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Prime Cordial Labeling For Some Cycle Related Graphs

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Abstract

We present here prime cordial labeling for the graphs obtained by some graph operations on cycle related graphs.

Keywords: *prime cordial labeling, prime cordial graph, duplication, path union, friendship graph.*

1 Introduction

If the vertices of the graph are assigned values subject to certain conditions is known as *graph labeling*. A dynamic survey on graph labeling is regularly updated by Gallian [2] and it is published by Electronic Journal of Combinatorics. Vast amount of literature is available on different types of graph labeling and more than 1000 reserch papers have been published so far in past three decades.

For any labeling problems following three characteristics are really noteworthy

◇ A set of numbers from which vertex labels are chosen;
◇ A rule that assigns a value to each edge;
◇ A condition that these values must satisfy.

The present work is aimed to discuss one such a labeling namely prime cordial labeling.

We begin with simple,finite,connected and undirected graph $G = (V(G), E(G))$

with p vertices and q edges. For all other terminology and notations in graph theory we follow West [6]. We will give brief summary of definitions which are useful for the present investigations.

Definition 1.1 *Let $G = (V(G), E(G))$ be a graph. A mapping $f : V(G) \longrightarrow \{0, 1\}$ is called* binary vertex labeling *of G and $f(v)$ is called the* label *of the vertex v of G under f.*

For an edge $e = uv$, the induced edge labeling $f^* : E(G) \longrightarrow \{0, 1\}$ is given by $f^*(e) = |f(u) - f(v)|$. Let $v_f(0), v_f(1)$ be the number of vertices of G having labels 0 and 1 respectively under f and let $e_f(0), e_f(1)$ be the number of edges having labels 0 and 1 respectively under f^*.

Definition 1.2 *A binary vertex labeling of a graph G is called a* cordial labeling *if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. A graph G is* cordial *if it admits cordial labeling.*

This concept was introduced by Cahit [1] as a weaker version of graceful and harmonious graphs. After this many researchers have investigated graph families or graphs which admit cordial labeling. Some labeling schemes are also introduced with minor variations in cordial theme. Some of them are product cordial labeling, total product cordial labeling and prime cordial labeling. The present work is focused on prime cordial labeling which is defined as follows.

Definition 1.3 *A prime cordial labeling of a graph G with vertex set $V(G)$ is a bijection $f : V(G) \longrightarrow \{1, 2, 3, \ldots, p\}$ defined by*
$f(e = uv) = 1;\quad$ *if* $gcd(f(u), f(v)) = 1$
$\qquad\qquad = 0;\quad$ *otherwise*
and $|e_f(0) - e_f(1)| \leq 1$. A graph which admits prime cordial labeling is called a prime cordial graph.

The concept of prime cordial labeling was introduced by Sundaram et al.[4] and in the same paper they investigate several results on prime cordial labeling. Vaidya and Vihol [5] have also discussed prime cordial labeling in the context of graph operations. In the present work we will investigate some new prime cordial graphs.

Definition 1.4 Duplication *of a vertex v_k by a new edge $e = v_k^{'} v_k^{''}$ in a graph G produces a new graph $G^{'}$ such that $N(v_k^{'}) \cap N(v_k^{''}) = v_k$.*

Definition 1.5 Duplication *of an edge $e = uv$ by a new vertex w in a graph G produces a new graph $G^{'}$ such that $N(w) = \{u, v\}$.*

Definition 1.6 *(Shee and Ho[3]) Let graphs $G_1,\ G_2, ..., G_n,\ n \geq 2$ be all copies of a fixed graph G. Adding an edge between G_i to G_{i+1} for $i = 1, 2, ..., n-1$ is called the* path union *of G.*

Definition 1.7 *A* Friendship graph *F_n is a one point union of n copies of cycle C_3.*

2 Main Results

Theorem 2.1 *The graph obtained by duplicating each edge by a vertex in cycle C_n admits prime cordial labeling except for $n = 4$.*

Proof: If $C_n^{'}$ be the graph obtained by duplicating an edge by a vertex in a cycle C_n then let $v_1,\ v_2,\ \ldots,\ v_n$ be the vertices of cycle C_n and $v_1^{'},\ v_2^{'},\ \ldots,\ v_n^{'}$ be the added vertices to obtain $C_n^{'}$ corresponding to the vertices $v_1,\ v_2,\ \ldots,\ v_n$ in C_n.
Define $f : V(C_n^{'}) \longrightarrow \{1, 2, 3\ldots, 2p\}$, we consider following two cases.
Case 1: n is odd
Sub Case 1: $n = 3, 5$
The prime cordial labeling of $C_n^{'}$ for $n = 3, 5$ is as shown in *Figure* 1.

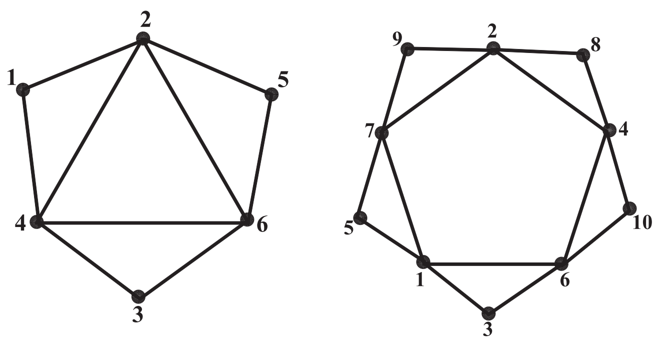

Fig 1 Prime cordial labeling of $C_3^{'}$ and $C_5^{'}$

Sub Case 2: $n \geq 7$
$f(v_1) = 2,\ f(v_2) = 4,$
$f(v_{2+i}) = 6 + 2i; \qquad 1 \leq i \leq \lfloor \frac{n}{2} \rfloor - 2$
$f(v_{\frac{n+1}{2}}) = 6,$
$f(v_{\frac{n+1}{2}+1}) = 1,$
$f(v_{\lfloor \frac{n}{2} \rfloor+2+i}) = 4i + 3; \qquad 1 \leq i \leq \lfloor \frac{n}{2} \rfloor - 1$
$f(v_i^{'}) = f(v_{\lfloor \frac{n}{2} \rfloor}) + 2i; \qquad 1 \leq i \leq \lfloor \frac{n}{2} \rfloor$
$f(v_{\lfloor \frac{n}{2} \rfloor+1}^{'}) = 3,$
$f(v_{\lfloor \frac{n}{2} \rfloor+1+i}^{'}) = 4i + 1; \qquad 1 \leq i \leq \lfloor \frac{n}{2} \rfloor$

In the view of the labeling pattern defined above we have
$e_f(0) + 1 = e_f(1) = 3\lfloor \frac{n}{2} \rfloor + 2$
Case 2: n is even
Sub Case 1: $n = 4$
For the graph $C_4^{'}$ the possible pairs of labels of adjacent vertices are $(1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (1, 7), (1, 8), (2, 3), (2, 4), (2, 5), (2, 6), (2, 7), (2, 8), (3, 4), (3, 5), (3, 6), (3, 7), (3, 8), (4, 5), (4, 6), (4, 7), (4, 8), (5, 6), (5, 7), (5, 8), (6, 7), (6, 8), (7, 8)$. Then obviously $e_f(0) = 5, e_f(1) = 7$. That is, $e_f(1) - e_f(0) = 2$ and in all other possible arrangement of vertex labels $|e_f(0) - e_f(1)| \geq 2$. Thus $C_4^{'}$ is not a prime cordial graph.
Sub Case 2: $n = 6, 8, 10$
The prime cordial labeling of $C_6^{'}$, $C_8^{'}$ and $C_{10}^{'}$ is as shown in *Figure* 2.

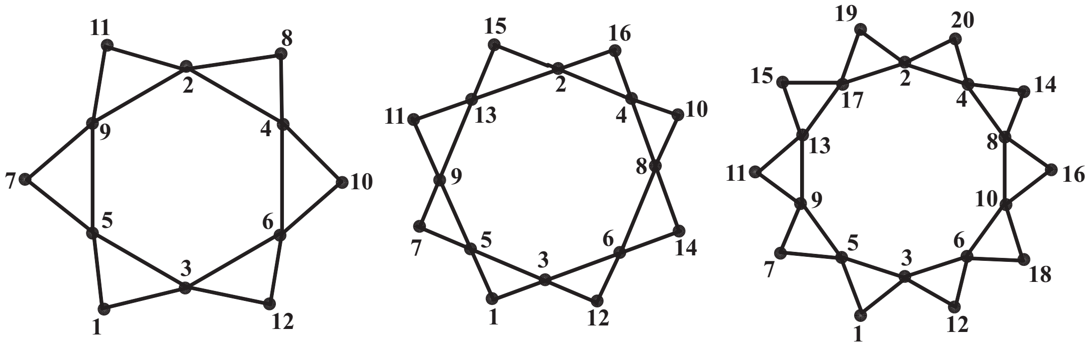

Fig 2 Prime cordial labeling of $C_6^{'}$, $C_8^{'}$ *and* $C_{10}^{'}$

Sub Case 3: $n \geq 12$
$f(v_1) = 2,\ f(v_2) = 4,\ f(v_3) = 8,\ f(v_4) = 10,\ f(v_5) = 14,$
$f(v_{5+i}) = 14 + 2i; \qquad 1 \leq i \leq \frac{n}{2} - 6$
$f(v_{\frac{n}{2}}) = 6,$
$f(v_{\frac{n+1}{2}+1}) = 3,$
$f(v_{\frac{n}{2}+1+i}) = 4i + 1; \qquad 1 \leq i \leq \frac{n}{2} - 1$
$f(v_1^{'}) = 2n$
$f(v_{1+i}^{'}) = f(v_{\frac{n}{2}-1}) + 2i; \quad 1 \leq i \leq \frac{n}{2} - 2$
$f(v_{\frac{n}{2}}^{'}) = 12$
$f(v_{\frac{n}{2}+1}^{'}) = 1$
$f(v_{\frac{n}{2}+1+i}^{'}) = 4i + 3; \qquad 1 \leq i \leq \frac{n}{2} - 1$
In the view of the labeling above defined we have
$e_f(0) = e_f(1) = \frac{3n}{2}$
Thus in the above two cases we have $|e_f(0) - e_f(1)| \leq 1$
Hence the graph obtained by duplicating each edge by a vertex in a cycle C_n admits prime cordial labeling except for $n = 4$.

Example 2.2 *Consider the graph C'_{12}. The labeling is as shown in Figure 3.*

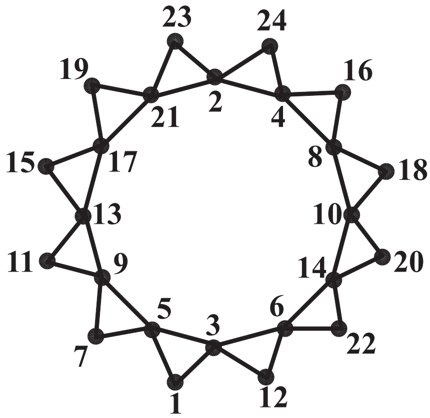

Fig 3 Prime cordial labeling of C'_{12}

Theorem 2.3 *The graph obtained by duplicating a vertex by an edge in cycle C_n is prime cordial graph.*

Proof: If C'_n be the graph obtained by duplicating a vertex by an edge in cycle C_n then let $v_1, v_2, \ldots, v_n$ be the vertices of cycle C_n and $v'_1, v'_2, \ldots, v'_{2n}$ be the added vertices to obtain C'_n corresponding to the vertices $v_1, v_2, \ldots, v_n$ in C_n.
To define $f : V(C'_n) \longrightarrow \{1, 2, 3 \ldots, 3p\}$, we consider following two cases.

<u>**Case 1:** n is odd</u>
<u>**Sub Case 1:** $n = 3, 5$</u>
The prime cordial labeling of C'_n for $n = 3, 5$ is shown in *Figure* 4.

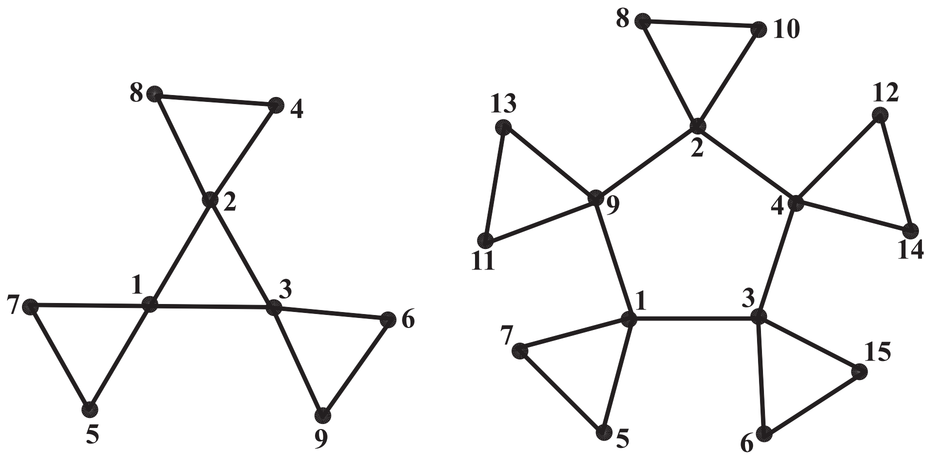

Fig 4 Prime cordial labeling of C'_3 and C'_5

Sub Case 2: $n \geq 7$
$f(v_1) = 2,\ f(v_2) = 4,$
$f(v_{2+i}) = 6 + 2i;$ $\quad 1 \leq i \leq \lfloor \frac{n}{2} \rfloor - 2$
$f(v_{\frac{n+1}{2}}) = 3,$
$f(v_{\frac{n+1}{2}+1}) = 1,$
$f(v_{\lfloor \frac{n}{2} \rfloor+2+i}) = 6i + 5;$ $\quad 1 \leq i \leq \lfloor \frac{n}{2} \rfloor - 1$
$f(v_i^{'}) = f(v_{\lfloor \frac{n}{2} \rfloor}) + 2i;$ $\quad 1 \leq i \leq 2\lfloor \frac{n}{2} \rfloor$
$f(v_{2\lfloor \frac{n}{2} \rfloor+1}^{'}) = 6,\ f(v_{2\lfloor \frac{n}{2} \rfloor+2}^{'}) = 9$
$f(v_{2\lfloor \frac{n}{2} \rfloor+3}^{'}) = 5,\ f(v_{2\lfloor \frac{n}{2} \rfloor+4}^{'}) = 7$
$f(v_{2\lfloor \frac{n}{2} \rfloor+4+2i-1}^{'}) = 6i + 7;$ $\quad 1 \leq i \leq \lfloor \frac{n}{2} \rfloor - 1$
$f(v_{2\lfloor \frac{n}{2} \rfloor+4+2i}^{'}) = 6i + 9;$ $\quad 1 \leq i \leq \lfloor \frac{n}{2} \rfloor - 1$
Case 2: n is even
Sub Case 1: $n = 4, 6$
The prime cordial labeling of $C_n^{'}$ for $n = 4, 6$ is shown in *Figure* 5.

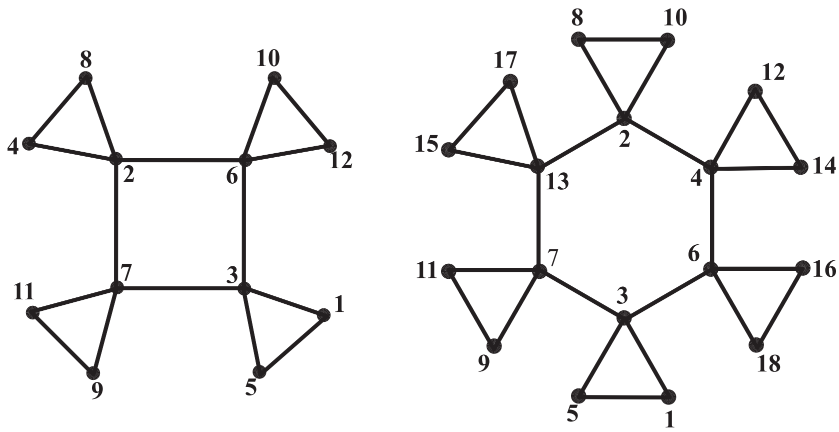

Fig 5 Prime cordial labeling of $C_4^{'}$ and $C_6^{'}$

Sub Case 2: $n \geq 8$
$f(v_1) = 2,\ f(v_2) = 4,$
$f(v_{2+i}) = 6 + 2i;$ $\quad 1 \leq i \leq \frac{n}{2} - 3$
$f(v_{\frac{n}{2}}) = 6,$
$f(v_{\frac{n}{2}+1}) = 3,$
$f(v_{\frac{n}{2}+1+i}) = 6i + 1;$ $\quad 1 \leq i \leq \frac{n}{2} - 1$
$f(v_i^{'}) = f(v_{\frac{n}{2}-1}) + 2i;$ $\quad 1 \leq i \leq n$
$f(v_{n+1}^{'}) = 1,\ f(v_{n+2}^{'}) = 5$
$f(v_{n+1+2i}^{'}) = 6i + 3;$ $\quad 1 \leq i \leq \frac{n}{2} - 1$
$f(v_{n+2+2i}^{'}) = 6i + 5;$ $\quad 1 \leq i \leq \frac{n}{2} - 1$

Thus in both the cases defined above we have
$e_f(0) = e_f(1) = 2n$
Hence $C_n^{'}$ admits prime cordial labeling.

Example 2.4 *Consider the graph C_7'. The labeling is as shown in Figure 6.*

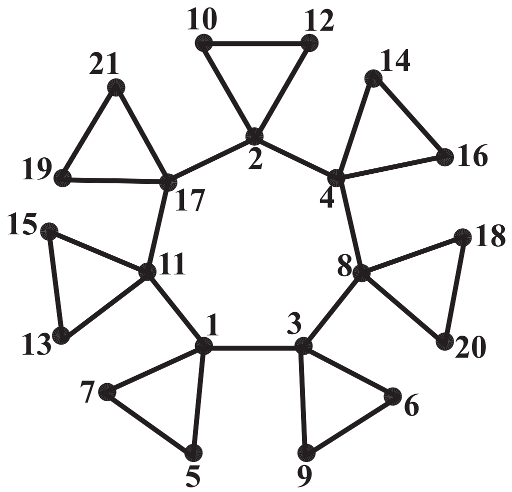

Fig 6 Prime cordial labeling of C_7'

Theorem 2.5 *The path union of m copies of cycle C_n is a prime cordial graph.*

Proof: Let G' be the path union of m copies of cycle C_n and v_1, v_2, v_3, $v_4 \ldots, v_{mn}$ be the vertices of G'.
To define $f : V(G') \longrightarrow \{1, 2, 3 \ldots, mn\}$ we consider following four cases.
Case 1: n even, m even
$f(v_i) = 2i; \quad 1 \le i \le \frac{mn}{2}$
$f(v_{\frac{mn}{2}+1}) = 1,$
$f(v_{\frac{mn}{2}+1+i}) = 4i - 1; \quad 1 \le i \le \frac{n}{2}$
$f(v_{\frac{mn}{2}+\frac{n}{2}+2}) = f(v_{\frac{mn}{2}+\frac{n}{2}+1}) - 2,$
$f(v_{\frac{mn}{2}+\frac{n}{2}+2+i}) = f(v_{\frac{mn}{2}+\frac{n}{2}+2}) - 4i; \quad 1 \le i \le \frac{n}{2} - 2$
$f(v_{\frac{mn}{2}+jn+i}) = f(v_{\frac{mn}{2}+(j-1)n+i}) + 2n; \quad 1 \le j \le \frac{m}{2} - 1, 1 \le i \le n$
Case 2: n odd, m even
$f(v_i) = 2i; \quad 1 \le i \le \frac{mn}{2}$
$f(v_{\frac{mn}{2}+1}) = 1,$
$f(v_{\frac{mn}{2}+1+i}) = 4i - 1; \quad 1 \le i \le \lfloor \frac{n}{2} \rfloor$
$f(v_{\frac{mn}{2}+\lfloor \frac{n}{2} \rfloor+2}) = f(v_{\frac{mn}{2}+\frac{n}{2}+1}) + 2,$
$f(v_{\frac{mn}{2}+\lfloor \frac{n}{2} \rfloor+2+i}) = f(v_{\frac{mn}{2}+\lfloor \frac{n}{2} \rfloor+2}) - 4i; \quad 1 \le i \le \lfloor \frac{n}{2} \rfloor - 1$
$f(v_{\frac{mn}{2}+jn+i}) = f(v_{\frac{mn}{2}+(j-1)n+i}) + 2n; \quad 1 \le j \le \frac{m}{2} - 1, 1 \le i \le n$
using above pattern we have $e_f(0) + 1 = e_f(1) = \frac{m(n+1)}{2}$
Case 3: n even, m odd
$f(v_1) = 4, \ f(v_2) = 8,$
$f(v_{2+i}) = 8 + 2i; \quad 1 \le i \le n\lfloor \frac{m}{2} \rfloor - 2$
$f(v_{n\lfloor \frac{m}{2} \rfloor+1}) = 2$

$f(v_{n\lfloor\frac{m}{2}\rfloor+1+i}) = f(v_{n\lfloor\frac{m}{2}\rfloor}) + 2i;$ $1 \leq i \leq \frac{n}{2} - 2$
$f(v_{n\lfloor\frac{m}{2}\rfloor+\frac{n}{2}}) = 6,$
$f(v_{n\lfloor\frac{m}{2}\rfloor+\frac{n}{2}+1}) = 3,\ f(v_{n\lfloor\frac{m}{2}\rfloor+\frac{n}{2}+2}) = 1$
$f(v_{n\lfloor\frac{m}{2}\rfloor+\frac{n}{2}+2+i}) = 2i + 3$ $1 \leq i \leq \frac{n}{2} - 2$
$f(v_{n\lfloor\frac{m}{2}\rfloor+n+1}) = f(v_{n\lfloor\frac{m}{2}\rfloor+n}) + 2$ or $f(v_{n\lfloor\frac{m}{2}\rfloor+n+1}) = f(v_{n\lfloor\frac{m}{2}\rfloor+n}) + 4$ for $n = 4$
$f(v_{n\lfloor\frac{m}{2}\rfloor+n+2}) = f(v_{n\lfloor\frac{m}{2}\rfloor+n+1}) + 2,$
$f(v_{n\lfloor\frac{m}{2}\rfloor+n+2+i}) = f(v_{n\lfloor\frac{m}{2}\rfloor+n+1}) + 4i,$ $1 \leq i \leq \frac{n}{2} - 1$
$f(v_{n\lfloor\frac{m}{2}\rfloor+n+\frac{n}{2}+2}) = f(v_{n\lfloor\frac{m}{2}\rfloor+n+\frac{n}{2}+1}) + 2$
$f(v_{n\lfloor\frac{m}{2}\rfloor+n+\frac{n}{2}+2+i}) = f(v_{n\lfloor\frac{m}{2}\rfloor+n+\frac{n}{2}+2}) - 4i;$ $1 \leq i \leq \frac{n}{2} - 2$
$f(v_{n\lfloor\frac{m}{2}\rfloor+(j+1)n+i}) = f(v_{n\frac{m}{2}+(j)n+i}) + 2n;$ $1 \leq j \leq \lfloor\frac{m}{2}\rfloor - 1,\ 1 \leq i \leq n$
using above pattern we have $e_f(0) = e_f(1) = \lfloor\frac{m}{2}\rfloor(n+1) + \frac{n}{2}$

Case 4: n odd, m odd

Sub Case 1: $n = 3$

$f(v_1) = 2,\ f(v_2) = 4,$
$f(v_{2+i}) = 6 + 2i;$ $1 \leq i \leq n\lfloor\frac{m}{2}\rfloor - 2$
$f(v_{n\lfloor\frac{m}{2}\rfloor+1}) = 6,$
$f(v_{n\lfloor\frac{m}{2}\rfloor+2}) = 3,$
$f(v_{n\lfloor\frac{m}{2}\rfloor+3}) = 5,$
$f(v_{n\lfloor\frac{m}{2}\rfloor+4}) = 1,$
$f(v_{n\lfloor\frac{m}{2}\rfloor+4+i}) = 2i + 3;$ $1 \leq i \leq n\lfloor\frac{m}{2}\rfloor - 1$
using above pattern we have $e_f(0) + 1 = e_f(1) = \lfloor\frac{m}{2}\rfloor(n+1) + 2$

Sub Case 2: $n \geq 5$

$f(v_1) = 4,\ f(v_2) = 8,$
$f(v_{2+i}) = 8 + 2i;$ $1 \leq i \leq n\lfloor\frac{m}{2}\rfloor - 2$
$f(v_{n\lfloor\frac{m}{2}\rfloor+1}) = 2,$
$f(v_{n\lfloor\frac{m}{2}\rfloor+1+i}) = f(v_{n\lfloor\frac{m}{2}\rfloor}) + 2i;$ $1 \leq i \leq \lfloor\frac{n}{2}\rfloor - 2$
$f(v_{n\lfloor\frac{m}{2}\rfloor+\lfloor\frac{n}{2}\rfloor}) = 6,$
$f(v_{n\lfloor\frac{m}{2}\rfloor+\lfloor\frac{n}{2}\rfloor+1}) = 3,$
$f(v_{n\lfloor\frac{m}{2}\rfloor+\lfloor\frac{n}{2}\rfloor+2}) = 1,$
$f(v_{n\lfloor\frac{m}{2}\rfloor+\lfloor\frac{n}{2}\rfloor+2+i}) = 2i + 3,$ $1 \leq i \leq \lfloor\frac{n}{2}\rfloor - 1$
$f(v_{n\lfloor\frac{m}{2}\rfloor+n+1}) = f(v_{n\lfloor\frac{m}{2}\rfloor+n}) + 2,$
$f(v_{n\lfloor\frac{m}{2}\rfloor+n+1+i}) = f(v_{n\lfloor\frac{m}{2}\rfloor+n+1}) + 2i,\ 1 \leq i \leq n - 1$
$f(v_{n\lfloor\frac{m}{2}\rfloor+(j+1)n+i}) = f(v_{n\frac{m}{2}+(j)n+i}) + 2n;$ $1 \leq j \leq \left\lfloor\frac{m}{2}\right\rfloor - 1,\ 1 \leq i \leq n$

using above pattern we have $e_f(0) + 1 = e_f(1) = \lfloor\frac{m}{2}\rfloor(n+1) + \lfloor\frac{n}{2}\rfloor + 1$

Thus in all the above cases we have $|e_f(0) - e_f(1)| \leq 1$.
Hence G' admits prime cordial labeling.

Example 2.6 *Consider the path union of three copies of C_7. The labeling is as shown in Figure 7.*

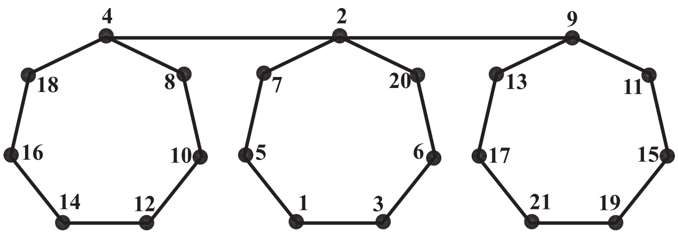

Fig 7 Prime cordial labeling of C_7'

Theorem 2.7 *The friendship graph F_n is a prime cordial graph for $n \geq 3$.*

Proof: Let v_1 be the vertex common to all the cycles. Without loss of generality we start the label assignment from v_1.
To define $f : V(F_n) \longrightarrow \{1, 2, 3 \ldots, 2n + 1\}$, we consider following two cases.
Case 1: n even
let p be the highest prime such that $3p \leq 2n + 1$,
$f(v_1) = 2p$,
now label the remaining vertices from 1 to $2n + 1$ except $2p$.
In the view of the labeling pattern defined above we have
$e_f(0) = e_f(1) = \frac{3n}{2}$

Case 2: n odd
let p be the highest prime such that $2p \leq 2n + 1$,
$f(v_1) = 2p$,
now label the remaining vertices from 1 to $2n + 1$ except $2p$.
In the view of the labeling above defined we have
$e_f(0) + 1 = e_f(1) = 3\lfloor \frac{n}{2} \rfloor + 2$
Thus in above two cases $|e_f(0) - e_f(1)| \leq 1$
Hence friendship graph admits prime cordial labeling.

Example 2.8 *Consider the friendship graph F_8. The labeling is as shown in figure 8.*

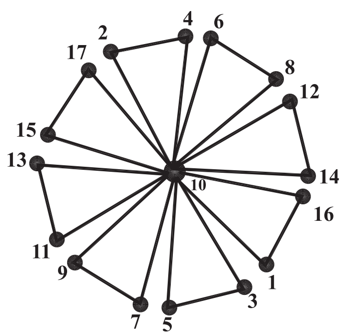

Fig 8 Prime cordial labeling of F_8

3 Concluding Remarks

Labeling of discrete structure is a potential area of research due to its diversified applications and it is very interesting to investigate whether any graph or graph family admit a particular labeling or not? Here we contribute four results in the context of prime cordial labeling. Shee and Ho[3] have proved that the path union of cycles admits cordial labeling while we show that the path union of cycles admits prime cordial labeling.

4 Open Problem

- Analogous results can be investigated for various graph families.

- Similar results can be obtained in the context of different graph labeling techniques

5 Acknowledgement

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