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Regular Quasi-Γ-absorbent in Γ-Groupoid-lattices

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Abstract

Otto. Steinfeld introduced the concept of quasi-ideal in rings and semigroups in his paper ([13], [14]). Much of steinfeld's contributions to quasi-ideals is contained in his monogragh [17]. In the Paper (with Rédei) [5] the authors generalize concepts from groups, rings, and semigroups to groupoid-lattices. In our paper [19], we have introduced the notion of Γ-absorbents in Γ-groupoid-lattices. Here in this paper we will discuss some properties of regular quasi-Γ-absorbent in Γ-groupoid-lattices.

Keywords: Γ-*groupoid-lattice,* Γ-*absorbent, Regular*-Γ-*absorbent.*

1 Introduction

The notion of regularity was introduced by J. Von Neumann is his paper [18]. Many authors, A.H. Clifford and G.B. Preston [1], J. Luh [2], L. Kovác [4] and S. Lajos ([8], [9], [10], [11], [12]), etc., have characterized many results on regular rings and semigroups by means of their left ideals, right-ideals and quasi-ideals. O. Steinfeld [17] also introduced the notion of regular elements in groupoid-lattices. The notion of partially ordered Γ-groupoid-lattice was introduced by Sen and Seth in [7]. Here in this paper we will consider only the groupoid-lattices of semigroups with 0 and associative rings, since all semigroups with 0 and associative rings satisfies the associativity and distributivity Conditions $A_1, A_2, D_v, D_v^*, A_2^+$, etc., given in [17].

A binary operation Γ on C is defined as a mapping from $C \times C \to C$ i.e., Γ assigns to each pair $(a, b) \in C \times C$ exactly one element $\Gamma(a, b) \in C$. Instead of $\Gamma(a, b)$ one mostly writes $a\Gamma b$. Let C and Γ be two non-empty sets. A mapping from $C \times \Gamma \times C \to C$ will be called a Γ-multiplication in C and is denoted by $(\cdot)_\Gamma$. The result of this Γ-multiplication for $(a, b) \in C$ and $\gamma \in \Gamma$ is denoted by $a\gamma b$. Let $C = \{x, y, z, \cdots\}$ and $\Gamma = \{\alpha, \beta, \gamma, \cdots\}$ be two non-empty sets. Then C is called a Γ-groupoid if it satisfies $x\alpha y \in C$ for all $x, y \in C$ and $\alpha, \beta \in \Gamma$. A Γ-semigroup is a Γ-groupoid such that the operation $(\cdot)_\Gamma$ is associative [8].

Example 1: Let $\mathcal{G}$ be a groupoid and Γ be any non-empty set. Let us define a mapping $\mathcal{G} \times \mathcal{G} \to \mathcal{G}$ by $x\alpha y = xy$ for all $x, y \in \mathcal{G}$ and $\alpha \in \Gamma$. Then $\mathcal{G}$ is a Γ-groupoid.

A partially ordered Γ-groupoid is a non-empty set C satisfying the following properties:

- (i) C is a groupoid w.r.t multiplication "$(\cdot)_\Gamma$";
- (ii) C is a partially ordered set w.r.t a partial ordering $'' \leq''$;
- (iii) If $a \leq b$ $(\forall a, b \in C)$, then $c\gamma a \leq c\gamma b$ and $a\gamma c \leq b\gamma c$ $\forall\ c \in C$ and $\gamma \in \Gamma$.

A Γ-groupoid-lattice is a partially ordered Γ-groupoid $\langle B, (\cdot)_\Gamma, \leq\rangle$ such that B is a complete lattice with respect to the partial ordering $\leq$ and it has the properties:

- (iv) $a\gamma a \leq a$ $\forall\ a \in B$;
- (v) $0\gamma e = e\gamma 0 = 0$.

for the greatest element e and least element 0 of B, where B denotes a Γ-groupoid-lattice.
(iii) and (v) implies,

$$0\gamma a = a\gamma 0 = 0 \qquad \forall a \in B.$$

An element b of B is called Γ-absorbent of the element a of B if

$$b \leq a \tag{1.1}$$

$$\text{and} \quad a\gamma b \leq b, \quad \forall \gamma \in \Gamma, \tag{1.2}$$

$$\text{and} \quad b\gamma a \leq b, \quad \forall \gamma \in \Gamma, \tag{1.3}$$

holds, b is a left-Γ-absorbent of a if (1.1) and (1.2) holds and right-Γ-absorbent if (1.1) and (1.3) holds.

An element k of B is called Quasi-Γ-absorbent of $a \in B$ if

$$k \leq a \quad \text{and} \quad k\gamma a \wedge a\gamma k \leq k \qquad \forall \gamma \in \Gamma \tag{1.4}$$

By a bi-Γ-absorbent of $a \in B$ we mean an element $b \in B$ such that

$$b \leq a \quad \text{and} \quad (b\gamma a)\gamma b \wedge b\gamma(a\gamma b) \leq b \qquad \forall \gamma \in \Gamma \tag{1.5}$$

An element x of Γ-groupoid-lattice B is called idempotent if the identity $x = x\gamma x$ holds for all $\gamma \in \Gamma$.

If $a_\lambda (\lambda \in \Lambda)$ are elements of the Γ-groupoid-lattices B, then from (iii):

$$b\gamma(\bigwedge_{\lambda\in\Lambda} a_\lambda) \leq \bigwedge_{\lambda\in\Lambda} b\gamma a_\lambda, (\bigwedge_{\lambda\in\Lambda} a_\lambda)\gamma b \leq \bigwedge_{\lambda\in\Lambda} a_\lambda\gamma b \qquad \forall b \in B \quad \text{and} \quad \gamma \in \Gamma \tag{1.6}$$

$$b\gamma(\bigvee_{\lambda\in\Lambda} a_\lambda) \geq \bigvee_{\lambda\in\Lambda} b\gamma a_\lambda, (\bigvee_{\lambda\in\Lambda} a_\lambda)\gamma b \geq \bigvee_{\lambda\in\Lambda} a_\lambda\gamma b \qquad \forall b \in B \quad \text{and} \quad \gamma \in \Gamma \tag{1.7}$$

Proposition 1.1(See Y. S. Singh and M.R. Khan [19]): Let $b_\lambda(\lambda \in \Lambda)$ are the Γ-absorbent (left-, right-, quasi-, bi-Γ-absorbent) of the element $a \in V$. Then the meet $\bigwedge_{\lambda\in\Lambda} b_\lambda$ is a Γ-absorbent (left-, right-, quasi-, bi-Γ-absorbent) of the element a.

Proposition 1.2(See Y. S. Singh and M.R. Khan [19]): If R and L are right- and left-Γ-absorbents of a, respectively, then $R\Gamma L \leq R \wedge L$, $R\Gamma L$ is a bi-Γ-absorbent and $R \wedge L$ is a Quasi-Γ-absorbents of $a \in B$.

2 Results

Proposition 2.1: Let us suppose that the element a of B satisfies Condition A_2 and D_v^* of [17]. If L and R are left- and right-Γ-absorbents of a, respectively, then $L \vee L\gamma a$ and $R \vee a\gamma R$ are the Γ-absorbents of a generated by L and R, for all $\gamma \in \Gamma$.

Proof: Since, $a\gamma(L \vee L\gamma a) = a\gamma L \vee a\gamma(L\gamma a) \leq L \vee L\gamma a \leq a.$

$$\text{and} \quad (L \vee L\gamma a)\gamma a = L\gamma a \vee (L\gamma a)\gamma a = L\gamma a \vee L\gamma(a\gamma a) \leq L \vee L\gamma a \leq a$$

hold, $L \vee L\gamma a$ is a Γ-absorbent of a such that $L \leq L \vee L\gamma a$. If b is the Γ-absorbent of a generated by L, then $L \leq b \leq L \vee L\gamma a$.

On the other hand, $L \leq b$ implies $L \vee L\gamma a \leq b \vee b\gamma a \leq b$.
Hence $b = L \vee L\gamma a$, i.e., $L \vee L\gamma a$ is the Γ-absorbent of a generated by L. Similarly, one can show that $R \vee a\gamma R$ is the Γ-absorbent of a generated by R.

Proposition 2.2: If the element a of B satisfies Conditions A_1 and D_v of [17], then

$$L = y \vee a\gamma y \quad \text{and} \quad R = y \vee y\gamma a \quad (\forall y \in B; \gamma \in \Gamma; y \leq a)$$

are the left and right-Γ-absorbent of a, respectively, generated by y.

Proof: Since $a\gamma(y \vee a\gamma y) = a\gamma y \vee a\gamma(a\gamma y) = a\gamma y \leq y \vee a\gamma y \leq a, \quad \forall \gamma \in \Gamma$
the element $L = y \vee a\gamma y$ is a left-Γ-absorbent of a such that $y \leq L$. If m is a left-Γ-absorbent of a generated by y, then $y \leq m \leq L$.

On the other hand, from $y \leq m$ it follows that $L = y \vee a\gamma y \leq m \vee a\gamma m \leq m$. Thus $L = y \vee a\gamma y$ is the left-Γ-absorbent of a generated by y, indeed.

Similarly, one can show that $R = y \vee y\gamma a$ is the right-Γ-absorbent of a generated by y for all $\gamma \in \Gamma$.

Theorem 2.1: Assume that Conditions A_2 and D_v^* of [17] hold for the element a of the Γ-groupoid-lattice B. The following assertions concerning the element a are equivalent:

(i) for every right-Γ-absorbent R and left-Γ-absorbent L of a,

$$R\gamma L = R \wedge L \quad \forall \gamma \in \Gamma; \tag{1.8}$$

(ii) for every right-Γ-absorbent R and left-Γ-absorbent L of a, we have

(a) $R\gamma R = R$ (b) $L\gamma L = L$ (c) $R\gamma L$ is a quasi-Γ-absorbent of a;

(iii) the quasi-Γ-absorbents of a form a regular (multiplicative) subsemigroup K of the Γ-groupoid-lattice $\langle B, (\cdot)_\Gamma \rangle$;

(iv) every quasi-absorbent k of a has the form $k = k\gamma a\gamma k \quad \forall \gamma \in \Gamma$.

Proof: $(i) \Rightarrow (ii)$ From (1.8) and proposition 1.2 imply property (c) in (iii).

Proposition 2.1, Conditions A_2 and D_v^* of [17], furthermore (1.8) imply:

$$\begin{aligned} R &= R \wedge (R \vee a\Gamma R) \\ &= R\Gamma(R \vee a\Gamma R) \\ &= R\Gamma R \vee R\Gamma(a\Gamma R) \\ &= R\Gamma R \vee (R\Gamma a)\Gamma R \\ &= R\Gamma R \end{aligned}$$

i.e., property (a) of (ii) holds. Property (b) of (ii) can be proved similarly.

$(ii) \Rightarrow (iii)$ First we show that every quasi-Γ-absorbent k of a can be written in the form

$$k = k\Gamma a \wedge a\Gamma k. \tag{1.9}$$

Because of (a) and Conditions A_2 and D_v^* of [17] we have

$$k = k \vee k\Gamma a = (k \vee k\Gamma a)^2 = k^2 \vee k^2\Gamma a \vee k\Gamma a\Gamma k \vee k\Gamma a\Gamma k\Gamma a \leq k\Gamma a.$$

Similarly, $k \leq a\Gamma k$.
Hence $k \leq k\Gamma a \wedge a\Gamma k \leq k$.
(1.9) and condition (c) imply:

$$R\Gamma L = (R\Gamma L)\Gamma a \wedge a\Gamma(R\Gamma L) \tag{1.10}$$

for every right-Γ-absorbent R and left-Γ-absorbent L of a.

Now we show that the product of the quasi-Γ-absorbents k_1 and k_2 is a quasi-Γ-absorbent of a.

Let us suppose that Condition A_2 of [17] holds for the element a of B. If k is the quasi-Γ-absorbent of a, then we have

$$a\Gamma(a\Gamma k) = (a\Gamma a)\Gamma k \leq a\Gamma k \quad \text{and} \quad (k\Gamma a)\Gamma a = k\Gamma(a\Gamma a) \leq k\Gamma a. \tag{1.11}$$

$$(k\Gamma a)\Gamma k = k\Gamma(a\Gamma k) \leq a\Gamma k \wedge k\Gamma a \leq k. \tag{1.12}$$

From (1.11) and Condition A_2 it follows that

$$a\Gamma(a\Gamma(k_1\Gamma k_2) = a\Gamma((a\Gamma k_1)\Gamma k_2$$

$$(a\Gamma(a\Gamma k_1)\Gamma k_2 \leq (a\Gamma k_1)\Gamma k_2$$
$$= a\Gamma(k_1\Gamma k_2)$$

is a left-Γ-absorbent of a.

Similarly, one can show that $(k_1\Gamma k_2)\Gamma a$ is a right-Γ-absorbent of a.
These results, property (a), (b) of (ii), (1.12) and (1.10) imply:

$$(k_1\Gamma k_2)\Gamma a \wedge a\Gamma(k_1\Gamma k_2)$$

$$= (k_1\Gamma k_2\Gamma a)(a\Gamma k_1\Gamma k_2)\Gamma a \wedge (a\Gamma k_1\Gamma k_2)\Gamma a(a\Gamma k_1\Gamma k_2)$$
$$= (k_1\Gamma k_2\Gamma a)\Gamma(a\Gamma k_1\Gamma k_2)$$
$$\leq k_1\Gamma(k_2\Gamma a\Gamma k_2) \leq k_1\Gamma k_2$$

i.e., $k_1\Gamma k_2$ is a quasi-Γ-absorbent of a, indeed.

Thus the set k of all quasi-Γ-absorbent of a is a multiplicative subsemigroup of $\langle B, (\cdot)_\Gamma\rangle$

Finally, we have to show that k is regular semigroup. Let $k \in K$, then from (1.9),(1.10),(1.11),(1.12) and condition (ii) it follows that

$$k = k\Gamma a \wedge a\Gamma k = (k\Gamma a\Gamma a\Gamma k)\Gamma a \wedge a\Gamma(k\Gamma a\Gamma a\Gamma k) = k\Gamma a\Gamma a\Gamma k = k\Gamma a\Gamma k \leq k,$$

that is,
$k = kak \in kKk.$

$(iii) - (iv)$. Let k be a quasi-Γ-absorbent of a. From (iii), there exists a quasi-Γ-absorbent x of a such that $k = k\Gamma x\Gamma k$. Hence

$$k = k\Gamma x\Gamma k \leq k\Gamma a\Gamma k \leq k\Gamma a \wedge a\Gamma k \leq k,$$

that is, $k = kak$.

$(iv) \Rightarrow (i)$. In view of proposition [1.2], the meet $R \wedge L$ of the right-Γ-absorbent R and left-Γ-absorbent L of a is a quasi-Γ-absorbent of a. Condition (iv) implies

$$R \wedge L = (R \wedge L)\Gamma a\Gamma(R \wedge L) \leq R\Gamma a\Gamma L \leq R\Gamma L \leq R \wedge L,$$

that is, $R\Gamma L = R \wedge L$.

An element a of a Γ-groupoid-lattice B is regular if a satisfies conditions A_2 and D_v^* of [17], furthermore one of the properties $(i), (ii), (iii)$ or (iv) of Theorem 2.1.

Corollary 2.1: Every quasi-Γ-absorbent k of a regular element a of B can be written in the form

$$k = R \wedge L = R\gamma L, \quad \forall \gamma \in \Gamma \tag{1.14}$$

where R and L are right- and left-Γ-absorbents of a, respectively. Furthermore, $k\gamma k = k\gamma k\gamma k \quad \forall \gamma \in \Gamma$.

Proof: Using (1.11) Conditions A_2 and D_v^* of [17], similar to proposition 2.2, one can show that $L = k \vee a\gamma k$ and $R = k \vee k\gamma a$ are the left and right-Γ-absorbents of a respectively, generated by the quas-Γ-absorbent k. Furthermore, from definition of quasi-Γ-absorbent, Conditions A_2 and D_v^* of [17] and relations (1.8), (1.11) we have

$$k \leq (k \vee k\gamma a) \wedge (k \vee a\gamma k) = R \wedge L = R\gamma L = (k \vee k\gamma a)\gamma(k \vee a\gamma k)$$

$$= (k \vee k)\gamma k \vee (k \vee k\gamma a)\gamma(a\gamma k)$$

$$= k\gamma k \vee (k\gamma a)\gamma k \vee k\gamma(a\gamma k) \vee (k\gamma a)(a)) \leq k\gamma a \wedge a\gamma k \leq k,$$

i.e., (1.14) holds.

On the other hand, from property (iii) in Theorem 2.1, it follows that k is a quasi-Γ-absorbent of a for all $\gamma \in \Gamma$. Then there exists a quasi-Γ-absorbent y of a such that $k\gamma k = (k\gamma k)\gamma y\gamma(k\gamma k)$. This, condition A_2 of [17] and relation (1.12) imply

$$k\gamma k = (k\gamma k)\gamma y\gamma(k\gamma k) \leq (k\gamma k)\gamma a\gamma(k) = k\gamma(k\gamma a\gamma k)\gamma k \leq k\gamma k\gamma k.$$

Since $k\gamma k\gamma k \leq k\gamma k$ always holds,the statement $k\gamma k = k\gamma k\gamma k$ is true.

Proposition 2.3: If the element a of a Γ-groupoid-lattice B satisfies condition $A_2]^+$ of [17], and if c is an arbitrary bi-Γ-absorbent of a, then $a\Gamma c$ and $c\Gamma a$ are left- and right-Γ-absorbents of a, respectively.

Proof: From definitions and from conditions A_2^+ of [17] it follows that

$$a\Gamma c \leq a$$

$$\text{and} \quad a\Gamma(a\Gamma c) \leq (a\Gamma a)\Gamma c \leq a\Gamma c.$$

$$\text{Similarly,} \quad c\Gamma a \leq a$$

$$\text{and} \quad (c\Gamma a)\Gamma a \leq c\Gamma a.$$

Proposition 2.4: If a regular element a of a Γ-groupoid-lattice B satisfies condition A_2^+ of [17], then every bi-Γ-absorbent c of a is a quasi-Γ-absorbent of a.

Proof: By proposition 2.3, the product $a\Gamma c$ and $c\Gamma a$ are left- and right-Γ-absorbents of a. Theorem 2.1 and condition A_2^+ of [17] imply:

$$c\Gamma a \wedge a\Gamma c = (c\Gamma a)\Gamma(a\Gamma c)$$

$$= (c\Gamma a\Gamma a)\Gamma c$$

$$= c\Gamma(a\Gamma a\Gamma c) \leq c.$$

Theorem 2.2: Any element of a Γ-groupoid-lattice $a \in B$ is regular if and only if $X \wedge Y = X\Gamma Y$ for every left-Γ-absorbent X and every right-Γ-absorbent Y of a.

Proof: Let a be a regular Γ-groupoid-lattice, and let $x \in X \wedge Y$, then there is an element y such that $x\gamma y\gamma x = x$; $\forall\gamma \in \Gamma$. Since X is a left-Γ-absorbent, $x\gamma y \in X$. Therefore $x = x\gamma(y\gamma x) \in X\Gamma Y$. This shows $X \wedge Y \leq X\Gamma Y$. But, $X\Gamma Y \leq X \wedge Y$. Hence $X\Gamma Y = X \wedge Y$.

Conversely, let x be an element of $a \in B$. Then $(x\gamma y \vee x)$ is the right-Γ-absorbent $\langle x\rangle$ of a generated by x for all $y \in a$. By the hypothesis, $\langle x\rangle = \langle x\rangle \wedge a = \langle x\rangle\gamma a = x\gamma a$.
Therefore, we have $x \in x\gamma a$. Similarly $x \in a\gamma x$.
Hence, $x \in x\gamma Y \wedge Y\gamma x = x\gamma Y\gamma Y\gamma x$,
and there is an element y such that $x = x\gamma y\gamma x$.

Theorem 2.3: An element x of a regular Γ-groupoid-lattice $a \in B$ is a quasi-Γ-absorbent if and only if $x\gamma a\gamma x \leq x$ for all $\gamma \in \Gamma$.

Proof: Suppose that x is a quasi-Γ-absorbent of a Γ-groupoid-lattice $a \in B$, then we have $x\gamma a\gamma x \leq a\gamma x$ and $x\gamma a\gamma x \leq x\gamma a$.
Therefore, by the definition of quasi-Γ-absorbent, $x\gamma a\gamma x \leq x\gamma a \wedge a\gamma x \leq x$.
This shows that $x\gamma a\gamma x \leq x$ holds.
Conversely, suppose that an element x is an element of regular Γ-groupoid-lattice $a \in B$ satisfies $x\gamma a\gamma x \leq x$. Then we have $a\gamma x$ is a left-Γ-absorbent of a and $x\gamma a$ is a right-Γ-absorbent of a.
Since $L\gamma R = L \wedge R$ for any right-Γ-absorbent R and left-Γ-absorbent L of a, we have $x\gamma a \wedge a\gamma x = (x\gamma a)\gamma(a\gamma x)$. Therefore, $(x\gamma a)\gamma(a\gamma x) \leq x\gamma a\gamma x$ and $x\gamma a\gamma x \leq x$.
Hence, $x\gamma a \wedge a\gamma x \leq x$.
This shows that the set x is a quasi-Γ-absorbent of $a \in B$.

Theorem 2.4: Let $a \in B$ is a Γ-groupoid-lattice. If L is a left-Γ-absorbent and R is a right-Γ-absorbent of a, then the product $R\Gamma L$ is a bi-absorbent of a.

Proof: Since $(R\Gamma L)\Gamma(R\Gamma L) \leq R\Gamma L$, the product $R\Gamma L$ is a subsemigroup of the Γ-groupoid-lattice $a \in B$.
On the other hand, $(R\Gamma L)\Gamma a\Gamma(R\Gamma L) \leq R\Gamma a\Gamma L \leq R\Gamma L$.
i.e., the product $R\Gamma L$ ia a bi-Γ-absorbent of a.

Corollary 2.2: Let a be a regular Γ-groupoid-lattice, then for each $x \in a$, the converse of the theorem is true.

Theorem 2.5: Let $a \in B$ is a Γ-groupoid-lattice. Then the following proposition concerning the element a are equivalent.
(i) a is a regular element in Γ-groupoid-lattice whose left-Γ-absorbents are two-sided.
(ii) $B \wedge L = B\Gamma L$ for every bi-Γ-absorbent B and every left-absorbent L of a.
(iii) $L \wedge Q = Q\Gamma L$ for each left-Γ-absorbent L and each quasi-Γ-absorbent Q of a.

Proof: $(i) \Rightarrow (ii)$ Let a be a regular Γ-groupoid-lattice whose left-Γ-absorbents are two-sided.
Then by Theorem 2.4 every bi-Γ-absorbent B of a may be represented as $B = R\Gamma I$ where R is a right-Γ-absorbent and I is a two-sided absorbent of a.
Now applying the regularity criterion, we obtain
$B \wedge L = R\Gamma I \wedge L = R\Gamma I\Gamma L = B\Gamma L$ for every bi-Γ-absorbent B and every left-Γ-absorbent L of a.

$(ii) \Rightarrow (iii)$ This is evident because of proposition 2.4.

$(iii) \Rightarrow (i)$ Let a be a Γ-groupoid-lattice with property (iii). Then in case $Q = R$, R is an arbitrary right-Γ-absorbent of a, (iii) implies that a is regular.

Secondly, in case of $L = a$, $Q = L$, L is an arbitrary left-Γ-absorbent of a, condition (iii) implies $L = L \wedge a = L\Gamma a$,
i.e., any left-Γ-absorbent L is also a right-Γ-absorbent of a.
Hence, a is a regular element in Γ-groupoid-lattice whose left-Γ-absorbents are two-sided.

Theorem 2.6: Every two-sided Γ-absorbent T of a regular Γ-groupoid-lattice $a \in B$ is a regular element of a.

Proof: Let x be an arbitrary element of T. Then by regularity of x there exist an element y in a such that $x = x\gamma y$ for all $\gamma \in \Gamma$. Hence $x = (x\gamma y\gamma x)\gamma(y\gamma x)$. Since $y\gamma x\gamma y$ is an element of T and $x = x\gamma(y\gamma x\gamma y)\gamma x$, the element x is regular in T. Hence proved.

Corollary 2.3: Let a be a regular element in Γ-groupoid-lattice B. Then the following assertions holds:
(a) Every quasi-Γ-absorbent k of a can be written in the form $k = R\wedge L = R\Gamma L$, where R is the right and L left absorbent of a generated by k.

(b) Every bi-Γ-absorbent of a is a quasi-Γ-absorbent of a.

(c) Every bi-Γ-absorbent of any two-sided absorbent of a is a quasi-Γ-absorbent of a.

Theorem 2.7: An element a of a Γ-groupoid-lattice B is a regular duo element if and only if $(L \vee L\Gamma a)^2 = L$, and $(R \vee a\Gamma R)^2 = R$ hold for every left and right-Γ-absorbent of a.

Proof: Let a be regular duo element of a Γ-groupoid-lattice. Then every

one-sided Γ-absorbent of a is two-sided, and $I \wedge J = I\Gamma J$ holds for any couple of Γ-absorbents of a. $I \wedge J = I\Gamma J$ implies that every Γ-absorbent I of a is idempotent, i.e., $I\Gamma I = I$ for any Γ-absorbent I of a. This implies both $(L \vee L\Gamma a)^2 = L$, and $(R \vee a\Gamma R)^2 = R$.
Conversely, suppose that a is Γ-groupoid-lattice with $(L \vee L\Gamma a)^2 = L$, and $(R \vee a\Gamma R)^2 = R$ for every left and right-Γ-absorbent of a, respectively. Then $(R \vee a\Gamma R)^2 = R$ implies that each right-Γ-absorbent R of a is also a left-Γ-absorbent, and $(L \vee L\Gamma a)^2 = L$ implies that every left-Γ-absorbent L of a is two-sided. Therefore a is a duo element. Finally $(L \vee L\Gamma a)^2 = L$ implies $I\Gamma I = I$ for any Γ-absorbent I of a, which is equivalent to the regularity of duo element of Γ-groupoid-lattices.

3 Open problem

Is it possible to define an inverse element a in a Γ-groupoid-lattice B. If it is possible than we can further generalize the concepts from inverse Γ-semigroup to Γ-groupoid-lattices.

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