t} \parallel U(t,t_0)x_0 \parallel,\end{aligned}$$

for all $(t, t_0, x_0) \in \Delta \times X$.

Remark 3.2 At this point, we do not know whether the sufficiency of the previous result is also valid.

We now come to our main result:

Theorem 3.3 *Let U be an evolution operator with exponential decay. Then U is exponentially unstable if and only if there are a function $R \in \mathcal{R}$ and constants $K \geq 1$, $\beta \geq 0$ and $\gamma > \beta$ such that*

$$\int_{t_0}^{t} R\left(e^{\gamma(t-\tau)} \parallel U(\tau,t_0)x_0 \parallel\right) d\tau \leq R\left(Ke^{\beta t} \parallel U(t,t_0)x_0 \parallel\right), \tag{5}$$

for all $(t, t_0, x_0) \in \Delta \times X$.

Proof. Necessity follows the same lines as in the proof of the previous proposition.

Sufficiency. Without loss of generality we may assume that R is a strictly non-decreasing function, otherwise we consider $\widetilde{R}(t) = R(t) + t,\ \forall\, t \geq 0$. Fix $t \geq s \geq t_0 \geq 0$. If $t \geq s + 1$ then

$$\begin{aligned} & R\left(\frac{1}{M} e^{-(\gamma+\omega)} e^{-\varepsilon s} e^{\gamma(t-s)} \parallel U(s,t_0)x_0 \parallel\right) = \\ & = \int_s^{s+1} R\left(\frac{1}{M} e^{-(\gamma+\omega)} e^{-\varepsilon s} e^{\gamma(t-s)} \parallel U(s,t_0)x_0 \parallel\right) d\tau \\ & \leq \int_s^{s+1} R\left(e^{-(\gamma+\omega)} e^{\omega(\tau-s)} e^{\gamma(t-\tau)} e^{\gamma(\tau-s)} \parallel U(\tau,t_0)x_0 \parallel\right) d\tau \\ & \leq \int_s^{s+1} R\left(e^{\gamma(t-\tau)} \parallel U(\tau,t_0)x_0 \parallel\right) d\tau \\ & \leq R\left(Ke^{\beta t} \parallel U(t,t_0)x_0 \parallel\right), \end{aligned}$$

and therefore

$$\parallel U(s,t_0)x_0 \parallel \leq MKe^{\gamma+\omega} e^{(\varepsilon+\beta)s} e^{-(\gamma-\beta)(t-s)} \parallel U(t,t_0)x_0 \parallel, \tag{6}$$

where $M \geq 1$, $\varepsilon \geq 0$ and $\omega > 0$ are given by Definition 2.11. On the other hand, it is clear that

$$\parallel U(s,t_0)x_0 \parallel \leq Me^{\gamma+\omega} e^{\varepsilon s} e^{-\gamma(t-s)} \parallel U(t,t_0)x_0 \parallel, \text{ for } t \in [s, s+1). \tag{7}$$

Hence the evolution operator U is exponentially unstable, which completes the proof.

A consequence of Theorem 3.3 is indicated in the next corollary:

Corollary 3.4 Let U be an evolution operator with exponential decay. Then U is exponentially unstable if and only if there are constants $p, K > 0$, $\beta \geq 0$ and $\gamma > \beta$ such that

$$\int_{t_0}^{t} e^{p\gamma(t-\tau)} \parallel U(\tau,t_0)x_0 \parallel^p d\tau \leq Ke^{p\beta t} \parallel U(t,t_0)x_0 \parallel^p, \tag{8}$$

for all $(t, t_0, x_0) \in \Delta \times X$.

Remark 3.5 The previous result can be considered a version for the case of exponential instability of a well-known result due to Datko [3].

4 Open Problem

An important property of exponential stability or exponential dichotomy is their roughness (we refer the reader to [2] and [8]). We address the question of roughness of the concept of exponential instability considered in this note.

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