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Fixed Point Theorems of Compatible Mappings of Type(R) in Metric Spaces

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Abstract

In this paper we proved some common fixed point theorems for compatible mappings of Type(R) introduced by Y. Rohen and M. R. Singh [7] in metric spaces.

Keywords: common fixed point, compatible mappings, compatible mappings of type(P), compatible mappings of type(R).

1 Introduction and Preliminaries

Jungck [1] introduced the concept of compatible mappings in 1986 by generalizing commuting mappings. Pathak, Chang and Cho [5] introduced the concept of compatible mappings of type (P). Y. Rohen and M. R. Singh[7] introduced the concept of compatible mappings of type (R) by combining the definitions of compatible and compatible mappings of type (P).

Following are the various types of compatible mappings.

Definition [1]: Let S and T be mappings from a complete metric space X into itself. The mappings S and T are said to be compatible if $\lim_{x\to\infty} d(STx_n, TSx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{x\to\infty} Sx_n = \lim_{x\to\infty} Tx_n = t$ for some $t \in X$.

Definition [5]: Let S and T be mappings from a complete metric space X into itself. The mappings S and T are said to be compatible of type (P) if

$$\lim_{n\to\infty} d(SSx_n, TTx_n) = 0$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.

Definition [7]: Let S and T be mappings from a complete metric space X into itself. The mappings S and T are said to be compatible of type (R) if

$$\lim_{n\to\infty} d(STx_n, TSx_n) = 0$$

and

$$\lim_{n\to\infty} d(SSx_n, TTx_n) = 0$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.

2 Main Result

We prove the following propositions.

Proposition 2.1. Let S, T : (X, d) $\rightarrow$ (X, d) be mappings. If S and T are compatible mappings of type (R) and Sz = Tz for some z $\in$ X, then

$$SSz = STz = TSz = TTz.$$

Proof: Let $\{x_n\}$ be a sequence in X defined by $x_n = z$, n = 1, 2, …, and Sz = Tz for some z $\in$ X. Then we have Sx_n, $Tx_n \rightarrow Sz$ as $n \rightarrow \infty$. Since S and T are compatible mappings of type (R), we have

$$d(SSz, TTz) = \lim_{n\to\infty} d(SSx_n, TTx_n) = 0.$$

Therefore, SSz = TTz. But Sz = Tz implies SSz = STz = TSz = TTz. This completes the proof.

Proposition 2.2. Let S, T : (X, d) $\rightarrow$ (X, d) be mappings. Let S and T are compatible mappings of type (R) and let Sx_n, $Tx_n \rightarrow z$ as $n \rightarrow \infty$ for some z $\in$ X. Then we have the followings:

(i) $\lim_{n\to\infty} TTx_n = Sz$ if S is continuous at z.

(ii) $\lim_{n\to\infty} SSx_n = Tz$ if T is continuous at z.

(iii) $\lim_{n\to\infty} STx_n = Tz$ if T is continuous.

(iv) $\lim_{n\to\infty} TSx_n = Sz$ if S is continuous.

(v) STz = TSz and Sz = Tz if S and T are continuous at z.

Proof. (i) Suppose that S is continuous at z. Since

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$$

for some $z \in X$, we have $SSx_n \to Sz$ as $n \to \infty$. Again, since S and T are compatible of type (R), we have $\lim_{n\to\infty} d(TTx_n, SSx_n) = 0$ and so, since we have

$$d(TTx_n, Sz) \le d(TTx_n, SSx_n) + d(SSx_n, Sz),$$

it follows that $TTx_n \to Sz$ as $n \to \infty$.

Proof of (ii), (iii) and (iv) can be done in the similar process as in (i).

(v) Suppose that S and T are continuous at z. Since $Tx_n \to z$ as $n \to \infty$ and S is continuous at z, by (i), $TTx_n \to Sz$ as $n \to \infty$. On the other hand, since $Tx_n \to z$ as $n \to \infty$ and T is also continuous at z, $TTx_n \to Tz$. Thus, we have $Sz = Tz$ by the uniqueness of the limit and so, by proposition 2.1, $TSz = STz$.

This completes the proof.

Theorem 2.3. Let (X, d) be a complete metric space and A, B, S and T be mappings from X into itself. Suppose that S and T are continuous mappings satisfying the following conditions:

$A(X) \subset T(X)$ and $B(X) \subset S(X)$, (1)

The pairs {A, S} and {B, T} are compatible of type (R), (2)

$$d(Ax, By) \le \Phi\left(\max\left\{d(Sx, Ty), d(Sx, Ax), d(Ty, By), \frac{1}{2}[d(Sx, By) + d(Ty, Ax)]\right\}\right) \quad (3)$$

for all $x, y \in X$, where $\Phi:[0, \infty) \to [0, \infty)$ is a non-decreasing and upper semi continuous function and $\Phi(t) < t$ for all $t > 0$. Then A, B, S and T have a unique common fixed point in X.

Proof. Since $A(X) \subset T(X)$ and $B(X) \subset S(X)$, we can choose a sequence $\{x_n\}$ in X such that $Sx_{2n} = Bx_{2n-1}$ and $Tx_{2n-1} = Ax_{2n-2}$ for n = 1, 2, 3, …. Suppose that

$$y_{2n-1} = Tx_{2n-1} = Ax_{2n-2} \text{ and } y_{2n} = Sx_{2n} = Bx_{2n-1} \quad (4)$$

for n = 1, 2, 3, …. By using the technique of Chang [8], we can prove that $\{y_n\}$ is a Cauchy sequence in X and so, since X is complete, it converges to a point z in X. On the other hand, the subsequences $\{Ax_{2n-2}\}$, $\{Bx_{2n-1}\}$, $\{Sx_{2n}\}$ and $\{Tx_{2n-1}\}$ of $\{y_n\}$ also converge to the point z.

Since {A, S} and {B, T} are compatible of type (R), it follows from the continuity of S and T, (4) and Proposition 2.2 that

$$\left.\begin{array}{ll} Ty_{2n} \to Tz, & By_{2n} = BBx_{2n-1} \to Tz, \\ Sy_{2n-1} \to Sz, & Ay_{2n-1} = AAx_{2n-2} \to Sz \end{array}\right\} \quad (5)$$

as $n \to \infty$. By (3) and (4), we have

$d(Ay_{2n-1}, By_{2n})$

$$\leq \Phi(\max\{d(Sy_{2n-1}, Ty_{2n}), d(Sy_{2n-1}, Ay_{2n-1}), d(Ty_{2n}, By_{2n}), \frac{1}{2}[d(Sy_{2n-1}, By_{2n}) + d(Ty_{2n}, Ay_{2n-1})]\}).$$

By the upper semicontinuity of $\Phi(t)$, (4) and (5), if $Sz \neq Tz$, then we have

$$d(Sz, Tz) \leq \Phi(\max\{d(Sz, Tz), 0, 0, d(Sz, Tz)\}) = \Phi(d(Sz, Tz)) < d(Sz, Tz),$$

which is contradiction. Thus it follows that $Sz = Tz$.

Similarly, from (3), (4), (5) and the upper semicontinuity of Φ, we can obtain $Sz = Bz$ and $Tz = Az$. Hence we have

$$Az = Bz = Sz = Tz. \quad (6)$$

From (3) and (4), we have also

$$d(Ax_{2n}, Bz) \leq \Phi(\max\{d(Sx_{2n}, Tz), d(Sx_{2n}, Ax_{2n}), d(Tz, Bz), \frac{1}{2}[d(Sx_{2n}, Bz) + d(Tz, Ax_{2n})]\}).$$

This implies that, if $Bz \neq z$, then

$$d(z, Bz) \leq \Phi(d(z, Bz)) < d(z, Bz),$$

which is a contradiction. Therefore, we have $z = Az = Bz = Sz = Tz$. The uniquencesss of the fixed point z is obvious from (2). This completes the proof.

From Theorem (2.3), we have the following:

Theorem 2.4. Let (X, d) be a complete metric space and A, B be mappings from X into itself satisfying the following condition

$$d(Ax, By) \leq \Phi(\max\{d(x, y), d(x, Ax), d(y, By), \frac{1}{2}[d(x, By) + d(y, Ax)]\}). \quad (7)$$

for all x, y in X, where $\Phi(t)$ is the same as in Theorem (2.3) then A and B have a unique common fixed point in X.

Proof. Define a sequence $\{x_n\}$ in X by

$$x_{2n-1} = Ax_{2n-2} \text{ and } x_{2n} = Bx_{2n-2} \quad (8)$$

for n = 1, 2, 3, …. Then it is easy to show that $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, letting $x_n \to z \in X$ as $n \to \infty$, we know that $\{x_{2n-1}\}$ and $\{x_{2n}\}$ converge to z, too. By (7) and (8), we have

$$d(Az, x_{2n}) \leq d(Az, Bx_{2n-2}) \leq \Phi(\max\{d(z, x_{2n-2}), d(z, Az), d(x_{2n-2}, x_{2n}), \frac{1}{2}[d(z, x_{2n}) + d(x_{2n-2}, Az)]\}).$$

By the upper semicontinuity of Φ (t), if Az $\neq$ z, then we have

$$d(Az, z) \leq \Phi (d(z, Az)) < d(z, Az),$$

which is contradiction and so z = Az. Similarly, we have z = Bz. This completes the proof.

The following result is an immediate consequence of Theorem 2.3.

Theorem 2.5. Let (X, d) be a complete metric space and S, T and A_n be mappings from X into itself, n = 1, 2,…. Suppose further that S and T are continuous and, for every $n \in N$, the pairs $\{A_{2n-1}, S\}$ and $\{A_{2n}, T\}$ are compatible of type (R), $A_{2n-1}(X) \subset T(X)$ and $A_{2n}(X) \subset S(X)$ and, for any $n \in N$, the set of positive integers, the following condition is satisfied:

$$d(A_n x, A_{n+1} y) \leq \Phi (\max \{d(Sx, Ty), d(Sx, A_n x), d(Ty, A_{n+1} y), \frac{1}{2}[d(Sx, A_{n+1} y) + d(Ty, A_n x)]\}). \qquad (9)$$

for all x, y $\in$ X, where Φ (t) is the same as in Theorem 2.3. Then S, T and $\{A_n\}$, $n \in N$, have a unique common fixed point in X.

Open Problem

It remains open to check the result using Biased mappings.

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