Regular Quasi-$\Gamma$-absorbent in $\Gamma$-Groupoid-lattices

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Abstract

Otto. Steinfeld introduced the concept of quasi-ideal in rings and semigroups in his paper ([13], [14]). Much of steinfeld’s contributions to quasi-ideals is contained in his monograph [17]. In the Paper (with Rédei) [5] the authors generalize concepts from groups, rings, and semigroups to groupoid-lattices. In our paper [19], we have introduced the notion of $\Gamma$-absorbents in $\Gamma$-groupoid-lattices. Here in this paper we will discuss some properties of regular quasi-$\Gamma$-absorbent in $\Gamma$-groupoid-lattices.

Keywords: $\Gamma$-groupoid-lattice, $\Gamma$-absorbent, Regular-$\Gamma$-absorbent.

1 Introduction

The notion of regularity was introduced by J. Von Neumann in his paper [18]. Many authors, A.H. Clifford and G.B. Preston [1], J. Luh [2], L. Kováč [4] and S. Lajos ([8], [9], [10], [11], [12]), etc., have characterized many results on regular rings and semigroups by means of their left ideals, right-ideals and quasi-ideals. O. Steinfeld [17] also introduced the notion of regular elements in groupoid-lattices. The notion of partially ordered $\Gamma$-groupoid-lattice was introduced by Sen and Seth in [7]. Here in this paper we will consider only the groupoid-lattices of semigroups with 0 and associative rings, since all semigroups with 0 and associative rings satisfies the associativity and distributivity Conditions $A_1, A_2, D_v, D'_v, A_2^\ast$, etc., given in [17].
A binary operation $\Gamma$ on $C$ is defined as a mapping from $C \times C \to C$ i.e., $\Gamma$ assigns to each pair $(a, b) \in C \times C$ exactly one element $\Gamma(a, b) \in C$. Instead of $\Gamma(a, b)$ one mostly writes $a \Gamma b$. Let $C$ and $\Gamma$ be two non-empty sets. A mapping from $C \times \Gamma \times C \to C$ will be called a $\Gamma$-multiplication in $C$ and is denoted by $(\cdot)_{\Gamma}$. The result of this $\Gamma$-multiplication for $(a, b) \in C$ and $\gamma \in \Gamma$ is denoted by $a \gamma b$. Let $C = \{x, y, z, \cdots\}$ and $\Gamma = \{\alpha, \beta, \gamma, \cdots\}$ be two non-empty sets. Then $C$ is called a $\Gamma$-groupoid if it satisfies $x \alpha y \in C$ for all $x, y \in C$ and $\alpha, \beta \in \Gamma$. A $\Gamma$-semigroup is a $\Gamma$-groupoid such that the operation $(\cdot)_{\Gamma}$ is associative \[8\].

**Example 1:** Let $G$ be a groupoid and $\Gamma$ be any non-empty set. Let us define a mapping $G \times G \to G$ by $x \alpha y = xy$ for all $x, y \in G$ and $\alpha \in \Gamma$. Then $G$ is a $\Gamma$-groupoid.

A partially ordered $\Gamma$-groupoid is a non-empty set $C$ satisfying the following properties:

(i) $C$ is a groupoid w.r.t multiplication “$(\cdot)_{\Gamma}$”;

(ii) $C$ is a partially ordered set w.r.t a partial ordering “$\leq$”;

(iii) If $a \leq b$ ($\forall a, b \in C$), then $c \gamma a \leq c \gamma b$ and $a \gamma c \leq b \gamma c \forall c \in C$ and $\gamma \in \Gamma$.

A $\Gamma$-groupoid-lattice is a partially ordered $\Gamma$-groupoid $(B, (\cdot)_{\Gamma}, \leq)$ such that $B$ is a complete lattice with respect to the partial ordering $\leq$ and it has the properties:

(iv) $a \gamma a \leq a \forall a \in B$;

(v) $0 \gamma e = e \gamma 0 = 0$.

for the greatest element $e$ and least element $0$ of $B$, where $B$ denotes a $\Gamma$-groupoid-lattice.

(iii) and (v) implies,

$$0 \gamma a = a \gamma 0 = 0 \quad \forall a \in B.$$  

An element $b$ of $B$ is called $\Gamma$-absorbent of the element $a$ of $B$ if

$$b \leq a \quad (1.1)$$

and $a \gamma b \leq b, \forall \gamma \in \Gamma$,  

$$b \gamma a \leq b, \forall \gamma \in \Gamma, \quad (1.2)$$

holds, $b$ is a left-$\Gamma$-absorbent of $a$ if (1.1) and (1.2) holds and right-$\Gamma$-absorbent if (1.1) and (1.3) holds.

An element $k$ of $B$ is called Quasi-$\Gamma$-absorbent of $a \in B$ if

$$k \leq a \quad \text{and} \quad k \gamma a \land a \gamma k \leq k \quad \forall \gamma \in \Gamma \quad (1.4)$$
By a bi-$\Gamma$-absorbent of $a \in B$ we mean an element $b \in B$ such that

$$b \leq a \quad \text{and} \quad (b \gamma a) \gamma b \land b \gamma (a \gamma b) \leq b \quad \forall \gamma \in \Gamma \quad (1.5)$$

An element $x$ of $\Gamma$-groupoid-lattice $B$ is called idempotent if the identity $x = x \gamma x$ holds for all $\gamma \in \Gamma$.

If $a_\lambda (\lambda \in \Lambda)$ are elements of the $\Gamma$-groupoid-lattices $B$, then from (iii):

$$b \gamma \left( \bigwedge \limits_{\lambda \in \Lambda} a_\lambda \right) \leq \bigwedge \limits_{\lambda \in \Lambda} b \gamma a_\lambda, \left( \bigwedge \limits_{\lambda \in \Lambda} a_\lambda \right) \gamma b \leq \bigwedge \limits_{\lambda \in \Lambda} a_\lambda \gamma b \quad \forall b \in B \quad \text{and} \quad \gamma \in \Gamma \quad (1.6)$$

$$b \gamma \left( \bigvee \limits_{\lambda \in \Lambda} a_\lambda \right) \geq \bigvee \limits_{\lambda \in \Lambda} b \gamma a_\lambda, \left( \bigvee \limits_{\lambda \in \Lambda} a_\lambda \right) \gamma b \geq \bigvee \limits_{\lambda \in \Lambda} a_\lambda \gamma b \quad \forall b \in B \quad \text{and} \quad \gamma \in \Gamma \quad (1.7)$$

Proposition 1.1 (See Y. S. Singh and M.R. Khan [19]): Let $b_\lambda (\lambda \in \Lambda)$ are elements of the $\Gamma$-groupoid-lattices $B$, then from (iii):

$$b \gamma \left( \bigwedge \limits_{\lambda \in \Lambda} a_\lambda \right) \leq \bigwedge \limits_{\lambda \in \Lambda} b \gamma a_\lambda, \left( \bigwedge \limits_{\lambda \in \Lambda} a_\lambda \right) \gamma b \leq \bigwedge \limits_{\lambda \in \Lambda} a_\lambda \gamma b \quad \forall b \in B \quad \text{and} \quad \gamma \in \Gamma$$

$$b \gamma \left( \bigvee \limits_{\lambda \in \Lambda} a_\lambda \right) \geq \bigvee \limits_{\lambda \in \Lambda} b \gamma a_\lambda, \left( \bigvee \limits_{\lambda \in \Lambda} a_\lambda \right) \gamma b \geq \bigvee \limits_{\lambda \in \Lambda} a_\lambda \gamma b \quad \forall b \in B \quad \text{and} \quad \gamma \in \Gamma$$

Proposition 1.2 (See Y. S. Singh and M.R. Khan [19]): If $R$ and $L$ are right- and left-$\Gamma$-absorbents of $a$, respectively, then $R \Gamma L \leq R \land L$, $R \Gamma L$ is a bi-$\Gamma$-absorbent and $R \land L$ is a Quasi-$\Gamma$-absorbents of $a$.

2 Results

Proposition 2.1: Let us suppose that the element $a$ of $B$ satisfies Condition $A_2$ and $D^*_v$ of [17]. If $L$ and $R$ are left- and right-$\Gamma$-absorbents of $a$, respectively, then $L \land L \gamma a$ and $R \lor a \gamma R$ are the $\Gamma$-absorbents of $a$ generated by $L$ and $R$, for all $\gamma \in \Gamma$.

Proof: Since, $a \gamma (L \lor L \gamma a) = a \gamma L \lor a \gamma (L \gamma a) \leq L \lor L \gamma a \leq a$.

and $(L \lor L \gamma a) \gamma a = L \gamma a \lor (L \gamma a) \gamma a = L \gamma a \lor L \gamma (a \gamma a) \leq L \lor L \gamma a \leq a$

hold, $L \lor L \gamma a$ is a $\Gamma$-absorbent of $a$ such that $L \leq L \lor L \gamma a$. If $b$ is the $\Gamma$-absorbent of $a$ generated by $L$, then $L \leq b \leq L \lor L \gamma a$.

On the other hand, $L \leq b$ implies $L \lor L \gamma a \leq b \lor b \gamma a \leq b$.

Hence $b = L \lor L \gamma a$, i.e., $L \lor L \gamma a$ is the $\Gamma$-absorbent of $a$ generated by $L$. Similarly, one can show that $R \lor a \gamma R$ is the $\Gamma$-absorbent of $a$ generated by $R$. 
Proposition 2.2: If the element $a$ of $B$ satisfies Conditions $A_1$ and $D_v$ of [17], then

$$L = y \lor a \gamma y \quad \text{and} \quad R = y \lor y \gamma a \quad (\forall y \in B; \gamma \in \Gamma; y \leq a)$$

are the left and right-$\Gamma$-absorbent of $a$, respectively, generated by $y$.

Proof: Since $a \gamma (y \lor a \gamma y) = a \gamma y \lor a \gamma (a \gamma y) = a \gamma y \leq y \lor a \gamma y \leq a$, $\forall \gamma \in \Gamma$

the element $L = y \lor a \gamma y$ is a left-$\Gamma$-absorbent of $a$ such that $y \leq L$. If $m$

is a left-$\Gamma$-absorbent of $a$ generated by $y$, then $y \leq m \leq L$.

On the other hand, from $y \leq m$ it follows that $L = y \lor a \gamma y \leq m \lor a \gamma m \leq m$.
Thus $L = y \lor a \gamma y$ is the left-$\Gamma$-absorbent of $a$ generated by $y$, indeed.

Similarly, one can show that $R = y \lor y \gamma a$ is the right-$\Gamma$-absorbent of $a$
generated by $y$ for all $\gamma \in \Gamma$.

Theorem 2.1: Assume that Conditions $A_2$ and $D_v^*$ of [17] hold for the element $a$ of the $\Gamma$-groupoid-lattice $B$. The following assertions concerning the element $a$ are equivalent:

(i) for every right-$\Gamma$-absorbent $R$ and left-$\Gamma$-absorbent $L$ of $a$,

$$R \gamma L = R \land L \quad \forall \gamma \in \Gamma; \quad (1.8)$$

(ii) for every right-$\Gamma$-absorbent $R$ and left-$\Gamma$-absorbent $L$ of $a$, we have

(a) $R \gamma R = R$ (b) $L \gamma L = L$ (c) $R \gamma L$ is a quasi-$\Gamma$-absorbent of $a$;

(iii) the quasi-$\Gamma$-absorbents of $a$ form a regular (multiplicative) subsemigroup $K$ of the $\Gamma$-groupoid-lattice $\langle B, (\cdot)_\Gamma \rangle$;

(iv) every quasi-absorbent $k$ of $a$ has the form $k = k \gamma a \gamma k \quad \forall \gamma \in \Gamma$.

Proof: (i) $\Rightarrow$ (ii) From (1.8) and proposition 1.2 imply property (c) in (iii).

Proposition 2.1, Conditions $A_2$ and $D_v^*$ of [17], furthermore (1.8) imply:

$$R = R \land (R \lor a \Gamma R)$$

$$= R \Gamma (R \lor a \Gamma R)$$

$$= R \Gamma R \lor R \Gamma (a \Gamma R)$$

$$= R \Gamma R \lor (R \Gamma a) \Gamma R$$

$$= R \Gamma R$$

i.e., property (a) of (ii) holds. Property (b) of (ii) can be proved similarly.
(ii) ⇒ (iii) First we show that every quasi-Γ-absorbent $k$ of $a$ can be written in the form
$$k = k \Gamma a \land a \Gamma k.$$  \hspace{1cm} (1.9)

Because of (a) and Conditions $A_2$ and $D_v^*$ of [17] we have
$$k = k \lor k \Gamma a = (k \lor k \Gamma a)^2 = k^2 \lor k^2 \Gamma a \lor k \Gamma a \Gamma k \lor k \Gamma a \Gamma k \Gamma a \leq k \Gamma a.$$  Similarly, $k \leq a \Gamma k$.

Hence $k \leq k \Gamma a \land a \Gamma k \leq k$.

(1.9) and condition (c) imply:
$$RL = (R \Gamma L) \Gamma a \land a \Gamma (R \Gamma L)$$  \hspace{1cm} (1.10)

for every right-Γ-absorbent $R$ and left-Γ-absorbent $L$ of $a$.

Now we show that the product of the quasi-Γ-absorbents $k_1$ and $k_2$ is a quasi-Γ-absorbent of $a$.

Let us suppose that Condition $A_2$ of [17] holds for the element $a$ of $B$. If $k$ is the quasi-Γ-absorbent of $a$, then we have
$$a \Gamma (a \Gamma k) = (a \Gamma a) \Gamma k \leq a \Gamma k \quad \text{and} \quad (k \Gamma a) \Gamma a = k \Gamma (a \Gamma a) \leq k \Gamma a.$$  \hspace{1cm} (1.11)

$$k \Gamma a \Gamma k = k \Gamma (a \Gamma k) \leq a \Gamma k \land k \Gamma a \leq k.$$  \hspace{1cm} (1.12)

From (1.11) and Condition $A_2$ it follows that
$$a \Gamma (a \Gamma (k_1 \Gamma k_2)) = a \Gamma ((a \Gamma k_1) \Gamma k_2)$$

$$(a \Gamma (a \Gamma k_1)) \Gamma k_2 \leq (a \Gamma k_1) \Gamma k_2$$

$$= a \Gamma (k_1 \Gamma k_2)$$

is a left-Γ-absorbent of $a$.

Similarly, one can show that $(k_1 \Gamma k_2) \Gamma a$ is a right-Γ-absorbent of $a$.

These results, property (a), (b) of (ii), (1.12) and (1.10) imply:

$$(k_1 \Gamma k_2) \Gamma a \land a \Gamma (k_1 \Gamma k_2)$$

$$= (k_1 \Gamma k_2 \Gamma a)(a \Gamma k_1 \Gamma k_2) \Gamma a \land (a \Gamma k_1 \Gamma k_2) \Gamma a (a \Gamma k_1 \Gamma k_2)$$

$$= (k_1 \Gamma k_2 \Gamma a) \Gamma (a \Gamma k_1 \Gamma k_2)$$

$$\leq k_1 \Gamma (k_2 \Gamma a \Gamma k_2) \leq k_1 \Gamma k_2$$

i.e., $k_1 \Gamma k_2$ is a quasi-Γ-absorbent of $a$, indeed.

Thus the set $k$ of all quasi-Γ-absorbent of $a$ is a multiplicative subsemigroup of $\langle B, \cdot \rangle_\Gamma$.
Finally, we have to show that \( k \) is regular semigroup. Let \( k \in K \), then from (1.9),(1.10),(1.11),(1.12) and condition (\( ii \)) it follows that

\[
k = k\Gamma a \land a\Gamma k = (k\Gamma a\Gamma a\Gamma k)\Gamma a \land a\Gamma(k\Gamma a\Gamma a\Gamma k) = k\Gamma a\Gamma a\Gamma k = k\Gamma a\Gamma k \leq k,
\]
that is,

\[
k = kak \in kKk.
\]

(iii) – (iv). Let \( k \) be a quasi-\( \Gamma \)-absorbent of \( a \). From (iii), there exists a quasi-\( \Gamma \)-absorbent \( x \) of \( a \) such that \( k = k\Gamma x\Gamma k \). Hence

\[
k = k\Gamma x\Gamma k \leq k\Gamma a\Gamma k \leq k\Gamma a \land a\Gamma k \leq k,
\]
that is, \( k = kak \).

(iv) \( \Rightarrow \) (i). In view of proposition [1.2], the meet \( R \land L \) of the right-\( \Gamma \)-absorbent \( R \) and left-\( \Gamma \)-absorbent \( L \) of \( a \) is a quasi-\( \Gamma \)-absorbent of \( a \). Condition (iv) implies

\[
R \land L = (R \land L)\Gamma a\Gamma(R \land L) \leq R\Gamma a\Gamma L \leq R\Gamma L \leq R \land L,
\]
that is, \( R\Gamma L = R \land L \).

An element \( a \) of a \( \Gamma \)-groupoid-lattice \( B \) is regular if \( a \) satisfies conditions \( A_2 \) and \( D_v^* \) of [17], furthermore one of the properties (i), (ii), (iii) or (iv) of Theorem 2.1.

**Corollary 2.1:** Every quasi-\( \Gamma \)-absorbent \( k \) of a regular element \( a \) of \( B \) can be written in the form

\[
k = R \land L = R\gamma L, \quad \forall \gamma \in \Gamma \tag{1.14}
\]

where \( R \) and \( L \) are right- and left-\( \Gamma \)-absorbents of \( a \), respectively. Furthermore, \( k\gamma k = k\gamma k\gamma k \quad \forall \gamma \in \Gamma \).

**Proof:** Using (1.11) Conditions \( A_2 \) and \( D_v^* \) of [17], similar to proposition 2.2, one can show that \( L = k \lor a\gamma k \) and \( R = k \lor k\gamma a \) are the left and right-\( \Gamma \)-absorbents of \( a \) respectively, generated by the quasi-\( \Gamma \)-absorbent \( k \). Furthermore, from definition of quasi-\( \Gamma \)-absorbent, Conditions \( A_2 \) and \( D_v^* \) of [17] and relations (1.8), (1.11) we have

\[
k \leq (k \lor k\gamma a) \land (k \lor a\gamma k) = R \land L = R\gamma L = (k \lor k\gamma a)\gamma(k \lor a\gamma k)
\]

\[
= (k \lor k)\gamma k \lor (k \lor k\gamma a)\gamma(a\gamma k)
\]

\[
= k\gamma k \lor (k\gamma a)\gamma k \lor k\gamma(a\gamma k) \lor (k\gamma a)(a) \leq k\gamma a \land a\gamma k \leq k,
\]
i.e., (1.14) holds.

On the other hand, from property \((iii)\) in Theorem 2.1, it follows that \(k\) is a quasi-\(\Gamma\)-absorbent of \(a\) for all \(\gamma \in \Gamma\). Then there exists a quasi-\(\Gamma\)-absorbent \(y\) of \(a\) such that \(k\gamma k = (k\gamma k)\gamma y\gamma(k\gamma k)\). This, condition \(A_2\) of [17] and relation (1.12) imply
\[
k\gamma k = (k\gamma k)\gamma y\gamma(k\gamma k) \leq (k\gamma k)\gamma a\gamma(k) = k\gamma(k\gamma a\gamma k)\gamma k \leq k\gamma k\gamma k.
\]
Since \(k\gamma k\gamma k \leq k\gamma k\) always holds, the statement \(k\gamma k = k\gamma k\gamma k\) is true.

**Proposition 2.3:** If the element \(a\) of a \(\Gamma\)-groupoid-lattice \(B\) satisfies condition \(A_2^+\) of [17], and if \(c\) is an arbitrary bi-\(\Gamma\)-absorbent of \(a\), then \(a\Gamma c\) and \(c\Gamma a\) are left- and right-\(\Gamma\)-absorbents of \(a\), respectively.

**Proof:** From definitions and from conditions \(A_2^+\) of [17] it follows that
\[
a\Gamma c \leq a
\]
and \(a\Gamma(a\Gamma c) \leq (a\Gamma a)\Gamma c \leq a\Gamma c\).

Similarly, \(c\Gamma a \leq a\)
and \((c\Gamma a)\Gamma a \leq c\Gamma a\).

**Proposition 2.4:** If a regular element \(a\) of a \(\Gamma\)-groupoid-lattice \(B\) satisfies condition \(A_2^+\) of [17], then every bi-\(\Gamma\)-absorbent \(c\) of \(a\) is a quasi-\(\Gamma\)-absorbent of \(a\).

**Proof:** By proposition 2.3, the product \(a\Gamma c\) and \(c\Gamma a\) are left- and right-\(\Gamma\)-absorbents of \(a\). Theorem 2.1 and condition \(A_2^+\) of [17] imply:
\[
c\Gamma a \land a\Gamma c = (c\Gamma a)\Gamma(a\Gamma c)
= (c\Gamma a\Gamma a)\Gamma c
= c\Gamma(a\Gamma a\Gamma c) \leq c.
\]

**Theorem 2.2:** Any element of a \(\Gamma\)-groupoid-lattice \(a \in B\) is regular if and only if \(X \land Y = X\Gamma Y\) for every left-\(\Gamma\)-absorbent \(X\) and every right-\(\Gamma\)-absorbent \(Y\) of \(a\).

**Proof:** Let \(a\) be a regular \(\Gamma\)-groupoid-lattice, and let \(x \in X \land Y\), then there is an element \(y\) such that \(x\gamma y \gamma x = x; \forall \gamma \in \Gamma\). Since \(X\) is a left-\(\Gamma\)-absorbent, \(x\gamma y \in X\). Therefore \(x = x\gamma(y\gamma x) \in X\Gamma Y\). This shows \(X \land Y \leq X\Gamma Y\). But, \(X\Gamma Y \leq X \land Y\). Hence \(X\Gamma Y = X \land Y\).
Conversely, let \( x \) be an element of \( a \in B \). Then \( (x \gamma y \lor x) \) is the right-\( \Gamma \)-absorbent \( \langle x \rangle \) of \( a \) generated by \( x \) for all \( y \in a \). By the hypothesis, \( \langle x \rangle = \langle x \rangle \land a = \langle x \rangle \gamma a = x \gamma a \).

Therefore, we have \( x \in x \gamma a \). Similarly \( x \in a \gamma x \).

Hence, \( x \in x \gamma Y \land Y \gamma x = x \gamma Y \gamma Y x \), and there is an element \( y \) such that \( x = x \gamma y \gamma x \).

**Theorem 2.3:** An element \( x \) of a regular \( \Gamma \)-groupoid-lattice \( a \in B \) is a quasi-\( \Gamma \)-absorbent if and only if \( x \gamma a \gamma x \leq x \) for all \( \gamma \in \Gamma \).

**Proof:** Suppose that \( x \) is a quasi-\( \Gamma \)-absorbent of a \( \Gamma \)-groupoid-lattice \( a \in B \), then we have \( x \gamma a \gamma x \leq a \gamma x \) and \( x \gamma a \gamma x \leq x \gamma a \).

Therefore, by the definition of quasi-\( \Gamma \)-absorbent, \( x \gamma a \gamma x \leq x \gamma a \land a \gamma x \leq x \).

This shows that \( x \gamma a \gamma x \leq x \) holds.

Conversely, suppose that an element \( x \) is an element of regular \( \Gamma \)-groupoid-lattice \( a \in B \) satisfies \( x \gamma a \gamma x \leq x \). Then we have \( a \gamma x \) is a left-\( \Gamma \)-absorbent of \( a \) and \( x \gamma a \) is a right-\( \Gamma \)-absorbent of \( a \).

Since \( L \gamma R = L \land R \) for any right-\( \Gamma \)-absorbent \( R \) and left-\( \Gamma \)-absorbent \( L \) of \( a \), we have \( x \gamma a \land a \gamma x = (x \gamma a) \gamma (a \gamma x) \). Therefore, \( (x \gamma a) \gamma (a \gamma x) \leq x \gamma a \gamma x \) and \( x \gamma a \gamma x \leq x \).

Hence, \( x \gamma a \land a \gamma x \leq x \).

This shows that the set \( x \) is a quasi-\( \Gamma \)-absorbent of \( a \in B \).

**Theorem 2.4:** Let \( a \in B \) be a \( \Gamma \)-groupoid-lattice. If \( L \) is a left-\( \Gamma \)-absorbent and \( R \) is a right-\( \Gamma \)-absorbent of \( a \), then the product \( R \Gamma L \) is a bi-\( \Gamma \)-absorbent of \( a \).

**Proof:** Since \( (R \Gamma L) \Gamma (R \Gamma L) \leq R \Gamma L \), the product \( R \Gamma L \) is a subsemigroup of the \( \Gamma \)-groupoid-lattice \( a \in B \).

On the other hand, \( (R \Gamma L) \Gamma a \Gamma (R \Gamma L) \leq R \Gamma a \Gamma L \leq R \Gamma L \).

i.e., the product \( R \Gamma L \) is a bi-\( \Gamma \)-absorbent of \( a \).

**Corollary 2.2:** Let \( a \) be a regular \( \Gamma \)-groupoid-lattice, then for each \( x \in a \), the converse of the theorem is true.

**Theorem 2.5:** Let \( a \in B \) be a \( \Gamma \)-groupoid-lattice. Then the following proposition concerning the element \( a \) are equivalent.

(i) \( a \) is a regular element in \( \Gamma \)-groupoid-lattice whose left-\( \Gamma \)-absorbents are two-sided.

(ii) \( B \land L = B \Gamma L \) for every bi-\( \Gamma \)-absorbent \( B \) and every left-absorbent \( L \) of \( a \).

(iii) \( L \land Q = Q \Gamma L \) for each left-\( \Gamma \)-absorbent \( L \) and each quasi-\( \Gamma \)-absorbent \( Q \) of \( a \).
Proof: $(i) \Rightarrow (ii)$ Let $a$ be a regular $\Gamma$-groupoid-lattice whose left-$\Gamma$-absorbs are two-sided. Then by Theorem 2.4 every bi-$\Gamma$-absorbent $B$ of $a$ may be represented as $B = R\Gamma I$ where $R$ is a right-$\Gamma$-absorbent and $I$ is a two-sided absorbent of $a$. Now applying the regularity criterion, we obtain $B \wedge L = R\Gamma I \wedge L = R\Gamma\Gamma L = B\Gamma L$ for every bi-$\Gamma$-absorbent $B$ and every left-$\Gamma$-absorbent $L$ of $a$.

$(ii) \Rightarrow (iii)$ This is evident because of proposition 2.4.

$(iii) \Rightarrow (i)$ Let $a$ be a $\Gamma$-groupoid-lattice with property $(iii)$. Then in case $Q = R$, $R$ is an arbitrary right-$\Gamma$-absorbent of $a$, $(iii)$ implies that $a$ is regular.

Secondly, in case of $L = a$, $Q = L$, $L$ is an arbitrary left-$\Gamma$-absorbent of $a$, condition $(iii)$ implies $L = L \wedge a = L\Gamma a$, i.e., any left-$\Gamma$-absorbent $L$ is also a right-$\Gamma$-absorbent of $a$. Hence, $a$ is a regular element in $\Gamma$-groupoid-lattice whose left-$\Gamma$-absorbs are two-sided.

**Theorem 2.6:** Every two-sided $\Gamma$-absorbent $T$ of a regular $\Gamma$-groupoid-lattice $a \in B$ is a regular element of $a$.

**Proof:** Let $x$ be an arbitrary element of $T$. Then by regularity of $x$ there exist an element $y$ in $a$ such that $x = x\gamma y$ for all $\gamma \in \Gamma$. Hence $x = (x\gamma y\gamma x)\gamma(y\gamma x)$. Since $y\gamma x\gamma y$ is an element of $T$ and $x = x\gamma(y\gamma x\gamma y)\gamma x$, the element $x$ is regular in $T$. Hence proved.

**Corollary 2.3:** Let $a$ be a regular element in $\Gamma$-groupoid-lattice $B$. Then the following assertions holds:

(a) Every quasi-$\Gamma$-absorbent $k$ of $a$ can be written in the form $k = R\wedge L = R\Gamma L$, where $R$ is the right and $L$ left absorbent of $a$ generated by $k$.

(b) Every bi-$\Gamma$-absorbent of $a$ is a quasi-$\Gamma$-absorbent of $a$.

(c) Every bi-$\Gamma$-absorbent of any two-sided absorbent of $a$ is a quasi-$\Gamma$-absorbent of $a$.

**Theorem 2.7:** An element $a$ of a $\Gamma$-groupoid-lattice $B$ is a regular duo element if and only if $(L \vee L\Gamma a)^2 = L$, and $(R \vee a\Gamma R)^2 = R$ hold for every left and right-$\Gamma$-absorbent of $a$.

**Proof:** Let $a$ be regular duo element of a $\Gamma$-groupoid-lattice. Then every
one-sided $\Gamma$-absorbent of $a$ is two-sided, and $I \wedge J = I \Gamma J$ holds for any couple of $\Gamma$-absorbents of $a$. $I \wedge J = I \Gamma J$ implies that every $\Gamma$-absorbent $I$ of $a$ is idempotent, i.e., $I \Gamma I = I$ for any $\Gamma$-absorbent $I$ of $a$. This implies both $(L \vee L \Gamma a)^2 = L$, and $(R \vee a \Gamma R)^2 = R$.

Conversely, suppose that $a$ is $\Gamma$-groupoid-lattice with $(L \vee L \Gamma a)^2 = L$, and $(R \vee a \Gamma R)^2 = R$ for every left and right-$\Gamma$-absorbent of $a$, respectively. Then $(R \vee a \Gamma R)^2 = R$ implies that each right-$\Gamma$-absorbent $R$ of $a$ is also a left-$\Gamma$-absorbent, and $(L \vee L \Gamma a)^2 = L$ implies that every left-$\Gamma$-absorbent $L$ of $a$ is two-sided. Therefore $a$ is a duo element. Finally $(L \vee L \Gamma a)^2 = L$ implies $I \Gamma I = I$ for any $\Gamma$-absorbent $I$ of $a$, which is equivalent to the regularity of duo element of $\Gamma$-groupoid-lattices.

3 Open problem

Is it possible to define an inverse element $a$ in a $\Gamma$-groupoid-lattice $B$. If it is possible than we can further generalize the concepts from inverse $\Gamma$-semigroup to $\Gamma$-groupoid-lattices.

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