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# The Line $n$-Sigraph of a Symmetric $n$-Sigraph-III

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**Abstract**

***An n-tuple** $(a_1, a_2, ..., a_n)$ **is symmetric, if** $a_k = a_{n-k+1}, 1 \leq k \leq n$**. Let** $H_n = \{(a_1, a_2, ..., a_n) : a_k \in \{+, -\}, a_k = a_{n-k+1}, 1 \leq k \leq n\}$ **be the set of all symmetric** $n$**-tuples. A symmetric n-sigraph (symmetric n-marked graph) is an ordered pair** $S_n = (G, \sigma)$ **(**$S_n = (G, \mu)$**), where** $G = (V, E)$ **is a graph called the underlying graph of** $S_n$ **and** $\sigma : E \to H_n$ **(**$\mu : V \to H_n$**) is a function. The path graph** $P_k(G)$ **of a graph** $G$ **is obtained by representing the paths** $P_k$ **in** $G$ **by vertices whenever the corresponding paths** $P_k$ **in** $G$ **from a path** $P_{k+1}$ **or a cycle** $C_k$**. In this paper, we introduce a natural extension of the notion of path graphs to the realm of symmetric** $n$**-sigraphs. It is shown that for any symmetric** $n$**-sigraph** $S_n$**,** $P_k(S_n)$ **is** $i$**-balanced. The notion** $P_k(S_n)$ **which generalizes the notion of line symmetric** $n$**-sigraph** $L(S_n)$ **introduced by E. Sampathkumar et al. (2010). Further, in this paper we discuss the structural characterization of path symmetric** $n$**-sigraphs. Also, we characterize symmetric** $n$**-sigraphs which are switching equivalent to their path symmetric** $n$**-sigraphs** $P_3(S_n)$ **(**$P_4(S_n)$**).***

**Keywords:** *Symmetric n-sigraphs, Symmetric n-marked graphs, Balance, Switching, Path symmetric n-sigraphs, Line symmetric n-sigraphs, Complementation.*

# 1 Introduction

For standard terminology and notion in graph theory we refer the reader to Harary [2]; the non-standard will be given in this paper as and when required. We treat only finite simple graphs without self loops and isolates.

Let $n \geq 1$ be an integer. An *n-tuple* $(a_1, a_2, ..., a_n)$ is *symmetric*, if $a_k = a_{n-k+1}, 1 \leq k \leq n$. Let $H_n = \{(a_1, a_2, ..., a_n) : a_k \in \{+, -\}, a_k = a_{n-k+1}, 1 \leq k \leq n\}$ be the set of all symmetric $n$-tuples. Note that $H_n$ is a group under coordinate wise multiplication, and the order of $H_n$ is $2^m$, where $m = \lceil \frac{n}{2} \rceil$.

A *symmetric n-sigraph (symmetric n-marked graph)* is an ordered pair $S_n = (G, \sigma)$ ($S_n = (G, \mu)$), where $G = (V, E)$ is a graph called the *underlying graph* of $S_n$ and $\sigma : E \to H_n$ ($\mu : V \to H_n$) is a function.

In this paper by an *n-tuple/n-sigraph/n-marked graph* we always mean a symmetric $n$-tuple/symmetric $n$-sigraph/symmetric $n$-marked graph.

An $n$-tuple $(a_1, a_2, ..., a_n)$ is the *identity n-tuple*, if $a_k = +$, for $1 \leq k \leq n$, otherwise it is a *non-identity n-tuple*. In an $n$-sigraph $S_n = (G, \sigma)$ an edge labelled with the identity $n$-tuple is called an *identity edge*, otherwise it is a *non-identity edge*.

Further, in an $n$-sigraph $S_n = (G, \sigma)$, for any $A \subseteq E(G)$ the *n-tuple* $\sigma(A)$ is the product of the $n$-tuples on the edges of $A$.

In [9], the authors defined two notions of balance in $n$-sigraph $S_n = (G, \sigma)$ as follows (See also R. Rangarajan and P. Siva Kota Reddy [6]):

**Definition.** Let $S_n = (G, \sigma)$ be an $n$-sigraph. Then,
(i) $S_n$ is *identity balanced* (or *i-balanced*), if product of $n$-tuples on each cycle of $S_n$ is the identity $n$-tuple, and
(ii) $S_n$ is *balanced*, if every cycle in $S_n$ contains an even number of non-identity edges.
**Note:** An $i$-balanced $n$-sigraph need not be balanced and conversely.
The following characterization of $i$-balanced $n$-sigraphs is obtained in [9].

**Proposition 1.1** (**E. Sampathkumar et al. [9]**) *An n-sigraph $S_n = (G, \sigma)$ is i-balanced if, and only if, it is possible to assign n-tuples to its vertices such that the n-tuple of each edge uv is equal to the product of the n-tuples of u and v.*

In [9], the authors also have defined switching and cycle isomorphism of an $n$-sigraph $S_n = (G, \sigma)$ as follows: (See also [5, 7, 8, 11])

Let $S_n = (G, \sigma)$ and $S'_n = (G', \sigma')$, be two $n$-sigraphs. Then $S_n$ and $S'_n$ are said to be *isomorphic*, if there exists an isomorphism $\phi : G \rightarrow G'$ such that if $uv$ is an edge in $S_n$ with label $(a_1, a_2, ..., a_n)$ then $\phi(u)\phi(v)$ is an edge in $S'_n$ with label $(a_1, a_2, ..., a_n)$.

Given an $n$-marking $\mu$ of an $n$-sigraph $S_n = (G, \sigma)$, *switching* $S_n$ with respect to $\mu$ is the operation of changing the $n$-tuple of every edge $uv$ of $S_n$ by $\mu(u)\sigma(uv)\mu(v)$. The $n$-sigraph obtained in this way is denoted by $\mathcal{S}_\mu(S_n)$ and is called the *$\mu$-switched $n$-sigraph* or just *switched $n$-sigraph*.

Further, an $n$-sigraph $S_n$ *switches* to $n$-sigraph $S'_n$ (or that they are *switching equivalent* to each other), written as $S_n \sim S'_n$, whenever there exists an $n$-marking of $S_n$ such that $\mathcal{S}_\mu(S_n) \cong S'_n$.

Two $n$-sigraphs $S_n = (G, \sigma)$ and $S'_n = (G', \sigma')$ are said to be *cycle isomorphic*, if there exists an isomorphism $\phi : G \rightarrow G'$ such that the $n$-tuple $\sigma(C)$ of every cycle $C$ in $S_n$ equals to the $n$-tuple $\sigma(\phi(C))$ in $S'_n$.

We make use of the following known result (see [9]).

**Proposition 1.2** (**E. Sampathkumar et al.** [**9**]) *Given a graph $G$, any two $n$-sigraphs with $G$ as underlying graph are switching equivalent if, and only if, they are cycle isomorphic.*

## 2 Path $n$-Sigraphs

Broersma and Hoede [1] generalized the concept of line graphs to that of path graphs. Let $P_k$ and $C_k$ denote a path and a cycle with $k$ vertices, respectively. Denote $\Pi_k(G)$ the set of all paths of $G$ on $k$ vertices ($k \geq 1$). The *path graph* $P_k(G)$ of a graph $G$ has vertex set $\Pi_k(G)$ and edges joining pairs of vertices that represent two paths $P_k$, the union of which forms either a path $P_{k+1}$ or a cycle $C_k$ in $G$. A graph is called a *$P_k$-graph*, if it is isomorphic to $P_k(H)$ for some graph $H$. If $k = 2$, then the $P_2$-graph is exactly the line graph. The way of describing a line graph stresses the adjacency concept, whereas the way of describing a path graph stresses concept of the path generation by consecutive paths.

For $P_3$-graphs, Broersma and Hoede [1] gave a solution to the characterization problem, which contained flaw. Later, Li and Lin [3] presented corrected form of the characterization of $P_3$-graphs. For $k \geq 4$, the problems becomes more difficult. Although the determination and characterization problems for $P_k$-graphs for $k \geq 4$ have not been completely solved.

We extend the notion of $P_k(G)$ to the realm of $n$-sigraphs. In an $n$-sigraph $S_n = (G, \sigma)$, for any $A \subseteq E(G)$ the $n$-tuple $\sigma(A)$ is the product of the $n$-tuples on the edges of $A$. The *path n-sigraph* $P_k(S_n) = (P_k(G), \sigma')$ of an $n$-sigraph $S_n = (G, \sigma)$ is an $n$-sigraph whose underlying graph is $P_k(G)$ called *path graph* and the $n$-tuple of any edge $e = P_k P'_k$ in $P_k(S_n)$ is $\sigma'(P_k P'_k) = \sigma(P_k)\sigma(P'_k)$. Further, an $n$-sigraph $S_n = (G, \sigma)$ is called *path n-sigraph*, if $S_n \cong P_k(S'_n)$, for some $n$-sigraph $S'_n$. We now gives a straightforward, yet interesting, property of path $n$-sigraphs.

**Proposition 2.1** *For any n-sigraph $S_n = (G, \sigma)$, its path n-sigraph $P_k(S_n)$ is i-balanced.*

**Proof.** Since the $n$-tuple of any edge $\sigma'(e = P_k P'_k)$ in $P_k(S_n)$ is $\sigma(P_k)\sigma(P'_k)$, where $\sigma$ is the $n$-marking of $P_k(S_n)$, by Proposition 1.1, $P_k(S_n)$ is $i$-balanced.
**Remark:** For any two $n$-sigraphs $S_n$ and $S'_n$ with same underlying graph, their path $n$-sigraphs are switching equivalent.

E. Sampathkumar et al. [10] introduced the notion of *line n-sigraph* $L(S_n)$ of a given $n$-sigraph $S_n$ as follows: The *line n-sigraph* of an $n$-sigraph $S_n = (G, \sigma)$ is an $n$-sigraph $L(S_n) = (L(G), \sigma')$, where for any edge $ee'$ in $L(S_n)$, $\sigma'(ee') = \sigma(e)\sigma(e')$. Hence, we shall call a given $n$-sigraph $S_n$ a *line n-sigraph* if it is isomorphic to the line $n$-sigraph $L(S'_n)$ of some $n$-sigraph $S'_n$. By the definition of path $n$-sigraphs, we observe that $P_2(S_n) = L(S_n)$.

**Corollary 2.2** (**E. Sampathkumar et al. [10]**) *For any n-sigraph $S_n = (G, \sigma)$, its $P_2(S_n)$ (=$L(S_n)$) is i-balanced.*

In [10], the authors obtain structural characterization of line $n$-sigraphs as follows:

**Proposition 2.3** (**E. Sampathkumar et al. [10]**)
*An n-sigraph $S_n = (G, \sigma)$ is a line n-sigraph (or $P_2$-n-sigraph) if, and only if, $S_n$ is i-balanced and G is a line graph (or $P_2$-graph).*

**Proof.** Suppose that $S_n$ is $i$-balanced and $G$ is a line graph. Then there exists a graph $H$ such that $L(H) \cong G$. Since $S_n$ is $i$-balanced, by Proposition 1.1, there exists an $n$-marking $\mu$ of $G$ such that each edge $uv$ in $S_n$ satisfies $\sigma(uv) = \mu(u)\mu(v)$. Now consider the $n$-sigraph $S'_n = (H, \sigma')$, where for any edge $e$ in $H$, $\sigma'(e)$ is the $n$-marking of the corresponding vertex in $G$. Then clearly, $L(S'_n) \cong S_n$. Hence $S_n$ is a line $n$-sigraph.

Conversely, suppose that $S_n = (G, \sigma)$ is a line $n$-sigraph. Then there exists an $n$-sigraph $S'_n = (H, \sigma')$ such that $L(S'_n) \cong S_n$. Hence $G$ is the line graph of $H$ and by Corollary 2.2, $S_n$ is $i$-balanced.

We now characterize those $n$-sigraphs that are switching equivalent to their $P_3$ ($P_4$)-$n$-sigraphs. In the case of graphs the following results is due to Broersma and Hoede [1] and Li and Zhao [4] respectively.

**Proposition 2.4** (**Broersma and Hoede [1]** )
*A connected graph $G$ is isomorphic to its path graph $P_3(G)$ if, and only if, $G$ is a cycle.*

**Proposition 2.5** (**Li and Zhao [4]** )
*A connected graph $G$ is isomorphic to its path graph $P_4(G)$ if, and only if, $G$ is a cycle of length at least 4.*

**Proposition 2.6** *For any connected $n$-sigraph $S_n = (G, \sigma)$ satisfies*

*(i) $S_n \sim P_3(S_n)$ if, and only if, $S_n$ is an $i$-balanced $n$-sigraph on a cycle.*
*(ii) $S_n \sim P_4(S_n)$ if, and only if, $S_n$ is an $i$-balanced $n$-sigraph on a cycle of length at least 4.*

**Proof.** (i) Suppose that $S_n \sim P_3(S_n)$. This implies, $G \cong P_3(G)$ and hence by Proposition 2.4, we see that the graph $G$ must be a cycle. Now, if $S_n$ is any $n$-sigraph on a cycle, Proposition 2.1 implies that $P_3(S_n)$ is $i$-balanced and hence if $S_n$ is $i$-unbalanced, $P_3(S_n)$ being $i$-balanced cannot be switching equivalent to $S_n$ in accordance with Proposition 1.2. Therefore, $S_n$ must be $i$-balanced.

Conversely, suppose that $S_n$ is an $i$-balanced $n$-sigraph on a cycle. Then, since $P_3(S_n)$ is $i$-balanced as per Proposition 2.1 and since $P_3(G) \cong G$, the result follows from Proposition 1.2.
Similarly, we can prove (ii) using Proposition 2.5.

# 3 Complementation

In this section, we investigate the notion of complementation of a graph whose edges have signs (a *sigraph*) in the more general context of graphs with multiple signs on their edges. For any $t \in H_n$, the *t-complement* of $a = (a_1, a_2, ..., a_n)$ is: $a^t = at$. For any $T \subseteq H_n$, and $t \in H_n$, the *t-complement* of $T$ is $T^t = \{a^t : a \in T\}$.

For any $t \in H_n$, the *t-complement* of an $n$-sigraph $S_n = (G, \sigma)$, written $(S_n)^t$, is the same graph but with each edge label $a = (a_1, a_2, ..., a_n)$ replaced by $a^t$.

For each $t \in H_n$, an $n$-sigraph $S_n = (G, \sigma)$ is *t-self complementary*, if $S_n \cong S_n^t$.

**Proposition 3.1** (**P. Siva Kota Reddy et al. [12]**) *For all $t \in H_n$, an $n$-sigraph $S_n = (G, \sigma)$ is t-self complementary if, and only if, $S_n^a$ is t-self complementary, for any $a \in H_n$.*

For an $n$-sigraph $S_n = (G, \sigma)$, the $P_k(S_n)$ is $i$-balanced (Proposition 2.1). We now examine, the condition under which $t$-complement of $P_k(S_n)$ (i.e., $(P_k(S))^t$) is balanced.

**Proposition 3.2** *Let $S_n = (G, \sigma)$ be an n-sigraph. If $P_k(G)$ is bipartite then $(P_k(S))^t$ is i-balanced.*

**Proof.** Since, by Proposition 2.1, $P_k(S_n)$ is $i$-balanced, for each $k$, $1 \leq k \leq n$, the number of $n$-tuples on any cycle $C$ in $P_k(S_n)$ whose $k^{th}$ co-ordinate are $-$ is even. Also, since $P_k(G)$ is bipartite, all cycles have even length; thus, for each $k$, $1 \leq k \leq n$, the number of $n$-tuples on any cycle $C$ in $P_k(S_n)$ whose $k^{th}$ co-ordinate are $+$ is also even. This implies that the same thing is true in any $t$-complement, where for any $t \in H_n$. Hence $(P_k(S_n))^t$ is $i$-balanced.

Proposition 9 provides easy solutions to three other $n$-sigraph switching equivalence relations, which are given in the following results.

**Corollary 3.3** *For any n-sigraph $S_n = (G, \sigma)$,*

*(i) $(S_n)^t \sim P_3(S_n)$ if, and only if, $S_n$ is an i-unbalanced n-sigraph on any odd cycle.*
*(ii) $(S_n)^t \sim P_4(S_n)$ if, and only if, $S_n$ is an i-unbalanced n-sigraph on any odd cycle of length at least 5.*

**Corollary 3.4** *For any n-sigraph $S_n = (G, \sigma)$ and for any integer $k \geq 1$, $P_k((S_n)^t) \sim P_k(S_n)$.*

# 4 Open Problem

We strongly believe that the Proposition 2.3 can be generalized to path $n$-sigraphs $P_k(S_n)$ for $k \geq 3$. Hence, we pose it as an open problem:

**Problem 4.1** *If $S_n = (G, \sigma)$ is an i-balanced n-sigraph and its underlying graph G is a path graph, then $S_n$ is a path n-sigraph.*

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