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Common Fixed Point Theorems for Hybrid Pairs of Occasionally Weakly Semi Compatible Maps

Yogesh Kumar Vijaywar[1], N.P.S. Bawa[2] and Praveen Kumar Shrivastava[3]

[1]Department of Mathematics, Govt.J. S.T.P.G. College Balaghat M.P.India
e-mail:yogesh_vijaywar@y ahoo.com

[2]Department of Mathematics, Govt..P.G.science College rewa M.P.India
e-mail:pspraveenshrivastava@yahoo.co.in

[3]Department of Mathematics Govt.J. S.T.P.G. College Balaghat M.P.India
e-mail:promptbawa@yahoo.com

Abstract

We obtain common fixed point theorems for occasionally weakly semi compatible maps of hybrid pair which is more general and minimal condition for existence of common fixed point theorems. We improve the results of jungck et al. [6], Bouhadjera et al. [3] and other results on occasionally weakly compatible maps of hybrid pairs.

Keywords: weakly commuting maps; common fixed point theorem; metric space, symmetric space.

1 Introduction

Let (X, d) denotes a metric space and CB(X) the family of all nonempty closed and bounded subsets of X. Let H be the Hausdorff metric on CB(X) induced by the metric d; i.e.,

$$H(A, B) = \max\{\sup_{x\in A} d(x, B), \sup_{y\in B} d(A, y)\}$$

for A,B in CB(X),where

$$d(x, A) = \inf \{d(x, y): y \in A\}$$

Let f, g be two self-maps of a metric space (X, d). In his paper [11], Sessa defind f and g to be weakly commuting if for all $x \in X$

$$d(fgx, gfx) \leq d(gx, fx),$$

It can be seen that two commuting maps ($f\,g\,x = g\,f\,x$, $\forall\ x \in X$) are weakly commuting, but the converse is false in general (see [11]).

Afterwards, Jungck [5] in 1986 extended the concepts of commutativity and weak commutativity by giving the notion of compatibility. Maps f and g above are compatible if

$$\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$$

whenever $\{ x_n \}$ is a sequence in X such that $\lim_{n\to\infty} f\,x_n = \lim_{n\to\infty} g\,x_n = t$ for some $t \in X$.

Obviously, weakly commuting maps are compatible, but the converse is not true in general (see [5]).

Further, Kaneko and Sessa [8] in 1989 extended the concept of compatibility for single valued maps to the setting of single and multi-valued maps as follows: $f : X \to X$ and $F : X \to CB(X)$ are said to be compatible if $fFx \in CB(X)$ for all $x \in X$ and

$$\lim_{n\to\infty} H(Ffx_n, fFx_n) = 0,$$

whenever $\{x_n\}$ is a sequence in X such that $Fx_n \to A \in CB(X)$ and $fx_n \to t \in A$.

Definition 1.1. Let (X,d) denotes a metric space and $f : X \to X$ and $F : X \to CB(X)$.Then a point $x \in X$ is called a coincidence point of f and F if $fx \in Fx$. We shall call $w = fx \in Fx$ a point of coincidence of f and F.

In 1998 Jungck and Rhoades[7], weakened the notion of compatibility for single and multi-valued maps by giving the concept of weak compatibility. They define maps f and F above to be weakly compatible if they commute at their coincidence points; i.e., if $fFx = Ffx$ whenever $fx \in Fx$.

In 2000 Shrivastava et al.[12] gave another generalization of compatibility for single and multi-valued maps by giving the concept of compatibility of type (N).They define the pair (f,F) is a compatible pair of type(N) iff $f(x) \in Fx$ implies $ff(x) \in Ff(x)$.

In 2007 Abbas and Rhoades [1] generalized the concept of weak compatibility in the setting of single and multi-valued maps by introducing the notion of occasionally weak compatibility (owc). Maps f and F are said to be owc if and only if there exists some point x in X such that fx □ Fx and fFx $\subseteq$ Ffx.

Our theorems are proved in symmetric spaces which are more general than metric spaces. For our main results we need the following definitions:

Definition1.2 . Let X be a non-empty set. A symmetric on X is a mapping $r : X \times X \to [0,\infty]$ such that $r(x,y) = 0$ iff $x = y$ & $r(x,y) = r(y,x)$ $\forall x,y \in X$. A set X together with a symmetric r is called a symmetric space .

Let B(X) be the class of all non-empty bounded subsets of X,denote $\delta(A,B) = \sup \{ r(a,b): a\in A, b\in B \}$ and $D(A,B) = \inf \{r(a,b): a\in A, b\in B \}$.

Example1.3.Let X = {0,1} and d be an usual metric.Define f0 = 1, f1 = 0 and Fx = {0,1}. Then (f, F) is owc hybrid pair.It is clear that f and F have no common fixed point.

Remark 1.4. We observed that all results in[1] are not valid in view of Example 1.3 However they are valid if Hausdroff metric H is replaced by δ.

Definition 1.5. Let (X,d) denotes a symmetric space and $f : X \to X$ and $F : X \to CB(X)$.Then a point $x\in X$ is called a common fixed point of f & F if $x = fx\in Fx$.

In 2007 Rao et al.[10] generalized the concept of occasionally weakly compatibility (owc) in the setting of single and multi-valued maps by introducing the notion of occasionally weakly semi-compatibility (owsc). Let (X,d) denotes a symmetric space and $f : X \to X$ and $F : X \to CB(X)$.The hybrid pair (f,F) is said to be occasionally weakly semi-compatibile (owsc) iff there exits some point $x\in X$ such that $f x\in Fx$ and $ffx\in Ffx$.

2 Main results

Concept of owsc maps is more general form weak -compatibility and occasionally weak- compatibility.

Lemma 2.1. Concept of wc maps and owc maps for hybrid pair implise owsc maps but not conversely.

Proof. Obviously since every pair of wc maps is pair of owc maps but not conversely. Let (f ,F) is occasionally Weakly compatibile pair. Pair (f,F) is owc iff there exists some point x in X such that $fx \in Fx$ and $fFx \subseteq Ffx$.

$$f f x \in f F x \qquad (\text{ since } fx \in Fx)$$
$$ffx \in Ffx \qquad (\text{ since } fFx \subseteq Ffx)$$

Thus $fx \in Fx$ and $ffx \in Ffx$
thus (f,F) is owsc.

Similarly if (f,F) is weak compatible pair. Then $fFx = Ffx$ whenever $fx \in Fx$ So we get $ffx \in Ffx$ (since $fx \in Fx$)
thus (f,F) is owsc.

But this example shows converse is not true.

Example 2.2. Let $X = [0,1]$ and d be an usual metric.

Define $fx = 1-x$ and $FX = [0, ½]$ Then $f(½) = ½ \in F(½)$
But $fF(½) = [½,1] \not\subset Ff(½) = [0, ½]$
And $ff(½) = ½ \in Ff(½)$. Thus pair (f,F) is neither wc nor owc but owsc.

Now we extend lemma 1 of Jungck et al. [6] for hybrid pair.

Lemma 2.3. Let (X,d) denotes a metric space and $f : X \to X$ and $F : X \to CB(X)$. Let (f,F) be owsc pair . If f and F have a unique point of coincidence $w = fx \in Fx$ then w is unique common fixed point of f and F.

Proof. Since (f, F) is of owsc pair. So there exists some point $x \in X$ such that $w = fx \in Fx$ and $ffx \in Ffx$.
Since $ffx \in Ffx$ which says that ffx is a point of coincidence of f & F.
Since the point of coincidence $w = fx$ of f & F is unique by hypothesis ,

$$w = fx = ffx \in Ffx$$

thus $w = fx$ is a common fixed point of f & F. Moreover, if z is any other common fixed point of f & F then
$w = z = fz \in Fz$ by the uniqueness of the point of coincidence.

Remark 2.4. In view of lemma 2.1 and example 2.2 all theorems and corollaries of Bouhadjera et al.[3] and Bouhadjera et al.[4] can generalized taking owsc in the place of o w c maps.

Remark 2.5. In view of lemma 2.1 and example 2.2 all theorems and corollaries of M.abbas et al.[1] can generalized taking owsc in the place of owc maps. Also due to remark 1.4 of Rao et al [10] hausdorff metric H is replaced by δ.
We introduce Multi-valued version of Theorem 1 of G. Jungck [6] .

Theorem 2.6. Let X be a set with a symmetric r . Suppose that f, g, are selfmaps of X and $S,T:X \to B(X)$ and the pairs (f, S) and (g, T) are owsc. If

$$\delta(Sx , Ty) < M(x , y) \qquad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

for all $x , y \in X$ for which $fx \neq gy$.
$M(x , y) = \max \{ r(fx , gy) ,D(Sx , fx) ,D(Ty,gy) ,D(Sx , gy) , D(Ty , fx)\}$.

Then there is a unique point $w \in X$ such that $w = fw \in Sw$ and a unique point $z \in X$, such that $z = gz \in Tz$. Moreover $z = w$, so that there is a unique common fixed point of f, g, S and T.

Proof. Since the pairs (f , S) and (g , T) are each owsc. So there exist points x, $y \in X$ such that $fx \in Fx$ and $gy \in Ty$.
We claim that fx = gy For other wise by (1). we get
$r(fx, gy) \leq \delta(Sx,Ty) < M(x, y)$ and
$M(x, y) = \max\{ r(fx, gy, D(Sx, fx), D(Ty, gy), D(Sx, gy), D(Ty, fx) \}$
$= \max\{r(fx, gy), r(fx, fx), r(gy, gy), r(fx, gy), r(gy, fx)\}$
$= r(fx, gy)$
Thus $r(fx, gy) < M(x, y) = r(fx, gy)$
which is a contradiction . Therefore fx = gy

i.e. $fx \in Sx$, $fx = gy$, $gy \in Ty$.

Moreover ,if there is another point z such that $fz \in Sz$ then using (1) it follows that $fz \in Sz$, $gy \in Ty$, $fz = gy$ or $fx = fz$ and $w = fx \in Sx$ is unique point of coincidence of f and S by Lemma (2) w is only common fixed point of f and S.i.e.$w = fw \in Sw$. by symmetry there is unique point $z \in X$ such that $z = gz \in Tz$.
Suppose that $w \neq z$ using (1) we get
$r(w, z) = r(fw, gz) < M(x, y) = r(w, z)$
which is a contradiction . Therefore w = z and w is common fixed point by lemma (2) it is clear that w is unique .

Corollary 2.7. Let X be a set with symmetric r. suppose that $f, g : X \to X$ and $S, T : X \to B(X)$ and pairs (f ,S) and (g ,T) are owsc maps. If

$$\delta(Sx, Ty) \leq hm(x, y) \quad \text{for all } x, y \in X \qquad \ldots\ldots\ldots\ldots\ldots.. (2)$$

Where
$m(x,y) = \max\{r(fx, gy), D(Sx, fx), D(Ty, gy), [D(Sx, gy) + D(Ty, fx)] / 2\}$
And $0 \leq h < 1$. Then there is a unique point $w \in X$ such that $w = fw \in Sw$ and a unique point $z \in X$, such that $z = gz \in Tz$. Moreover $z = w$, so that there is a unique common fixed point of f, g, S and T.
then f, g, S, T have unique common fixed point .

Proof. Since (2) is special case of (1) the result follows immediately from Theorem 2.6.

Remark 2.8. Similarly we can generalized all theorems and corollaries of M. Abbas et al. [2] by taking owsc maps in the place of owc maps.

3. Open Problem

Which is more general,'hybrid pair of owsc maps' or 'hybrid pair of compatible type (N) maps' or both independent and How?

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