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On Equivalence of p-Adic 2-Norms in p-Adic Linear 2-Normed Spaces

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Abstract

In this paper, we investigate some properties of p-adic linear 2-normed spaces and obtained necessary and sufficient conditions for p-adic 2-norms to be equivalent on p-adic linear 2-normed spaces.

Keywords: 2-*normed spaces, p-adic numbers, p-adic norm, p-adic* 2-*norm, p-adic linear* 2-*normed space.*

1 Introduction

Kummer, in 1850, first introduced to p-adic numbers. Then the German Mathematician, Kurt Hensel (1861-1941) developed the p-adic numbers in a paper which was concerned with the development of algebraic numbers in power series, around the end of the nineteenth century, in 1897. Then p-adic numbers were generalized to ordinals (or valuation) by Kürschak in 1913, and Minkowski (1884), Tate (1960), Kubota-Leopoldt (1964), Iwasawa, Serre, Mazur, Manin, Katz, and the others. There are numbers of all kinds such as rational, real, complex, p-adic numbers. Hensel's p-adic's numbers are now widely used in many fields such as analysis, physics and computer science. The p-adic numbers are less well known than the others, but they play a fundamental role in number theory in other parts of mathematics. Although, they have penatrated several mathematical fields, among them, number theory, algebraic geometry, algebraic topology and analysis. These numbers are now well-established in mathematical world and used more and more by physicists as well. Over the last century p-adic numbers and p-adic analysis have come to play an important role in number theory. They have many applications in mathematics, for example: Representation theory, algebraic geometry, and modern number theory and many applications in mathematical physics since 1897, for example; String theory, QFT, quantum mechanics, dynamical systems, complex systems, etc. Recently, Branko Dragovich in his study ([5]) he constructed p-adic approach to the genetic code and the genome and gave a new approach between p-adic fields and biology with chemistry, especially organic chemistry. The other researchers gave the different approach with p-adic on various disciplines of mathematics and its allied subjects.

The concept of linear 2-normed spaces has been investigated by Gähler in 1965 ([9]) and has been developed extensively in different subjects by others. Lewandowska published a series of papers on 2-normed sets and generalized 2-normed spaces in 1999-2003 ([15]-[17]). In this paper we will not give a detailed information about p-adic number fields but we shall start with a review of p-adic numbers (see ([1], [2], [3], [7], [8], [10], [11], [12], [13], [14], [20]) for more details) and 2-normed spaces and related concepts such as generalized 2-normed spaces, convergent sequences, 2-Banach spaces, etc., (see ([4], [6], [9], [15], [16], [17], [21]) for more details).

Mehmet Acikgoz ([18]) introduced a very understandable and readable connection between the concepts in p-adic numbers, p-adic analysis and linear 2-normed spaces.

The main aim of this paper is to investigate some properties of p-adic linear 2-normed spaces and obtain necessary and sufficient conditions for p-adic 2-norms to be equivalent on p-adic linear 2-normed spaces.

2 Preliminaries

In this paper, we will use the notations; p for a prime number, Z - the ring of rational integers, Z^+ - the positive integers, Q - the field of rational numbers, R - the field of real numbers, R^+ - the positive real numbers, Z_p - the ring of p-adic rational integers, Q_p - the field of p-adic rational numbers, C - the field of complex numbers and C_p - the p-adic completion of the algebraic closure of Q_p. For each $x \in R$, the absolute value of x is denoted by $|x|$ and defined as $|x|=x$ if $x \geq 0$ and $|x|=-x$ if $x \leq 0$. Thus $|0| = 0$ and $|x| > 0$ if $x \neq 0$. It is not difficult to check that $|x+y| \leq |x|+|y|$ and $|xy| = |x||y|$ for every $x, y \in R$ as usual.

A valuation $v_p : F \longrightarrow R \cup \{\infty\}$ is a function from any field F to the extended real line such that (i) $v_p(ab) = v_p(a) + v_p(b)$ (ii) $v_p(a+b) \geq min\{v_p(a), v_p(b)\}$ (iii) $v_p(0) = \infty$. A consequence of the first property is $v_p(\frac{a}{b}) = v_p(a) - v_p(b)$.

An absolute value is a function $|\bullet|_p : F \longrightarrow R_+$ (where F is any field) such that (i) $|x|_p = 0$ iff $x = 0$ (ii) $|xy|_p = |x|_p|y|_p$ (iii) and one of the following (a) $|x+y|_p \leq |x|_p + |y|_p$ or (b) $|x+y|_p \leq max\{|x|_p, |y|_p\}$.

If an absolute value satisfies the triangle inequality ((iii)(a)) then it is said to be Archimedean and non-Archimedean if it satisfies ultrametric inequality ((iii)(b)).

If $x = p^n \frac{u}{v} \in Q$ and uv is not divisible by p then $v_p(x) = n$. The p-adic absolute value of x is defined by $|x|_p = p^{-v_p(x)}$.

A completion of with respect to $|\bullet|_p$ is a new field denoted Q_p such that every Cauchy sequence with respect to $|\bullet|_p$ converges.

In ([22]), some inequalities are defined as follows:

Suppose that $N(x)$, a non-negative real valued function defined on Q such that $N(0) = 0$, $N(x)$ is positive if $x \neq 0$ and $N(xy) = N(x)N(y)$ for all $x, y \in Q$ and

$$N(x+y) \leq K(N(x) + N(y)) \tag{1}$$

for $K \geq 1$ and $x, y \in Q$. The well-known triangle inequality satisfies for $K = 1$. The other version of the triangle inequality, is the ultrametric which is stronger and is shown by

$$N(x+y) \leq max\{N(x), N(y)\} \tag{2}$$

for all $x, y \in Q$. By using the equation (1), we have

$$N\left(\sum_{k=1}^{2^n} x_k\right) \leq K^n \sum_{k=1}^{2^n} N(x_k) \tag{3}$$

where $n \in Z^+$ and x_k's are in Q. The proof can be easily made by induction over n. The usual absolute value function $|x|$ satisfies these conditions with the well-known triangle inequality. For $x = 0$ and $x \neq 0$, if we have $N(x) = 0$ and $N(x) = 1$ respectively. In this case, $N(x)$ satisfies these conditions with the ultrametric type of the triangle inequality.

If $q \in C$ we assume that $|q| < 1$. If $q \in C_p$, it will be assumed that $N(1-q)_p < p^{-\frac{1}{p-1}}$ with $N(p)_p < p^{-ord_p(p)} = p^{-1}$, where $ord_p(p)$ be the normalized exponential valuation of C_p. We use the function

$$[x] = [x:q] = \frac{1-q^x}{1-q} \ and \ \lim_{q \to 1} \frac{1-q^x}{1-q} = x$$

for any x in the complex case and any x with $N(x)_p \leq 1$ in the p-adic case.

Every $x \in Q$ with $|x|_p \leq 1$ is the limit of a sequence of integers in the p-adic metric. That is; $\{x \in Q : |x|_p \leq 1\}$ is the same as the closure of Z in Q with respect to the p-adic metric. Set $Z_p = \{x \in Q_p : |x|_p \leq 1\}$. Every $x \in Q_p$ is the limit of a sequence of rational numbers in the p-adic metric. Because $\overline{Q} = Q_p$. ($\overline{Q}$ is the closure of Q). It is also that for $x \in Z_p$ in the p-adic metric. So we have $\overline{Z} = Z_p$ in Q_p.

Now let us give a brief knowledge about linear 2-normed spaces by starting their definitions and related facts.

Definition 2.1 *Let X be a linear space of dimension greater than 1 over K, where K is the real or complex numbers field. Suppose $N(\bullet,\bullet)$ be a non-negative real valued function on $X\times X$ satisfying the following conditions:*
$(2N1)$: $N(x,y)>0$ and $N(x,y)=0$ if and only if x and y are linearly dependent vectors,
$(2N2)$: $N(x,y)=N(y,x)$ for all $x,y\in X$,
$(2N3)$: $N(\lambda x,y)=|\lambda|N(x,y)$ for all $\lambda\in K$ and $x,y\in X$,
$(2N4)$: $N(x+y,z)\leq N(x,z)+N(y,z)$ for all $x,y,z\in X$.
Then $N(\bullet,\bullet)$ is called a 2-norm on X and the pair $(X,N(\bullet,\bullet))$ is called a linear 2-normed space.

In addition, for all scalars α and all $x,y,z\in X$, we have the following three properties of 2-norms:
(P_1) They are non-negative,
(P_2) $N(x,y)=N(x,y+\alpha x)$,
(P_3) $N(x-z,y-z)=N(x-y,x-z)$.

Every 2-normed space is a locally convex topological vector space. In fact for a fixed $b\in X$, $p_b(x)=N(x,b)$ for all $x\in X$, is a seminorm and the family $P=\{p_b : b\in X\}$ generates a locally convex topology on X. Such a topology is called the natural topology induced by 2-norm $N(\bullet,\bullet)$.

Definition 2.2 *A sequence $\{x_n\}_{n\geq1}$ in a linear 2-normed space $(X,N(\bullet,\bullet))$ is called Cauchy sequence if there exists two linearly independent elements y and z in X such that $\{N(x_n,y)\}$ and $\{N(x_n,z)\}$ are real Cauchy sequences.*

Definition 2.3 *A sequence $\{x_n\}_{n\geq1}$ in a linear 2-normed space $(X,N(\bullet,\bullet))$ is called convergent if there exists $x\in X$ such that $\{N(x_n-x,y)\}_{n\geq1}$ tends to zero for all $y\in X$.*

Definition 2.4 *A linear 2-normed space $(X,N(\bullet,\bullet))$ is called 2-Banach space if every Cauchy sequence is convergent.*

Lemma 2.5 *(i) Every linear 2-normed space of dimension 2 is a 2-Banach space, when the underlying field is complete.*
(ii) If $\{x_n\}_{n\geq1}$ is a sequence in 2-normed space $(X,N(\bullet,\bullet))$ and $\lim\limits_{n\to\infty}N(x_n-x,y)=0$ then $\lim\limits_{n\to\infty}N(x_n,y)=N(x,y)$.

3 p-adic ordinal, p-adic norm, p-adic metric and p-adic expansion

In this section, we introduce the notions of p-adic ordinal, p-adic norm, p-adic metric (distatnce), p-adic expansion and some related concepts. Let Q be the field of rational numbers, $0 \neq x \in Q$ and p is a fixed prime number. Every rational number can be represented in the form $x = p^n \frac{a}{b}$ with $gcd(a, b) = 1$, $a, n \in Z$, $b \in Z^+$ and neither a nor b is divisible by p (i.e., $(p, a) = 1$, $(p, b) = 1$). The integer n and the rational number $\frac{a}{b}$ are well defined by Fundamental Theorem of Arithmetic.

Now, let us give the definition of p-adic ordinal as follows:

Definition 3.1 *The p-adic ordinal (or valuation) is the function* $ord_p : Q \longrightarrow Z \cup \{\infty\}$ *with* $ord_p(x) = n$ *for* $0 \neq x \in Q$ *and* $ord_p(0) = \infty$.

For all $x, y \in Q$, we have the following some basic facts :
(i) $ord_p(xy) = ord_p(x) + ord_p(y)$,
(ii) $ord_p(x + y) \geq min\{ord_p(x), ord_p(y)\}$ and with equality when $ord_p(x) \neq ord_p(y)$,
(iii) $ord_p(0) = \infty$,
(iv) A clear consequence of the first property is that $ord_p(\frac{x}{y}) = ord_p(x) - ord_p(y)$

By using the ordinal (valuation) function, we can define p-adic norm function on Q as follows:

Definition 3.2 *For* $x \in Q$*, let the p-adic norm of* x *be given by*
$N(x)_p = p^{-ord_p(x)}$, *if* $x \neq 0$
$= p^{-\infty} = 0$, *if* $x = 0$.

The p-adic norm satisfies the following relations :
(i) $N(x)_p \geq 0$ for all x,
(ii) $N(x)_p = 0$ if and only if $x = 0$,
(iii) $N(xy)_p = N(x)_p \; N(y)_p$ for all x and y,
(iv) $N(x + y)_p \leq N(x)_p + N(y)_p$ for all x and y,
(v) $N(x + y)_p \leq max\{N(x)_p, N(y)_p\}$ for all x and y,
(vi) If $N(x)_p \neq N(y)_p$ then $N(x - y)_p = max\{N(x)_p, N(y)_p\}$,
(vii) If $N(x)_p = N(y)_p$ then $N(x - y)_p = N(x)_p$.

In the above, the properties (iv) and (v) are called the triangle inequality and the strong triangle inequality (ultrametric version of the triangle inequality) respectively. We observe that the relation (iv) follows from the relation (v).

By $d_p(x, y) = N(x - y)_p$, we define the p-adic metric (distance) on Q, for fixed a prime number p and $x, y \in Q$, as follows:

Definition 3.3 *For all $x, y, z \in Q$,*
D_1) $d_p(x,y) = N(x-y)_p > 0$ for $x \neq y$ and $d_p(x,x) = 0$,
D_2) $d_p(x,y) = d_p(y,x)$,
D_3) $d_p(x,z) \leq d_p(x,y) + d_p(y,z)$, (the triangle inequality)
D_4) $d_p(x,z) \leq max\{d_p(x,y), d_p(y,z)\}$, (the ultrametric inequality)

The properties given above from (D_1) to (D_3), they are called the axioms of p-adic metric and the pair (X, d_p) is called a p-adic metric space. If the metric also satisfies the (D_4) property then this metric is called a p-adic ultrametric space. Two points are p-adically closer as long as r is higher, such that p^r divides $N(x-y)_p$.

Definition 3.4 *A sequence $\{x_n\}_{n=1}^{\infty}$ of rational numbers converges to $x \in Q$ in p-adic metric if for every $\varepsilon > 0$, there is an $\ell \geq 1$ such that $d_p(x_n, x) = |x_n - x|_p < \varepsilon$ for every $n \geq \ell$.*

For the given two sequences of rational numbers which are $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ converges to $x, y \in Q$ in the p-adic metric respectively, then the sequence of sums $x_n + y_n$ and the product $x_n y_n$ converges to the sum $x+y$ and to the product xy of the limits of initial sequences.

Definition 3.5 *A sequence $\{x_n\}_{n=1}^{\infty}$ of rational numbers is a Cauchy sequence with respect to the p-adic metric if for each $\varepsilon > 0$, there is an $\ell \geq 1$ such that $d_p(x_n, x_m) = |x_n - x_m|_p < \varepsilon$, for every $n, m \geq \ell$.*

Every convergent sequence in Q is a Cauchy sequence. If $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in Q with respect to the p-adic metric, then the limit $\lim\limits_{n\to\infty}(x_n - x_{n+1}) = 0$ in p-adic metric. We know that the analogous statement also works for the standard metric $|x-y|$. For the p-adic metric, the converse holds because of the ultrametric version of the triangle inequality.

Definition 3.6 *A p-adic number α can be uniquely written in the canonical series form $\alpha = \sum\limits_{j=n}^{\infty} a_j p^j$, where each of $0 \leq a_j \leq p-1$ and the p-adic norm of the number α is defined as $N(\alpha)_p = p^{-n}$. Note that the series $1+p+p^2+p^3+\ldots$ converges to $\frac{1}{1-p}$ in the p-adic norm.*

Definition 3.7 *(i) The p-adic ordinal (valuation) of x and y, for $0 \neq x, y \in Z$ is*

$$ord_p(x,y) = max\{r : p^r/x \ \ and \ \ p^r/y\} \geq 0$$

(ii) For $\frac{a}{b}, \frac{c}{d} \in Q$, the p-adic value of $\frac{a}{b}$ and $\frac{c}{d}$ is

$$ord_p\left(\frac{a}{b}, \frac{c}{d}\right) = ord_p(a,c) - ord_p(a,d) - ord_p(b,c) + ord_p(b,d)$$

(iii) For $\frac{a}{b}$, $c \in Q$, with $d = 1$, the p-adic value of $\frac{a}{b}$ and c is

$$ord_p\left(\frac{a}{b}, c\right) = ord_p(a, c) - ord_p(b, c)$$

Notice that in all cases, ord_p in 2-norm, gives an integer and that for rational number $\frac{a}{b}$ and $\frac{c}{d}$ the value of $ord_p\left(\frac{a}{b}, \frac{c}{d}\right)$ is well defined. i.e., if $\frac{a}{b} = \frac{a'}{b'}$ and $\frac{c}{d} = \frac{c'}{d'}$ then

$$ord_p\left(\frac{a}{b}, \frac{c}{d}\right) = ord_p\left(\frac{a'}{b'}, \frac{c'}{d'}\right).$$

We also introduce the convention that $ord_p(0, y) = ord_p(x, 0) = \infty$.

The p-adic valuation has the following properties:

Proposition 3.8 *For all $x, y \in Q$, we have for ord_p;*
(i) $ord_p(x, y) = \infty$ iff $x = 0$ or $y = 0$,
(ii) $ord_p(xz, y) = ord_p(x, y) + ord_p(z, y)$,
(iii) $ord_p(x + z, y) \geq min\{ord_p(x, y), ord_p(z, y)\}$ and with equality when $ord_p(x, y) \neq ord_p(z, y)$.

Definition 3.9 *For all $x, y \in Q$, let the p-adic norm of x, y be given by*
$N(x, y)_p = p^{-ord_p(x,y)}$, if $x \neq 0$ and $y \neq 0$
$= p^{-\infty} = 0$, if $x = 0$ or $y = 0$
where $ord_p(x, y) = max\{r : p^r/x$ and $p^r/y\}$.

Proposition 3.10 *Let the function $N(\bullet, \bullet)_p$ be a non-negative real valued function on $Q \times Q$ satisfying the following conditions:*

$$N(\bullet, \bullet)_p : Q \times Q \longrightarrow R^+ \cup \{0\} = \{r : r \geq 0\}$$

(i) $N(x, z)_p = 0$ if and only if $x = 0$ or $z = 0$,
(ii) $N(xy, z)_p = N(x, z)_p\ N(y, z)_p$ for all x, y and $z \in Q$,
(iii) $N(x+y, z)_p \leq max\{N(x, z)_p, N(y, z)_p\}$ and with equality when $N(x, z)_p \neq N(y, z)_p$.
where $N(\bullet, \bullet)_p$ is a non-Archimedean norm on Q.

Let $N(x, z)_p$ be a non-negative real valued function defined on the rational numbers $Q \times Q$ such that $N(x, z)_p = 0$ for $x = 0$ or $z = 0$, $N(x, z)_p > 0$ when $x \neq 0$, $z \neq 0$. $N(xy, z)_p = N(x, z)_p\ N(y, z)_p$ for all $x, y, z \in Q$ and

$$N(x + y, z)_p \leq K(N(x, z)_p + N(y, z)_p) \tag{4}$$

for some $K \geq 1$ and all $x, y, z \in Q$. For the usual triangle inequality one ask that this condition holds with $K = 1$, i.e.,

$$N(x + y, z)_p \leq (N(x, z)_p + N(y, z)_p) \tag{5}$$

for all $x, y, z \in Q$.

The ultrametric version of the triangle inequality is stronger still and asks that

$$N(x+y,z)_p \leq max(N(x,z)_p, N(y,z)_p) \tag{6}$$

for all $x, y, z \in Q$. If $N(\bullet,\bullet)_p$ satisfies the equation (4), n is a positive integer and $x_1, x_2, x_3, \ldots, x_{2^n} \neq 0$, $z \in Q$, then

$$N\left(\sum_{k=1}^{2^n} x_k, z\right)_p \leq K^n \sum_{k=1}^{2^n} N(x_k, z)_p \tag{7}$$

as one can check using induction on n. For all $a > 0$, $N(x,z)_p^a$ is a non-negative real valued function on $Q \times Q$ which vanished at 0, is positive at all nonzero $x \in Q$ and sends products to products. If $N(x,z)_p$ satisfies the equation (4), then

$$N(x+y,z)_p^a \leq K^a\Big(N(x,z)_p^a + N(y,z)_p^a\Big) \tag{8}$$

when $0 < a \leq 1$ and

$$N(x+y,z)_p^a \leq 2^{a-1}K^a\Big(N(x,z)_p^a + N(y,z)_p^a\Big) \tag{9}$$

when $a \geq 1$.

In particular, if $N(x,z)_p$ satisfies the well-known triangle inequality (5) and $0 < a \leq 1$, then $N(x,z)_p^a$ also satisfies the the well-known triangle inequality. If $N(x,z)_p$ satisfies the ultrametric version (6) of the triangle inequality, then $N(x,z)_p^a$ satisfies the ultrametric version of the triangle inequality for all $a \geq 0$.

Definition 3.11 *Let X be a linear space of dimension greater than 1 over K, where K is the real or complex numbers field. Suppose $N(\bullet,\bullet)_p$ be a non-negative real valued function on $X \times X$ satisfying the following conditions:*
$(2-pN_1)$: $N(x,z)_p = 0$ if and only if x and z are linearly dependent vectors,
$(2-pN_2)$: $N(xy,z)_p = N(x,z)_p\ N(y,z)_p$ for all $x, y, z \in X$,
$(2-pN_3)$: $N(x+y,z)_p \leq N(x,z)_p + N(y,z)_p$ for all $x, y, z \in X$,
$(2-pN_4)$: $N(\lambda x,z)_p = |\lambda|\ N(x,z)_p$ for all $\lambda \in K$ and $x, z \in X$.
Then $N(\bullet,\bullet)_p$ is called a p-adic 2-norm on X and the pair $(X, N(\bullet,\bullet)_p)$ is called p-adic linear 2-normed space.

Definition 3.12 *A sequence $\{x_n\}_{n\geq 1}$ in a p-adic linear 2-normed space $(X, N(\bullet,\bullet)_p)$ is called convergent if there exists an $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, z)_p = 0$ for all $z \in X$.*

Definition 3.13 *A sequence $\{x_n\}_{n\geq 1}$ in a p-adic linear 2-normed space $(X, N(\bullet,\bullet)_p)$ is called Cauchy sequence if for each $\varepsilon > 0$, there is an $\ell \geq 1$ such that $N(x_n - x_m, z)_p < \varepsilon$, for all $n, m \geq \ell$ and for all $z \in X$.*

Definition 3.14 *A p-adic linear* 2*-normed space* $(X, N(\bullet,\bullet)_p)$ *is called complete if every Cauchy sequence is convergent in p-adic linear* 2*-normed space.*

Definition 3.15 *A p-adic linear* 2*-normed space* $(X, N(\bullet,\bullet)_p)$ *is called p-adic* 2*-Banach space if every p-adic linear* 2*-normed space is complete.*

Proposition 3.16 *If* $\lim_{n\to\infty} N(x_n, z)_p$ *exists then we say that* $\{x_n\}_{n\geq 1}$ *is a Cauchy sequence with respect to* $N(\bullet,\bullet)_p$.

Proof. Let us suppose that $\lim_{n\to\infty} N(x_n, z)_p = x$. Then we can obtain a constant M_1 such that $n > M_1 \Rightarrow N(x - x_n, z)_p < \frac{\varepsilon}{2}$. If $m, n > M_1$ then $N(x - x_n, z)_p < \frac{\varepsilon}{2}$ and $N(x - x_m, z)_p < \frac{\varepsilon}{2}$, hence by using the triangle inequality, we have

$$N(x_m - x_n, z)_p = N(x_m - x + x - x_n, z)_p$$

$$\leq N(x_m - x, z)_p + N(x - x_n, z)_p < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Definition 3.17 *A sequence* $\{x_n\}_{n\geq 1}$ *is called a null sequence in p-adic linear* 2*-normed space if* $\lim_{n\to\infty} N(x_n, z)_p = 0$ *for all* $z \in X$.

Example 3.18 *Let* $x_n = p^n$ *and* $z = p^r$ *with* $r < n$ *in the p-adic* 2*-norm over* $X = Q$. *Then*

$N(p^n, p^r)_p = p^{-ord_p(p^n, p^r)}$, *if* $p^n \neq 0$ *and* $p^r \neq 0$

$= p^{-\infty} = 0$, *if* $p^n = 0$ *or* $p^r = 0$.

In this case $N(p^n, p^r)_p = p^{-n} = \frac{1}{p^n} = 0$, *as* $n \to \infty$. *Therefore,* $\lim_{n\to\infty} N(x_n, z)_p = 0$ *for all* $z \in X$. *Hence this sequence is a null sequence with respect to the p-adic* 2*-norm.*

Definition 3.19 *A p-adic number* (α, β) *can be uniquely written in the form*

$$(\alpha, \beta) = \sum_{i=n,\ j=m}^{\infty} (a_i p^i, b_j p^j)$$

where each $0 \leq a_i, b_j \leq p - 1$ *and p-adic* 2*-norm of the number* (α, β) *is defined as* $N(\alpha, \beta)_p = n$, $(n \in R)$ *and the double series* $(1 + p + p^2 + p^3 + \ldots, 1 + p + p^2 + p^3 + \ldots)$ *converges to* $\frac{1}{1-p}$ *in the p-adic* 2*-norm.*

4 Main Results

In this section, we investigate some properties of p-adic linear 2-normed spaces and obtain necessary and sufficient conditions for p-adic 2-norms to be equivalent on p-adic linear 2-normed spaces.

Proposition 4.1 *Let $(X, N(\bullet, \bullet)_p)$ be a p-adic linear 2-normed space.*
(i) If $\{x_n\}_{n\geq 1}$ is a Cauchy sequence in X then $\{N(x_n, z)_p : z \in X\}_{n\geq 1}$ is a Cauchy sequence of non-negative reals.
(ii) If $\{x_n\}_{n\geq 1}$ and $\{y_n\}_{n\geq 1}$ are Cauchy sequences in X and $\{\alpha_n\}_{n\geq 1}$ is a Cauchy sequence of reals then $\{x_n + y_n\}_{n\geq 1}$ and $\{\alpha_n x_n\}_{n\geq 1}$ are Cauchy sequences in X.

Proof. (i) Let $\{x_n\}_{n\geq 1}$ is a Cauchy sequence in X then $\lim\limits_{n,m\to\infty} N(x_n - x_m, z)_p = 0$ for all $z \in X$. We have
$N(x_n, z)_p = N(x_n - x_m + x_m, z)_p \leq N(x_n - x_m, z)_p + N(x_m, z)_p$
$\Rightarrow \quad N(x_n, z)_p - N(x_m, z)_p \leq N(x_n - x_m, z)_p$.
Similarly $N(x_m, z)_p - N(x_n, z)_p \leq N(x_n - x_m, z)_p$.
Combining the above inequalities, we have
$|N(x_n, z)_p - N(x_m, z)_p| \leq N(x_n - x_m, z)_p \longrightarrow 0$ as $n, m \longrightarrow \infty$. Therefore $|N(x_n, z)_p - N(x_m, z)_p| \longrightarrow 0$ as $n, m \longrightarrow \infty$. Hence $\{N(x_n, z)_p : z \in X\}_{n\geq 1}$ is a Cauchy sequence of non-negative reals.
(ii) Let $\{x_n\}_{n\geq 1}$ and $\{y_n\}_{n\geq 1}$ be two Cauchy sequences in X then $\lim\limits_{n,m\to\infty} N(x_n - x_m, z)_p = 0$ for all $z \in X$ and $\lim\limits_{n,m\to\infty} N(y_n - y_m, z)_p = 0$ for all $z \in X$. Now $N((x_n+y_n)-(x_m+y_m), z)_p = N(x_n-x_m+y_n-y_m, z)_p \leq N(x_n - x_m, z)_p + N(y_n - y_m, z)_p \longrightarrow 0$ as $n, m \longrightarrow \infty$. Hence $\{x_n+y_n\}_{n\geq 1}$ is a Cauchy sequence in X. Let $\{\alpha_n\}_{n\geq 1}$ be a Cauchy sequence of reals. Also from (i) we have $\{N(x_n, z)_p : z \in X\}_{n\geq 1}$ is a Cauchy sequence of reals. Hence they are bounded. We can find $K_1, K_2 \geq 0$ such that $|\alpha_n| \leq K_1$ and $N(x_n, z)_p \leq K_2$ for all $z \in X$. We have $N(\alpha_n x_n - \alpha_m x_m, z)_p = N(\alpha_n x_n - \alpha_n x_m + \alpha_n x_m - \alpha_m x_m, z)_p$
$\leq N(\alpha_n x_n - \alpha_n x_m, z)_p + N(\alpha_n x_m - \alpha_m x_m, z)_p$
$\leq |\alpha_n|\ N(x_n - x_m, z)_p + |\alpha_n - \alpha_m|\ N(x_m, z)_p$
$\leq K_1\ N(x_n - x_m, z)_p + K_2\ |\alpha_n - \alpha_m| \longrightarrow 0$ as $n, m \longrightarrow \infty$. Thus $N(\alpha_n x_n - \alpha_m x_m, z)_p \longrightarrow 0$ as $n, m \longrightarrow \infty$. Hence $\{\alpha_n x_n\}_{n\geq 1}$ is a Cauchy sequence in X.

Proposition 4.2 *In any p-adic linear 2-normed space $(X, N(\bullet, \bullet)_p)$, we have the following*
(i) If $\lim\limits_{n\to\infty} N(x_n - x, z)_p = 0$ and $\lim\limits_{n\to\infty} N(y_n - y, z)_p = 0$ then $\lim\limits_{n\to\infty} N((x_n + y_n) - (x + y), z)_p = 0$ for all $z \in X$.
(ii) $\lim\limits_{n\to\infty} N(x_n - x, z)_p = 0$ and $\lim\limits_{n\to\infty} N(\alpha_n - \alpha, z)_p = 0$ then $\lim\limits_{n\to\infty} N(\alpha_n x_n - \alpha x, z)_p = 0$ for all $z \in X$.
(iii) If $dim X \geq 2$ and $\lim\limits_{n\to\infty} N(x_n - x, z)_p = 0$ and $\lim\limits_{n\to\infty} N(x_n - y, z)_p = 0$, for all $z \in X$ then $x = y$.

Proof. (i) We have $N((x_n + y_n) - (x + y), z)_p = N(x_n - x + y_n - y, z)_p$
$\leq N(x_n - x, z)_p + N(y_n - y, z)_p \longrightarrow 0$ as $n \longrightarrow \infty$ for all $z \in X$.
Therefore $N((x_n + y_n) - (x + y), z)_p \longrightarrow 0$ as $n \longrightarrow \infty$.

Hence $\lim_{n\to\infty} N((x_n + y_n) - (x + y), z)_p = 0$ for all $z \in X$.
(ii) Using the fact that a real convergent sequence is boudned, we have
$N(\alpha_n x_n - \alpha x, z)_p = N(\alpha_n x_n - \alpha_n x + \alpha_n x - \alpha x, z)_p$
$\leq N(\alpha_n x_n - \alpha_n x, z)_p + N(\alpha_n x - \alpha x, z)_p \leq |\alpha_n|\ N(x_n - x, z)_p + |\alpha_n - \alpha| N(x, z)_p$
$\leq K\ N(x_n - x, z)_p + |\alpha_n - \alpha| N(x, z)_p$ for some $K \geq 0$.
Therefore $\lim_{n\to\infty} N(\alpha_n x_n - \alpha x, z)_p = 0$ for all $z \in X$, since $\lim_{n\to\infty} N(x_n - x, z)_p = 0$ and $\lim_{n\to\infty} N(\alpha_n - \alpha, z)_p = 0$ and $N(x, z)_p$ is finite.
(iii) We have $N(x - y, z)_p = N(x - x_n + x_n - y, z)_p \leq N(x - x_n, z)_p + N(x_n - y, z)_p$.
It follows that $N(x-y, z)_p = 0$, since $\lim_{n\to\infty} N(x_n - x, z)_p = 0$ and $\lim_{n\to\infty} N(x_n - y, z)_p = 0$ for all $z \in X$. Hence $x - y$ and z are linearly dependent vectors. Since $dim X \geq 2$, the only way that $x - y$ can be linearly dependent with all $z \in X$ is for $x - y = 0 \Rightarrow x = y$

Proposition 4.3 *Let $(X, N(\bullet, \bullet)_p)$ be a p-adic linear 2-normed space. If $\{x_n\}_{n\geq 1}$ is a Cauchy sequence in X then $\{N(x_n - x, z)_p : z \in X\}_{n\geq 1}$ is a Cauchy sequence of non-negative reals for each $x \in X$.*

Proof. We have $N(x_n - x, z)_p = N(x_n - x_m + x_m - x, z)_p$
$\leq N(x_n - x_m, z)_p + N(x_m - x, z)_p$
$\Rightarrow \quad N(x_n - x, z)_p - N(x_m - x, z)_p \leq N(x_n - x_m, z)_p$.
Similarly $N(x_m - x, z)_p - N(x_n - x, z)_p \leq N(x_n - x_m, z)_p$.
Combining the above inequalities, we have
$|N(x_n - x, z)_p - N(x_m - x, z)_p| \leq N(x_n - x_m, z)_p \longrightarrow 0$ as $n, m \longrightarrow \infty$.
Therefore $|N(x_n - x, z)_p - N(x_m - x, z)_p| \longrightarrow 0$ as $n, m \longrightarrow \infty$. Hence $\{N(x_n - x, z)_p : z \in X\}_{n\geq 1}$ is a Cauchy sequence of non-negative reals for each $x \in X$.

Proposition 4.4 *If $\lim_{n\to\infty} N(x_n - x, z)_p = 0$ then $\lim_{n\to\infty} N(x_n, z)_p = N(x, z)_p$ for each $z \in X$.*

Proof. Let $\lim_{n\to\infty} N(x_n - x, z)_p = 0$, we have $|N(x_n, z)_p - N(x, z)_p| \leq N(x_n - x, z)_p \longrightarrow 0$ as $n \longrightarrow \infty$. It follows that, $|N(x_n, z)_p - N(x, z)_p| \longrightarrow 0$ as $n \longrightarrow \infty$. Hence, $\lim_{n\to\infty} N(x_n, z)_p = N(x, z)_p$ for each $z \in X$.

Proposition 4.5 *Limit of every convergent sequence in p-adic linear 2-normed space is unique.*

Proof. The proof is easy, so omitted.

Proposition 4.6 *Every convergent sequence in p-adic linear 2-normed space is a Cauchy sequence.*

Proof. The proof is easy, so omitted.

Now we are ready to give the main Theorem of this paper.

Definition 4.7 *Two p-adic 2-norms $N_1(\bullet,\bullet)_p$ and $N_2(\bullet,\bullet)_p$ on p-adic linear 2-normed space X are said to be equivalent if there exists constants $\alpha > 0$ and $\beta > 0$ such that*
$$\alpha\ N_1(x,z)_p \leq N_2(x,z)_p \leq \beta\ N_1(x,z)_p \text{ for all } x,z \in X.$$

Theorem 4.8 *Two p-adic 2-norms $N_1(\bullet,\bullet)_p$ and $N_2(\bullet,\bullet)_p$ are equivalent on p-adic linear 2-normed space X if and only if every Cauchy sequence with respect to one of the p-adic 2-norm is a Cauchy sequence with respect to other p-adic 2-norm.*

Proof. Suppose that two p-adic 2-norms $N_1(\bullet,\bullet)_p$ and $N_2(\bullet,\bullet)_p$ are equivalent on p-adic linear 2-normed space X. Then there exists constants $\alpha > 0$ and $\beta > 0$ such that
$$\alpha\ N_1(x,z)_p \leq N_2(x,z)_p \leq \beta\ N_1(x,z)_p \text{ for all } x,z \in X.$$
For a sequence $\{x_n\}_{n\geq 1}$ in X, we have
$$\alpha\ N_1(x_n - x_m, z)_p \leq N_2(x_n - x_m, z)_p \leq \beta\ N_1(x_n - x_m, z)_p \quad for\ all\ z \in X \tag{10}$$
The second inequality shows that if $\{x_n\}_{n\geq 1}$ is Cauchy sequence with respect to $N_1(\bullet,\bullet)_p$ if and only if it is a Cauchy sequence with respect to $N_2(\bullet,\bullet)_p$. For the converse part, suppose that the p-adic 2-norms are not equivalent. Then without loss of generality we can assume the following two cases.
Case (i) we can not find α such that $\alpha\ N_1(x,z)_p \leq N_2(x,z)_p$ for all $x,z \in X$.
Case (ii) we can not find β such that $N_2(x,z)_p \leq \beta\ N_1(x,z)_p$ for all $x,z \in X$.
In case (i) for $n = 1,2,\ldots,$ there exists x_n in X such that
$$\frac{1}{n}\ N_1(x_n,z)_p > N_2(x_n,z)_p \tag{11}$$
Let $y_n = \frac{1}{\sqrt{n}}\frac{1}{N_2(x_n,z)_p}x_n$, for each n. Then $N_2(y_n,z)_p = \frac{1}{\sqrt{n}} \longrightarrow 0$ as $n \longrightarrow \infty$ and using equation (11) we get
$$N_1(y_n,z)_p = \frac{1}{\sqrt{n}}\frac{1}{N_2(x_n,z)_p}N_1(x_n,z)_p > \frac{n}{\sqrt{n}} = \sqrt{n} \longrightarrow \infty,$$
as $n \longrightarrow \infty$ So, using Proposition 4.6, $\{y_n\}_{n\geq 1}$ is a Cauchy sequence with respect $N_2(\bullet,\bullet)_p$ but not with respect $N_1(\bullet,\bullet)_p$.
Similarly, we can prove case (ii). Hence the theorem.

Corollary 4.9 *Let $N_1(\bullet,\bullet)_p$ and $N_2(\bullet,\bullet)_p$ be two equivalent p-adic 2-norms on p-adic linear 2-normed space X, then $x_n \longrightarrow x$ with respect $N_1(\bullet,\bullet)_p$ if and only if $x_n \longrightarrow x$ with respect $N_2(\bullet,\bullet)_p$.*

Proof. By replacing "$x_n - x_m$" with "$x_n - x$" in equation (10) of Theorem 4.8, we get the result.

5 Open Problem

It can be easily introduce the notion of statistically convergence and statistically Cauchy sequence in p-adic linear 2-normed spaces.

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