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New Families of Odd Graceful Graphs

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Abstract

In this work some new odd graceful graphs are investigated. We prove that the graph obtained by fusing all the n vertices of C_n of even order with the apex vertices of n copies of $K_{1,m}$ admits odd graceful labeling. In addition to this we show that the shadow graphs of path P_n and star $K_{1,n}$ are odd graceful graphs.

Keywords: *Graceful graph,Graceful labeling,Odd graceful graph, Odd graceful labeling,Shadow graph.*

1 Introduction

We begin with simple,finite,connected and undirected graph $G = (V(G), E(G))$ with p vertices and q edges. For standard terminology and notations we follow Harary [4]. We will provide brief summary of definitions and other information which are prerequisites for the present investigations.

Definition 1.1 *If the vertices are assigned values subject to certain conditions then it is known as graph labeling.*

For detailed survey on graph labeling we refer to Gallian [2]

Definition 1.2 *A function f is called graceful labeling of graph G if $f : V \rightarrow \{0, 1, \ldots, q\}$ is injective and the induced function $f^* : E \rightarrow \{1, 2, \ldots, q\}$ defined as $f^*(e = uv) = |f(u) - f(v)|$ is bijective.A graph which admits graceful labeling is called a graceful graph.*

The study of graceful graphs and graceful labeling methods was introduced by Rosa[6]. While the graceful labeling of graphs was perceived to be primarily theoretical subject in the field of graph theory and discrete mathematics. But it is now accepted that gracefully labeled graphs often serve as models in a wide range of applications. Such applications including coding theory and communication net work addressing. The brief survey on applications of gracefully labeled graphs is reported in Bloom and Golomb [1]. The famous graceful tree conjecture and many illustrious works brought a tide of labeling scheme having graceful theme. The present work is targeted to discuss one such labeling known as odd graceful labeling which is defined as follows.

Definition 1.3 *A graph G=(V(G),E(G)) with p vertices and q edges is said to admit an odd graceful labeling if* $f : V(G) \rightarrow \{0, 1, 2, \ldots, 2q-1\}$ *injective and the induced function* $f^* : E(G) \rightarrow \{1, 3, 5, \ldots, 2q-1\}$ *defined as* $f^*(e = uv) = |f(u) - f(v)|$ *is bijective. The graph which admits odd graceful labeling is called an odd graceful graph.*

The above concept was introduced by Gnanajothi [3] and in the same paper she investigated several results on this newly defined concept.
Gracefulness and odd gracefulness are two entirely different concepts. A graph may posses one or both of these or neither. Kathiresan [5] has discussed odd gracefulness of ladders and the graphs obtained from them by subdividing each step exactly once. Sekar [7] has proved that the splitting graph of path P_n and the splitting graph of even cycle C_n are odd graceful graphs. In the present work we investigate odd graceful labeling for some cycle related and star related graphs generated by various graph operations.

2 Main Results

Theorem 2.1 *The graph obtained by fusing all the n vertices of cycle* C_n *of even order with the apex vertices of n copies of* $K_{1,m}$ *admits odd graceful labeling.*

Proof: Let C_n be a cycle of even order with v_1,v_2,. . . , v_n be its vertices and G be the graph obtained by fusing all the n vertices v_i of C_n with the apex vertices of star $K_{1,m}$. Denote the pendant vertices of $K_{1,m}$ by v_{ij} where $1 \leq i \leq n$ and $1 \leq j \leq m$. Then G is a graph with $|V(G)| = n + nm$ and $|E(G)| = n + nm$. To define $f : V(G) \rightarrow \{0, 1, 2, \ldots, 2q-1\}$ we consider following two cases.

Case 1: $n \equiv 0(mod 4)$.

For $1 \leq i \leq \frac{n}{2}$

$f(v_i) = (m+1)(2n-i)+1$; i is even
$\quad = (m+1)(i-1)$; i is odd

For $\frac{n}{2}+1 \leq i \leq n$
$f(v_i) = (m+1)(2n-i)+1$; i is even
$\quad = (m+1)(i-3)+2(m+2)$; i is odd

For $1 \leq i \leq \frac{n}{2}$; $1 \leq j \leq m$
$f(v_{ij}) = (m+1)(2n-i+1)-2j+1$; if i is odd
$\quad = (m+1)(i-2)+2j$; if i is even

For $\frac{n}{2}+1 \leq i \leq n$; $1 \leq j \leq m$
$f(v_{ij}) = (m+1)(2n-i+1)-2j+1$; if i is odd
$\quad = (m+1)(i-4)+2(m+j+2)$; if i is even

Case 2: $n \equiv 2(mod 4)$.

For $1 \leq i \leq \frac{n}{2}+1$
$f(v_i) = (m+1)(2n-i)+1$; i is even
$\quad = (m+1)(i-1)$; i is odd

For $\frac{n}{2}+2 \leq i \leq n-1$
$f(v_i) = (m+1)(2n-i)+1$; i is even
$\quad = (m+1)(i-3)+2(m+2)$; i is odd

$f(v_n) = (m+1)(2n-i)-1$

For $1 \leq i \leq \frac{n}{2}$; $1 \leq j \leq m$
$f(v_{ij}) = (m+1)(2n-i+1)-2j+1$; if i is odd
$\quad = (m+1)(i-2)+2j$; if i is even

For $\frac{n}{2}+1 \leq i \leq n-1$; $1 \leq j \leq m$
$f(v_{ij}) = (m+1)(2n-i+1)-2j+1$; if i is odd
$\quad = (m+1)(i-2)+2(j+1)$; if i is even

$f(v_{nj}) = (m+1)(n-2)+2(j+1)$; for $1 \leq j \leq m-1$
$\quad = (m+1)(n-2)+2(2j+1)$; for $j = m$

The above defined function f exhausts all the possibilities and the graph under consideration is an odd graceful graph.

Illustration 2.2 The following Figure 1 shows the labeling pattern of the graph obtained by fusing each vertex of C_8 with the apex vertices of eight copies of star $K_{1,3}$.

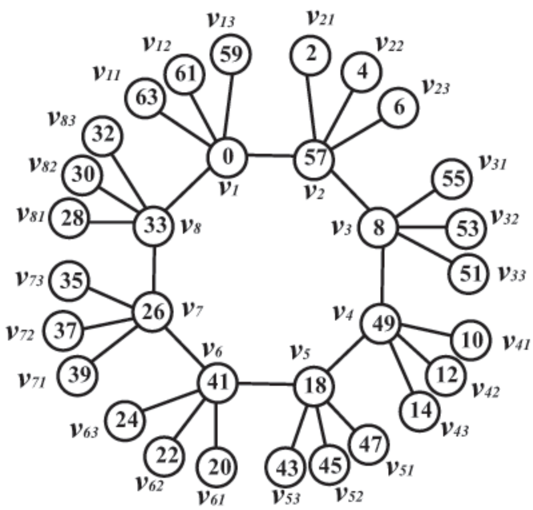

Fig 1

Definition 2.3 *Shadow graph* $D_2(G)$ of a connected graph G is obtained by taking two copies of G say G' and G'', join each vertex u' in G' to the neighbors of the corresponding vertex u'' in G''.

Theorem 2.4 *The graph* $D_2(P_n)$ *is an odd graceful graph.*

Proof: Let G be $D_2(P_n)$ then $|V(G)| = 2n$ and $|E(G)| = 4(n-1)$ and let v_1,v_2,..., v_n be the vertices of first copy of path P_n and v'_1,v'_2,..., v'_n be the vertices of the second copy of path P_n. Define $f\colon V(G) \rightarrow \{0, 1, 2, \ldots, 2q-1\}$ as follows.

$f(v_i) = 4(i-1)$; i is odd
$\quad = 4(2n-i) - 1$; i is even
$f(v'_i) = 4(i-1) + 2$; i is odd
$\quad = 4(2n-i) - 5$; i is even

The above defined function f provides graceful labeling for $D_2(P_n)$.

Illustration 2.5: The odd graceful labeling of $D_2(P_6)$ is given in Figure 2.

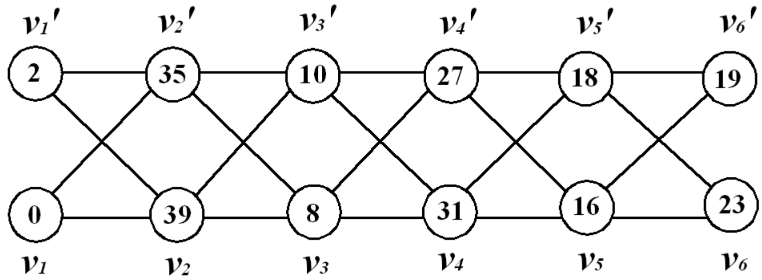

Fig 2

Theorem 2.6 *The graph $D_2(K_{1,n})$ is an odd graceful graph.*

Proof: Let G be $D_2(K_{1,n})$ and v,v_1,v_2,..., v_n be the vertices of first copy of star $K_{1,n}$ and v',v'_1,v'_2,..., v'_n be the vertices of the second copy of star $K_{1,n}$. Define $f\colon V(G) \to \{0, 1, 2, \ldots, 2q-1\}$ as follows.

$f(v) = 0$
$f(v_i) = 8n - 4i + 3$; for$1 \leq i \leq n$
$f(v') = 2$
$f(v'_i) = 4i - 1$; for$1 \leq i \leq n$

In view of above defined labeling pattern G admits odd graceful labeling.

Illustration 2.7: Consider the shadow graph of $K_{1,4}$. The labeling pattern is as shown in Figure 3.

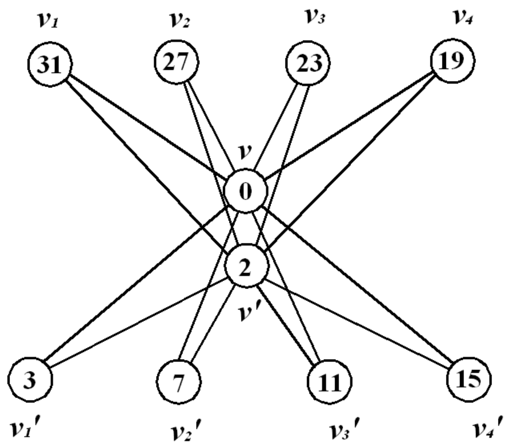

Fig 3

3 Conclusion

As every graph does not admit odd graceful labeling it is very interesting to investigate graphs or graph families which admit odd graceful labeling. In the present work we investigate three new families of odd graceful graphs.

4 Open Problems

- Investigate graphs which admit odd graceful labeling.
- Investigate similar results for other graph families and in the context of various graph labeling problems.
- Investigate characterizations for odd gracefulness.

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