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Some Generalized Difference Sequence Spaces Defined by Orlicz Function in a Seminormed Space

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Abstract

In this paper we introduce some new generalized difference sequence spaces using by an Orlicz function and examine some properties of these sequence spaces.

Keywords: *Complete, difference sequence space, Orlicz function, seminorm.*

1 Introduction and Preliminaries

Throughout the paper w denotes the space of all sequences and let l_∞, c and c_o be the linear spaces of bounded, convergent, and null spaces $x = (x_k)$ with complex terms, respectively, normed by $\|x\|_\infty = \sup_k |x_k|$ where $k \in \mathbb{N}$, the set of positive integers.

Kızmaz [5] introduced the notion of difference sequence spaces as follows

$$X(\Delta) = \{x = (x_k) \in w : \ (\Delta x_k) \in X\}$$

for $X = l_\infty$, c and c_o, where

$$\Delta x_k = (\ \Delta x_k)_{k=1}^\infty = (x_k - x_{k+1})_{k=1}^\infty .$$

Later on, the notion was generalized by Et and Çolak [9] as follows:

$$X(\Delta^m) = \{x = (x_k) \in w : \ (\Delta^m x_k) \in X\}$$

for $X = l_\infty$, c, c_o, where $\Delta^m x = (\Delta^m x_k)^\infty = (\Delta^{m-1} x_k - \Delta^{m-1} x_{k+1})$, $\Delta^o x = (x_k)$ and $m \geq 0$ is a fixed integer.

Recently, these sequence spaces were more generalized by Et and Esi [8] to the following sequences spaces

$$X(\Delta_v^m) = \{x = (x_k) \in w : \ (\Delta_v^m x_k) \in X\}$$

for $X = l_\infty$, c, and c_o, where $\Delta_v^m = (v_k x_k)$, $\Delta_v x_k = (v_k x_k - v_{k+1} x_{k+1})$, $\Delta_v^m x_k = (\Delta_v^{m-1} x_k - \Delta_v^{m-1} x_{k+1})$ such that $\Delta_v^m x_k = \sum\limits_{i=1}^{m} (-1)^i \binom{m}{i} v_{k+i} x_{k+i}$ and $v = (v_k)$ be any fixed sequence of non-zero complex numbers.

The idea of Kızmaz [5] was applied for introducing different type of difference sequence spaces and for studying their different algebraic and topological properties by Esi [1, 2], Esi and Tripathy [3], Tripathy et.al [4] and many others.

The following inequality will be used throughout the paper: Let $p = (p_k)$ be a positive sequence of real numbers with $0 < \inf p_k = h \leq p_k \leq \sup_k p_k = H < \infty$ and $K = \max(1, 2^{H-1})$. Then for $a_k, b_k \in \mathbb{C}$, we have,

$$|a_k + b_k|^{p_k} \leq K(|a_k|^{p_k} + |b_k|^{p_k}), \text{ for all } k \in \mathbb{N}.$$

An Orlicz function is a function $M : [0, \infty) \to [0, \infty)$ which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$.

An Orlicz function is a function M is said to satisfy the Δ_2- condition for all values of t, if there exist a constant $K > 0$ such that $M(2t) \leq KM(t)$, $(t \geq 0)$.

Lindenstrauss and Tzafriri [7] used the Orlicz function and introduced the sequence l_M as follows:

$$l_M = \left\{ x = (x_k) : \sum_k M\left(\frac{|x_k|}{r}\right) < \infty, \text{ for some } r > 0 \right\}.$$

They proved that l_M is a Banach space normed by

$$\|x\| = \inf \left\{ r > 0 : \ \sum_k M\left(\frac{|x_k|}{r}\right) \leq 1 \right\}.$$

Remark 1.1 *An Orlicz function M satisfies the inequality $M(\lambda x) \leq \lambda M(x)$ for all λ with $0 < \lambda < 1$.*

Let q_1 and q_2 be seminorms on a linear space X. Then q_1 is said to be stronger than q_2 if whenever $x = (x_k)$ is a sequence such that $q_1(x_k) \to 0$, than also $q_2(x_k) \to 0$.If each is stronger than the others, q_1 and q_2 are said to be equivalent, [10].

Lemma 1.2 ([10]) *Let q_1 and q_2 be seminorms on a linear space X. Then q_1 is said to be stronger than q_2 if and only if there exists a constant T such that $q_2(x) \le Tq_1(x)$, for all $x \in X$.*

Let $p = (p_k)$ be a sequence of strictly positive real number and $s \ge 0$. Let X be a seminormed spaces over the field $\mathbb{C}$ of complex numbers with the seminorm q. The symbol $w(X)$ denotes the spaces of all sequences defined over X. Let $v = (v_k)$ be any fixed sequence of non-zero complex numbers. Let M be an Orlicz function, we define the following sequence spaces as follows:

$$c\left[\Delta_v^m, M, p, q, s\right] = \left\{ x = (x_k) \in w(X) : \lim_k k^{-s} \left[M \left(\frac{q(\Delta_v^m x_k - l)}{r} \right) \right]^{p_k} = 0, \right.$$
$$\left. \text{for some } l \in X \text{ and } r > 0 \right\},$$

$$c_o\left[\Delta_v^m, M, p, q, s\right] = \left\{ x = (x_k) \in w(X) : \lim_k k^{-s} \left[M \left(\frac{q(\Delta_v^m X_k)}{r} \right) \right]^{p_k} = 0, \right.$$
$$\left. \text{for some } r > 0 \right\},$$

$$l_\infty\left[\Delta_v^m, M, p, q, s\right] = \left\{ x = (x_k) \in w(X) : \sup_k k^{-s} \left[M \left(\frac{q(\Delta_v^m X_k)}{r} \right) \right]^{p_k} < \infty, \right.$$
$$\left. \text{for some } r > 0 \right\}.$$

Some well-known spaces are obtained by specializing, M, p, q, s, v and m.

(a) If $M(x) = x$, $m = 1$, $v = (v_k) = (1, 1, 1, ...)$, $q(x) = |x|$, $s = 0$ and $p_k = 1$ for all $k \in \mathbb{N}$, then we obtain the spaces $c(\Delta)$, $c_o(\Delta)$ and $l_\infty(\Delta)$ which were defined and studied by Kızmaz [5].

(b) If $M(x) = x$, $m = 0$, $v = (v_k) = (1, 1, 1, ...)$, $s = 0$ and $q(x) = |x|$, then we obtain the spaces $c(p)$, $c_o(p)$ and $l_\infty(p)$ which were defined and studied by Maddox [6].

(c) If $M(x) = x$, $q(x) = |x|$, $s = 0$ and $p_k = 1$ for all $k \in \mathbf{N}$, then we obtain the spaces $c(\Delta_v^m)$, $c_o(\Delta_v^m)$ and $l_\infty(\Delta_v^m)$ which were defined by Et and Esi [7].

(d) If $M(x) = x$, $m = s = 0$, $v = (v_k) = (1, 1, 1, ...)$, $q(x) = |x|$ and $p_k = 1$ for all $k \in \mathbb{N}$, then we obtain classical sequence spaces c, c_o and l_∞.

2 Main Results

We prove the following theorems:

Theorem 2.1 *Let $p=(p_k)$ be a bounded sequence. Then $c_o\left[\Delta_v^m, M, p, q, s\right]$, $c\left[\Delta_v^m, M, p, q, s\right]$ and $l_\infty\left[\Delta_v^m, M, p, q, s\right]$ are linear spaces over the complex field $\mathbb{C}$.*

Proof. We give the proof only for $c_o\left[\Delta_v^m, M, p, q, s\right]$ the others can be treated similarly. Let $x, y \in c_o\left[\Delta_v^m, M, p, q, s\right]$ for $\lambda, \mu \in \mathbb{C}$. Then there exist positive numbers r_1 and r_2 such that

$$k^{-s}\left[M\left(q\left(\frac{\Delta_v^m(x_k)}{r_1}\right)\right)\right]^{p_k} \rightarrow 0 \text{ as } k \rightarrow \infty$$

and

$$k^{-s}\left[M\left(q\left(\frac{\Delta_v^m(y_k)}{r_2}\right)\right)\right]^{p_k} \rightarrow 0 \text{ as } k \rightarrow \infty .$$

Let $r_3=\max(2\left|\lambda\right| r_1, 2\left|\mu\right| r_2)$. Since M is non-decreasing convex function and q is a seminorm, we have

$$k^{-s}\left[M\left(q(\frac{\Delta_v^m\left(\lambda x_k+\mu y_k\right)}{r_3})\right)\right]^{p_k} \leq k^{-s}\left[M\left(q\frac{(\lambda \Delta_v^m x_k)}{r_3}\right)+q\left(\frac{\mu \Delta_v^m y_k}{r_3}\right)\right]^{p_k}$$
$$\leq K k^{-s}\left[M\left(q\frac{(\Delta_v^m x_k)}{r_1}\right)\right]^{p_k}+K k^{-s}\left[M\left(q\frac{(\Delta_v^m y_k}{r_2}\right)\right]^{p_k} \rightarrow 0 \text{ as } k \rightarrow \infty .$$

Therefore $\lambda x_k+\mu y_k \in c_o\left[\Delta_v^m, M, p, q, s\right]$. Hence $c_o\left[\Delta_v^m, M, p, q, s\right]$ is a linear space. ∎

Theorem 2.2 *The spaces $c_o\left[\Delta_v^m, M, p, q, s\right]$, $c\left[\Delta_v^m, M, p, q, s\right]$ and $l_\infty\left[\Delta_v^m, M, p, q, s\right]$ are paranormed space, paranormed by*

$$h(x)=\inf\left\{r^{p_n/H}>0: \sup_k k^{-s}\left[M\left(\frac{q(\Delta_v^m x_k)}{r}\right)\right] \leq 1,\ s \geq 0, \text{ for some } r>0,\ n \in \mathbb{N}\right\},$$

where $H=\max(1, \sup_k p_k)$.

Proof. We prove the theorem for the space $c_o\left[\Delta_v^m, M, p, q, s\right]$. The proof for the other spaces can be proved by the same way. Clearly $h(x)=h(-x)$ for all $x \in c_o\left[\Delta_v^m, M, p, q, s\right]$, and $h(\theta)=0$. Let $x, y \in c_o\left[\Delta_v^m, M, p, q, s\right]$. Then we have $r_1, r_2>0$ such that

$$\sup_k k^{-s}\left[M\left(q\left(\frac{\Delta_v^m x_k}{r_1}\right)\right)\right] \leq 1$$

and

$$\sup_k k^{-s}\left[M\left(q\left(\frac{\Delta_v^m x_k}{r_2}\right)\right)\right] \leq 1 .$$

Let $r = r_1 + r_2$. Then by the convexity of M, we have

$$\begin{aligned}
&\sup_k k^{-s}\left[M\left(q\left(\frac{\Delta_v^m(x_k+y_k)}{r}\right)\right)\right] \\
&\le \sup_k k^{-s}\left[M\left(\frac{r_1}{r_1+r_2}q\left(\frac{\Delta_v^m x_k}{r_1}\right)+\frac{r_2}{r_1+r_2}q\left(\frac{\Delta_v^m y_k}{r_2}\right)\right)\right] \\
&\le \frac{r_1}{r_1+r_2}\sup_k k^{-s}M\left(q\left(\frac{\Delta_v^m x_k}{r_1}\right)+\frac{r_2}{r_1+r_2}\sup_k k^{-s}M\left(q\left(\frac{\Delta_v^m y_k}{r_2}\right)\right)\right) \\
&\le 1.
\end{aligned}$$

Hence from above inequality, we have

$$\begin{aligned}
h(x+y) &= \inf\left\{r^{p_n/H} : \sup_k k^{-s}\left[M\left(q\left(\frac{\Delta_v^m(x_k+y_k)}{r}\right)\right)\right] \le 1,\ r>0,\ n\in\mathbb{N}\right\} \\
&\le \inf\left\{r_1^{p_n/H} : \sup_k k^{-s}\left[M\left(q\left(\frac{\Delta_v^m x_k}{r_1}\right)\right)\right] \le 1,\ s\geqslant 0,\ r_1>0,\ n\in\mathbb{N}\right\} \\
&+ \inf\left\{r_2^{p_n/H} : \sup_k k^{-s}\left[M\left(q\left(\frac{\Delta_v^m y_k}{r_2}\right)\right)\right] \le 1,\ s\geqslant 0,\ r_2>0,\ n\in\mathbb{N}\right\} \\
&= h(x)+h(y).
\end{aligned}$$

For the continuity of scalar multiplication let $\lambda \neq 0$ be any complex number. Then by the definition of h, we have

$$\begin{aligned}
h(\lambda x) &= \inf\left\{r^{P_n/H} : \sup_k k^{-s}\left[M\left(q\left(\frac{\Delta_v^m \lambda x_k}{r}\right)\right)\right] \le 1,\ s\ge 0,\ r>0,\ n\in\mathbb{N}\right\} \\
&= \inf\left\{(t\,|\lambda|)^{P_n/H} : \sup_k k^{-s}\left[M\left(q\left(\frac{\Delta_v^m x_k}{t}\right)\right)\right] \le 1, s\ge 0, t>0, n\in\mathbb{N}\right\},
\end{aligned}$$

where $t = \frac{r}{|\lambda|}$. Since $|\lambda|^{p_n} \le \max(1,|\lambda|^H)$, we have $|\lambda|^{p_n/H} \le (\max(1,|\lambda|^H))^{1/H}$. Then

$$\begin{aligned}
h(\lambda x) &\le (\max(1,|\lambda|^H))^{1/H}.\inf\left\{t^{P_n/H} : \sup_k k^{-s}M\left(q\left(\frac{\Delta_v^m x_k}{t}\right)\right) \le 1,\ s\ge 0,\right. \\
&\qquad\qquad\qquad\qquad \left. t>0,\ n\in\mathbb{N}\right\} \\
&= (\max(1,|\lambda|^H))^{1/H}.h(x),
\end{aligned}$$

and therefore $h(\lambda x)$ converges to zero when $h(x)$ converges to zero. Hence the space $c_o\left[\Delta_v^m, M, p, q, s\right]$ is paranormed by h. ■

Theorem 2.3 *Let (X,q) be complete seminormed space, then the spaces $c_o\left[\Delta_v^m, M, p, s\right]$,$c\left[\Delta_v^m, M, p, q, s\right]$ and $l_\infty\left[\Delta_v^m, M, p, q, s\right]$ are complete with paranorm h defined in Theorem 2.2.*

Proof. We prove it for the case $c_o\left[\Delta_v^m, M, p, q, s\right]$ and the other cases can be establish similarly. Let (x^i) be a Cauchy sequence in $c_o\left[\Delta_v^m, M, p, q, s\right]$. Then

$$h(x^i - x^j) \to 0 \text{ as } i, j \to \infty.$$

Let $x_o > 0$ be fixed and $t > 0$ be such that for a given $0 < \varepsilon < 1$, $\frac{\varepsilon}{x_o t} > 0$ and $x_o t \geq 1$. Then there exist a positive integer n$_o$ such that

$$h(x^i - x^j) < \frac{\varepsilon}{t\delta},\ i, j \geqslant n_0.$$

Using definition of paranorm, we get

$$\inf\left\{ r^{P_n/H} : \sup_k k^{-s} M\left[q\left(\frac{\Delta_v^m\left(x_k^i - x_k^j\right)}{r} \right) \right] \leq 1,\ s \geq 0,\ r > 0,\ n \in \mathbb{N} \right\} < \frac{\varepsilon}{x_o t}$$

and

$$\sup_k k^{-s} M\left[q\left(\frac{\Delta_v^m\left(x_k^i - x_k^j\right)}{h(x^i - x^j)} \right) \right] \leq 1, \text{ for all } i, j \geq n_0.$$

It follows that

$$M\left[q\left(\frac{\Delta_v^m\left(x_k^i - x_k^j\right)}{h(x^i - x^j)} \right) \right] \leq 1, \text{ for all } i, j \geq n_0.$$

For $t > 0$ with $M\left(\frac{tx_0}{2}\right) \geqslant 1$, we have

$$M\left[q\left(\frac{\Delta_v^m\left(x_k^i - x_k^j\right)}{h(x^i - x^j)} \right) \right] \leq M\left(\frac{tx_0}{2}\right).$$

Since M is continuous, then we obtain

$$q(\Delta_v^m x_k^i - \Delta_v^m x_k^j) < \frac{tx_0}{2}.\frac{\varepsilon}{tx_0} = \frac{\varepsilon}{2}.$$

Hence $(\Delta_v^m x^i)$ is a Cauchy sequence in (X, q). Since (X, q) is complete it is convergent in X. Suppose that $\Delta_v^m x_k^i \to x_k$ as $i \to \infty$, for all $k \in \mathbb{N}$. Now we have for all $i, j \geqslant n_o$.

$$\inf\left\{ r^{P_n/H} : \sup_k k^{-s} M\left[q\left(\frac{\Delta_v^m\left(x_k^i - x_k^j\right)}{r} \right) \right] \leq 1,\ s \geq 0,\ r > 0,\ n \in \mathbb{N} \right\} < \varepsilon$$

This implies that

$$\lim_{j\to\infty} \inf\left\{ r^{P_n/H} : \sup_k k^{-s} M\left[q\left(\frac{\Delta_v^m\left(x_k^i - x_k^j\right)}{r} \right) \right] \leq 1, s \geq 0, r > 0, n \in \mathbb{N} \right\} < \varepsilon$$

for all $i \geq n_o$. Taking infimum of r's we get

$$\inf\left\{r^{p_n/H} : \sup_k k^{-s} M\left(q\left(\frac{\Delta_v^m\left(x_k^i - x_k\right)}{r}\right)\right) \leq 1,\ s \geqslant 0,\ r > 0,\ n \in \mathbb{N}\right\} < \varepsilon$$

for all $i \geqslant n_0$. It follows that $(x^i - x) \in c_o\left[\Delta_v^m, M, p, q, s\right]$. Since $(x^i) \in c_o\left[\Delta_v^m, M, p, q, s\right]$ and $c_o\left[\Delta_v^m, M, p, q, s\right]$ is linear space, so we have $x = x^i - (x^i - x) \in c_o\left[\Delta_v^m, M, p, q, s\right]$. This completes the proof. ■

Theorem 2.4 *Let M_1 and M_2 be two Orlicz functions. Then*
(a) $Z\left[\Delta_v^m, M_1, p, q, s\right] \subset Z\left[\Delta_v^m, M_2 o M_1, p, q, s\right]$,
(b) $Z\left[\Delta_v^m, M_1, p, q, s\right] \bigcap Z\left[\Delta_v^m, M_2, p, q, s\right] \subset Z\left[\Delta_v^m, M_1 + M_2, p, q, s\right]$, *where* $Z = c_o,\ c,\ l_\infty$.

Proof. (a): We prove this part for $Z = c_o$ and the rest of the cases will follow similarly. Let $x = (x_k) \in c_o\left[\Delta_v^m, M_1, p, q, s\right]$. Then for given $0 < \varepsilon < 1$, there exists $r > 0$ such that there exist a subset A of $\mathbb{N}$, where

$$A = \left\{k \in \mathbb{N} : k^{-s}\left[M_1\left(q\left(\frac{\Delta_v^m x_k}{r}\right)\right)\right]^{p_k} < \frac{\varepsilon}{B}\right\},$$

$$B = \max\left(1, \sup_k\left[M_2\left(\frac{1}{(k^{-s})^{1/P_k}}\right)\right]^{p_k}\right).$$

If we take

$$y_k = (k^{-s})^{1/p_k} M_1\left(q\left(\frac{\Delta_v^m x_k}{r}\right)\right)$$

Then $y_k^{p_k} < \frac{\varepsilon}{B} < 1$ implies $y_k < 1$. Hence we have by Remark,

$$\begin{aligned}(M_2 o M_1)\left(q\left(\frac{\Delta_v^m x_k}{r}\right)\right) &= M_2\left(\frac{y_k}{(k^{-s})^{1/p_k}}\right)\\ &\leq y_k M_2\left(\frac{1}{(k^{-s})^{1/p_k}}\right).\end{aligned}$$

Thus

$$\begin{aligned}k^{-s}\left[M_2(y_k)\right]^{p_k} &\leq k^{-s}\left[M_2\left(\frac{y_k}{(k^{-s})^{1/P_k}}\right)\right]^{p_k}\\ &\leq k^{-s} B y_k^{p_k} \leq B y_k^{p_k} < \varepsilon.\end{aligned}$$

Then

$$k^{-s}(M_2 o M_1)\left(q\left(\frac{\Delta_v^m x_k}{r}\right)\right) < \varepsilon.$$

Hence $x = (x_k) \in c_o\left[\Delta_v^m, M_2 o M_1, p, q, s\right]$.

(b) It follows from the following inequality

$$k^{-s}\left[(M_1+M_2)\left(q\left(\frac{\Delta_v^m x_k}{\rho}\right)\right)\right]^{p_k}$$
$$\leq Kk^{-s}\left[M_1\left(q\left(\frac{\Delta_v^m x_k}{r}\right)\right)\right]^{p_k}+Kk^{-s}\left[M_2\left(q\left(\frac{\Delta_v^m x_k}{r}\right)\right)\right]^{p_k}.$$

■

Theorem 2.5 *Let M be an Orlicz function, then $c_o\left[\Delta_v^m, M, p, q, s\right] \subset c\left[\Delta_v^m, M, p, q, s\right] \subset l_\infty\left[\Delta_v^m, M, p, q, s\right]$ and the inclusions are strict.*

Proof. The first inclusions is obvious. We proof the second inclusion. Let $x=(x_k)\in c\left[\Delta_v^m, M, p, q, s\right]$. Since M is non-decreasing and convex function and q is a seminorm, we obtain

$$k^{-s}\left[M\left(\frac{q(\Delta_v^m x_k)}{r}\right)\right]^{p_k} \leq Kk^{-s}\left[M\left(\frac{q(\Delta_v^m x_k-l)}{r}\right)\right]^{p_k}+Kk^{-s}\left[M\left(q\left(\frac{l}{r}\right)\right)\right]^{p_k}.$$

Then there exist an integer K_l such that $q(l)\leq K_l$. Hence we have $k^{-s}\left[M\left(\frac{q(\Delta_v^m x_k)}{r}\right)\right]^{P_k} \leq Kk^{-s}\left[M\left(\frac{q(\Delta_v^m x_k-l)}{r}\right)\right]^{p_k}+Kk^{-s}\left[M\left(q\left(\frac{K_l}{r}\right)\right)\right]^{p_k}$. Hence $x=(x_k)\in l_\infty\left[\Delta_v^m, M, p, q, s\right]$. ■

To show that the inclusions are strict, consider the following example.

Example 2.6 *Let $X=\mathbf{C}$, $M(x)=x$, $q(x)=|x|$, $s=0$, $v=(v_k)=(1,1,...)$ and $p_k=1$ for all $k\in\mathbb{N}$. Then $x=(k^m)\in l_\infty\left[\Delta_v^m, M, p, q, s\right]$ but $x=(k^m)\notin c_o\left[\Delta_v^m, M, p, q, s\right]$, since $\Delta_v^m k^m=(-1)^m m!$. Under these restrictions on M, q, s, v and p_k's consider the sequences $x=(-1)^k$. Then $x\in l_\infty\left[\Delta_v^m, M, p, q, s\right]$, but $x\notin c\left[\Delta_v^m, M, p, q, s\right]$.*

Theorem 2.7 *For $Z=c_o$, c and l_∞, $Z\left[\Delta_v^{m-1}, M, p, q, s\right]\subset Z\left[\Delta_v^m, M, p, q, s\right]$ and also in general $Z\left[\Delta_v^i, M, p, q, s\right]\subset Z\left[\Delta_v^m, M, p, q, s\right]$ for all $i=1,2,...,m-1$. The inclusions are strict.*

Proof. We give the proof for $Z=l_\infty$. The other cases can be prove using the similar arguments. Let $x=(x_k)\in l_\infty\left[\Delta_v^{m-1}, M, p, q, s\right]$. Then we have

$$\sup_k k^{-s}\left[M\left(\frac{q(\Delta_v^{m-1}x_k)}{r}\right)\right]^{p_k}<\infty.$$

Since M is non-decreasing and convex function, and q is a seminorm and Δ_v^m is linear, we have

$$k^{-s}\left[M\left(\frac{q(\Delta_v^m x_k)}{r}\right)\right]^{p_k}=k^{-s}\left[M\left(\frac{q(\Delta_v^{m-1}x_k-\Delta_v^{m-1}x_{k+1})}{r}\right)\right]^{p_k}$$
$$\leq Kk^{-s}\left[M\left(\frac{q(\Delta_v^{m-1}x_k)}{r}\right)\right]^{p_k}+Kk^{-s}\left[M\left(\frac{q(\Delta_v^{m-1}x_{k+1})}{r}\right)\right]^{p_k}<\infty.$$

So, $x = (x_k) \in l_\infty [\Delta_v^m, M, p, q, s]$ and thus $l_\infty [\Delta_v^{m-1}, M, p, q, s] \subset l_\infty [\Delta_v^m, M, p, q, s]$. Proceeding in this way one will have $l_\infty [\Delta_v^i, M, p, q, s] \subset l_\infty [\Delta_v^m, M, p, q, s]$ for $i = 1, 2, ..., m-1$. ■

To show that the inclusions are strict, consider the following example.

Example 2.8 *Let $X = \mathbf{C}$, $M(x) = x$, $q(x) = |x|$, $s = 0$, $p_k = 1$ for all k and $v = (v_k) = (1, 1, ...)$. Consider the sequence $x = (k^{m-1})$, for example belongs to $Z [\Delta_v^m, M, p, q, s]$ but does not belong to $Z [\Delta_v^{m-1}, M, p, q, s]$ for $Z = c_o$ since $\Delta^m x_k = 0$ and $\Delta^{m-1} x_k = (-1)^{m-1}(m-1)!$ for all $k \in \mathbb{N}$. Under the above restrictions,consider the sequence $x = (k^m)$. Then $x \in Z [\Delta_v^m, M, p, q, s]$ but $x = (k^{m-1}) \notin Z [\Delta_v^{m-1}, M, p, q, s]$ for $Z = c$ and l_∞.*

Theorem 2.9 *For any two sequences $p = (p_k)$ and $u = (u_k)$ of strictly positive real numbers and for any two seminorms q_1 and q_2 on X, we have $Z [\Delta_v^m, M, p, q_1, s] \cap Z [\Delta_v^m, M, u, q_2, s] \neq \varnothing$, for $Z = c_o$, c and l_∞.*

Proof. Since the zero element belongs to each of the above sequence spaces, the intersection is non-empty. ■

Theorem 2.10 *Let M be an Orlicz function. Then*

(a) *For two seminorms q_1 and q_2, if q_1 is stronger than q_2, then $Z [\Delta_v^m, M, p, q_1, s] \subset Z [\Delta_v^m, M, p, q_2, s]$,*

(b) *Let $0 < \inf p_k \leq p_k \leq 1$. Then $Z [\Delta_v^m, M, p, q, s] \subset Z [\Delta_v^m, M, q, s]$,*

(c) *Let $1 \leq p_k \leq \sup_k p_k \leq \infty$. Then $Z [\Delta_v^m, M, q, s] \subset Z [\Delta_v^m, M, p, q, s]$,*

(d) *Let $s_1 \leq s_2$. Then $Z [\Delta_v^m, M, p, q, s_1] \subset Z [\Delta_v^m, M, p, q, s_2]$ for $Z = c_o$, c or l_∞.*

Proof. Proof the theorem is easy, so we omit it. ■

3 Open Problem

In this paper, we introduce some new generalized difference sequence spaces using by Orlicz function. We propose to study various some topological properties and establish some inclusion relations between these spaces. The researchers can be characterize different paranorms on these spaces and try to completeness of these spaces.

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