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A Mixed Integer Convexity Result with an Application to an M/M/s Queueing System

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Abstract

Many optimization problems in queueing theory have functions with integer and real variables. Convexity results for these functions can be obtained by fixing either the integer or the real variable where bounds of the considered function play an important role. Tokgöz, Maalouf and Kumin [11] ***introduces the concept of mixed convexity for functions with real and integer variables and obtain convexity results without fixing any variables. The motivation for this paper is based on a conjecture of Kumin*** [6] ***in regard to the convexity of an objective function corresponding to an*** $M/M/s$ ***queueing system. In addition, generalized convexity results for a set of functionals with domain*** $\mathbb{Z}^n \times \mathbb{R}^m$ ***are obtained and the conditions for a local minimum point to be the unique global minimum are stated.***

Keywords: Hessian matrix, queueing systems, $M/M/s$ queue, real convex functions, discrete convex functions, optimization, local minimum, global minimum.

1 Introduction

In any optimization problem, it is useful to know whether or not a local minimum point is also a global minimum point. For functions with real variables, the Hessian matrix can be used to determine whether or not this is so. Various Hessian matrices have been defined for functions defined on the integer lattice $\mathbb{Z}^n$ (see for example Hirai and Murota [5], Moriguchi and Murota [9], and Yüceer [12]). A Hessian matrix, with properties similar to the Hessian matrix corresponding to functions with real variables, called a mixed Hessian matrix, is introduced by Tokgöz et al. [11]. This mixed Hessian matrix can be used to determine convexity conditions for functionals with domain $\mathbb{Z}^n \times \mathbb{R}^m$. An objective function associated with an $M/E_k/1$ queueing system is shown to have a positive semi-definite mixed Hessian matrix in [11] which implies its mixed convexity. In this paper, new mixed convexity results are proven and the mixed convexity of an objective function corresponding to an $M/M/s$ queueing system is shown by applying these results.

Several convexity results are known for the $M/M/s$ queueing system; however, they are generally given with respect to a single real variable such as the arrival or service rate, or a single integer variable such as the number of servers. Dyer and Proll [2], and Grassmann [3] showed that the expected number of customers is a convex function of the service rate. Lee and Cohen [7] proved the same result and a third proof was given by Mehrez and Brimberg [8].

Convexity analysis of queueing systems can also be done by fixing either the integer or real variable of the functions corresponding to the queueing systems. Finding simple upper and lower bounds for such functions is an important technique in determining of the convexity properties of queueing systems (see for example Berezner, Krezinski and Taylor [1] and Harel [4]). Using the concept of mixed convexity, it is not necessary to fix any variables or to find any upper or lower bounds. In addition, an algorithm is presented which makes it computationally fast to determine whether a mixed function associated with an $M/M/s$ queueing system is mixed convex or not.

In section 2, a conjecture with regard to the convexity of an objective function corresponding to an $M/M/s$ queueing system is stated and this conjecture is resolved in section 5. In section 3, basic definitions and results with regard to mixed convexity are provided. In section 4, we obtain mixed convexity results for a set of functionals with domain $\mathbb{Z}^n \times \mathbb{R}^m$ and also obtain conditions for a local minimum point to be the unique global minimum points.

2 An Optimization Problem Associated with an M/M/s Queueing System

Consider a parallel channel queueing system in which customers arrive according to a Poisson process with rate λ. In each channel, service follows the same negative exponential distribution with parameter μ. When a customer arrives he enters service if there is a free channel. Otherwise he joins the queue and waits for service. The queue discipline is " first come, first served ". It is well known that steady-state exists for this system if $\rho = \frac{\lambda}{s\mu} < 1$. Assuming this to be the case, the expected number in the system is found to be

$$E(n) = s\rho + \frac{\rho P_s}{(1-\rho)^2}$$

where

$$P_s = \frac{\left(\frac{\lambda}{\mu}\right)^s P_0}{s!}$$

and

$$P_0 = \left[\left(\sum_{j=0}^{s-1} \frac{\left(\frac{\lambda}{\mu}\right)^j}{j!}\right) + \frac{\left(\frac{\lambda}{\mu}\right)^s}{s!}\left(\frac{1}{1-\rho}\right)\right]^{-1}$$

Let the decision variables be the number of servers s, and the service rate μ. Assume that there are linear costs c_1, c_2,and c_3 associated with the number of servers, the service rate, and the expected number of customers in the system. Thus, we can define the following optimization problem:

Choose (s,μ) to minimize

$$\begin{aligned}\Psi(s,\mu) &= c_1 s + c_2\mu + c_3 E(n)\\ &= c_1 s + c_2\mu + c_3\left\{\frac{\lambda}{\mu} + \frac{\lambda\mu\left(\frac{\lambda}{\mu}\right)^s}{(s-1)!\,(s\mu-\lambda)^2\left[\left(\sum_{j=0}^{s-1}\frac{\left(\frac{\lambda}{\mu}\right)^j}{j!}\right) + \frac{\left(\frac{\lambda}{\mu}\right)^s}{s!}\left(\frac{s\mu}{s\mu-\lambda}\right)\right]}\right\}\end{aligned} \tag{2.1}$$

$$subject\ to: \mu > 0, \lambda > 0, s = 2, 3, ...$$

where c_1, c_2 and c_3 are arbitrary positive constants and $\frac{\lambda}{s\mu} < 1$.

Kumin [6] shows that if a function has one variable with positive second derivative then it has its second difference positive and conjectures that the same result may not be practical for functions with more than one variable such as the objective function given in (2.1) . In section 5 we show that the mixed convexity introduced by Tokgöz et al. [11] resolves this conjecture.

3 Real, Integer and Mixed Convexity

A C^2 function $\Omega : \mathbb{R}^m \to \mathbb{R}$ is convex if its corresponding Hessian matrix is positive semi-definite. Ω is convex if and only if for any two points α_1 and α_2 in the domain of Ω and for a real number γ where $0 < \gamma < 1$,

$$\Omega(\alpha_1\gamma + (1-\gamma)\alpha_2) \leq \gamma\Omega(\alpha_1) + (1-\gamma)\Omega(\alpha_2)$$

Now, consider a function $\Phi : \mathbb{Z}^n \to \mathbb{R}$ such that $\nabla_i(\Phi)$ and $\nabla_{ij}(\Phi)$ denote the first and second differences of Φ which are defined by

$$\nabla_i(\Phi) = \Phi(\alpha + e_i) - \Phi(\alpha)$$

and

$$\nabla_{ij}(\Phi) = \Phi(\alpha + e_i + e_j) - \Phi(\alpha + e_j) - \Phi(\alpha + e_i) + \Phi(\alpha)$$

where e_i denotes the unit length integer vector at the i^{th} position of the function Φ.

Denote by $\widetilde{\Phi}^{\mathbb{Z}^n}$ the restriction of the domain of the function $\widetilde{\Phi} : \mathbb{R}^n \to \mathbb{R}$ to $\mathbb{Z}^n$. The Hessian matrix corresponding to an integer function Φ can be obtained from a real valued function $\widetilde{\Phi} : \mathbb{R}^n \to \mathbb{R}$ which satisfies $\widetilde{\Phi}^{\mathbb{Z}^n}(x) = \Phi(x)$ on the integer lattice. This can be done by considering the Hessian matrix of $\widetilde{\Phi}$, $\left[\frac{\partial^2\widetilde{\Phi}}{\partial\beta_i\partial\beta_j}\right]_{n\times n}$, and restricting the domain of $\widetilde{\Phi}$ to the domain of Φ which gives the discrete Hessian matrix $\left[\nabla_{ij}(\Phi)\right]_{n\times n}$.

Ψ is called a mixed function if it has integer and real variables. The following definitions follow from Tokgöz et al. [11].

Definition 3.1 A function $\Psi : \mathbb{Z}^n \times \mathbb{R}^m \to \mathbb{Z}$ is a (strict) mixed convex function if $\widetilde{\Psi} : \mathbb{R}^{n+m} \to \mathbb{R}$ is a real (strict) convex function on $\mathbb{R}^{n+m}$ such that $\Psi(x)$ is obtained from $\widetilde{\Psi}$ with the property that $\widetilde{\Psi}(x) = \Psi(x)$ for $\forall x \in \mathbb{Z}^n \times \mathbb{R}^m$.

This definition of mixed convexity is for mixed functions which have a real convex function extension. A mixed function Ψ is a 2-smooth mixed convex function if its real extension $\widetilde{\Psi}$ is a C^2 convex function, and Ψ is k-smooth if $\widetilde{\Psi}$ is a C^k convex function.

We now construct a convex function Ψ with domain $\mathbb{Z}^n \times \mathbb{R}^m$ and its corresponding Hessian matrix. Let $\widetilde{\Psi} : \mathbb{R}^{n+m} \to \mathbb{R}$ be a convex function. The corresponding Hessian matrix of Ψ is an $(n+m) \times (n+m)$ matrix. For our purpose, restrict the n-dimensional portion of the domain to integer space. This changes the domain of $\widetilde{\Psi}$ into $\mathbb{Z}^n \times \mathbb{R}^m$. To determine whether or not such a function is mixed convex, we define a new mixed Hessian matrix. Restricting n-dimensional real space to n-dimensional integer space in the domain of $\widetilde{\Psi}$ changes the differentials of the n components of the function $\widetilde{\Psi}$ to the differences of the n components in $\widetilde{\Psi}$. That is, the equality

$$\frac{\partial\widetilde{\Psi}(\alpha,\beta)}{\partial\alpha_j} = \lim_{h_j\to 0}\frac{\widetilde{\Psi}(\alpha + h_j e_j, \beta) - \widetilde{\Psi}(\alpha,\beta)}{h_j},\ 1 \leq j \leq n,$$

becomes

$$\nabla_j \Psi(\alpha, \beta) = \Psi(\alpha + e_j, \beta) - \Psi(\alpha, \beta),\ 1 \le j \le n,$$

by the choice of $h_j = 1$ for all j, $1 \le j \le n$. Similarly, the second differential of $\widetilde{\Psi}$ becomes the second difference $\nabla_{ij}\Psi(\alpha, \beta)$ by the choice of $h_i = h_j = 1$ for all $i,\ j$, $1 \le i,\ j \le n$. Throughout this paper, Ψ is defined to be $\Psi : \mathbb{Z}^n \times \mathbb{R}^m \to \mathbb{R}$ and $\widetilde{\widetilde{\Psi}}$ will be defined by $\widetilde{\Psi} : \mathbb{R}^{n+m} \to \mathbb{R}$ unless stated otherwise. An $(n+m) \times (n+m)$ mixed matrix H that corresponds to a map Ψ has the form

$$\begin{aligned} H &= \begin{bmatrix} \left[\nabla_{ij}\Psi(\alpha, \beta)\right] & \left[\nabla_i(\frac{\partial}{\partial \beta_k}\Psi(\alpha, \beta)\right] \\ \left[\frac{\partial}{\partial \beta_k}(\nabla_j \Psi(\alpha, \beta)\right] & \left[\frac{\partial^2}{\partial \beta_k \partial \beta_t}(\Psi(\alpha, \beta))\right] \end{bmatrix} \\ &= \begin{bmatrix} (H_{11})_{n \times n} & (H_{12})_{n \times m} \\ (H_{21})_{m \times n} & (H_{22})_{m \times m} \end{bmatrix}_{(n+m)\times(n+m)} \end{aligned} \tag{3.1}$$

where $\alpha \in \mathbb{Z}^n, \beta \in \mathbb{R}^m, 1 \le i, j \le n$ and $1 \le k, t \le m$. In H, $\frac{\partial}{\partial \beta_k}$ and $\frac{\partial^2}{\partial \beta_k \partial \beta_t}$ denote the first and second partial derivatives of the real variable β of Ψ while ∇_i and ∇_{ij} denote the first and the second differences of the integer variable α respectively.

Tokgöz et al. [11] define a mixed Hessian matrix for a given mixed function Ψ and prove that the matrix has the following properties:

- If Ψ is an affine mixed function (i.e. Ψ is linear with respect to both integer and real variables) then H as given above in (3.1) vanishes,
- Suppose $\Psi_1 : \mathbb{Z}^n \times \mathbb{R}^m \to \mathbb{R}$ and $\Psi_2 : \mathbb{Z}^n \times \mathbb{R}^m \to \mathbb{R}$ are two 2-smooth mixed functions. If $\Psi = \Psi_1 + \Psi_2$ then $H(\Psi) = H(\Psi_1) + H(\Psi_2)$. (i.e H is linear with respect to the mixed functions)
- H as given in (3.1) is symmetric.

Let $\Psi_{\mathbb{Z}^n}$ denote the integer variable function when the real variables of Ψ are fixed and $\Psi_{\mathbb{R}^m}$ denote the real variable function when the integer variables of Ψ are fixed. Tokgöz et al. [11] also show that:

- A function Ψ is 2-smooth strict mixed convex if and only if the mixed Hessian matix for Ψ is strictly positive.
- Let Ψ be a strict mixed convex function. Then there exists a unique global minimum value of Ψ.

4 Main Results

For a certain class of mixed convex functions the following result holds.

Lemma 4.1 Let Ψ be a 2-smooth strict mixed function where the first partial derivative of Ψ does not change when there is a change in the integer component of Ψ. Then Ψ is strict mixed convex if and only if the mixed Hessian matrix H corresponding to Ψ is positive definite.

Proof Suppose that Ψ is a 2-smooth strict mixed convex function. Then, by the definition of integer convexity, the Hessian matrix that corresponds to the integer convex function is positive definite, i.e.($(H_{11})_{n\times n}$ is positive definite). Also by the definition of real convexity, $(H_{22})_{m\times m}$ is positive definite.

Note that by the assumption

$$\frac{\partial}{\partial\beta_i}\Psi(\alpha+e_j,\beta)=\frac{\partial}{\partial\beta_i}\Psi(\alpha,\beta)$$

which gives

$$\frac{\partial}{\partial\beta_i}\Psi(\alpha+e_j,\beta)-\frac{\partial}{\partial\beta_i}\Psi(\alpha,\beta)=0$$

and hence

$$\frac{\partial}{\partial\beta_i}(\nabla_j\Psi(\alpha,\beta))=0$$

Similarly, it can be easily seen from the symmetry of (3.1) that

$$\nabla_i(\frac{\partial}{\partial\beta_j}(\Psi(\alpha,\beta)))=\frac{\partial}{\partial\beta_j}(\nabla_i(\Psi(\alpha,\beta)))$$

hence the submatrices H_{12} and H_{21} of the mixed matrix H are zero. This leaves H_{11} and H_{22} of the mixed matrix H to be nontrivial.

If we assume that H is positive definite, by the nature of a mixed Hessian matrix and the assumption, we find the second differences and second differentials of H to be positive which implies that Ψ is integer convex and real convex.

In the case where we consider all the integer components pointwise one has the following result:

Theorem 4.2 If $\Psi:\mathbb{Z}^n\times\mathbb{R}^m\to\mathbb{R}$ is a 2-smooth strict mixed convex function then for each fixed point of the strict integer convex function $\Psi_{\mathbb{Z}^n}$, $\Psi_{\mathbb{R}^m}$ is a C^2 strict convex function.

Proof Suppose that $\Psi:\mathbb{Z}^n\times\mathbb{R}^m\to\mathbb{R}$ is a strict mixed convex function. By Tokgöz et al. [11], Ψ is 2-smooth strict mixed convex if and only if the mixed Hessian matrix H for Ψ is strictly positive definite. Suppose that $c\in\mathbb{Z}^n$ is a fixed vector in $\mathbb{Z}^n$. Therefore, by the hyphothesis of the theorem,

$\Psi(c,\beta)$ is a strict mixed convex function which is a function of m real variables. Considering the mixed Hessian matrix defined by (3.1), we have the block diagonal matrices H_{12}, H_{21} and H_{11} zero and H_{22} non-zero. Following this, the positive definiteness of H implies the positive definiteness of H_{22}. This proves that $\Psi_{\mathbb{R}^m}$ is a C^2 strict convex function.

It is important to note that if for each fixed point of the strict integer convex function $\Psi_{\mathbb{Z}^n}$, $\Psi_{\mathbb{R}^m}$ is a C^2 strict convex function then it does not mean that Ψ is a 2-smooth strict mixed convex function. A simple counter example can be seen by choosing $\Psi : \mathbb{Z}\times \mathbb{R} \to \mathbb{R}$, $\Psi(\alpha,\beta) = (\alpha^2+0.5)(\beta^2+1)$. In this case,

$$H = \begin{bmatrix} 2(\beta^2+1) & 2\beta\,(2\alpha+1) \\ 2\beta\,(2\alpha+1) & 2\,(\alpha^2+0.5) \end{bmatrix}$$

and the choice of $\alpha=\beta=1$ gives $\det(H) = -24$ which shows that Ψ is not a strict mixed convex function; however, it is easy to see that for each fixed α, (β^2+1) is strict convex.

Theorem 4.3 Let $\Psi : \mathbb{Z}^n\times \mathbb{R}^m \to \mathbb{R}$ be a 2-smooth mixed convex function. Then the set of local minimums of Ψ form a set of global minimums and vice versa.

Proof First we extend the 2-smooth mixed convex function $\Psi : \mathbb{Z}^n\times \mathbb{R}^m \to \mathbb{R}$ to its real extension $\widetilde{\Psi} : \mathbb{R}^{n+m} \to \mathbb{R}$. It is well kown that for a real convex function every local minimum is also a global minimum and vice versa. Suppose the global minimum value of $\widetilde{\Psi}$ is obtained when $(\alpha^0,\beta^0) = (\alpha_1^0,\alpha_2^0,...,\alpha_n^0,\beta_1^0,\beta_2^0,...,\beta_m^0)$. This point exists in a local neighborhood U of $\widetilde{\Psi}$. Define a set

$$\Sigma_i = \left\{x_i^0 : x_i^0 \in \left\{\left\lfloor \alpha_1^0\right\rfloor,\ \alpha_1^0,\ \left\lceil \alpha_1^0\right\rceil\right\}\right\}\ for\ all\ 1\leq i\leq n,$$

and the singleton set $F = \{(\beta_1,\beta_2,\beta_3,...,\beta_m)\}$. Let $\Sigma = \Sigma_1\times\Sigma_2\times\Sigma_3\times...\times\Sigma_n$. The local minimum value $\Psi(x^0,\beta^0) = \Psi(x_1^0,x_2^0,...,x_n^0,\beta_1^0,\beta_2^0,...,\beta_m^0)$ of the 2-smooth mixed convex function Ψ exists for some $(x,\beta) = (x_1,x_2,...,x_n,\beta_1,\beta_2,$ $...,\beta_m) \in \widetilde{U} \subset \Sigma\times F$ where $\widetilde{U}$ is an extended local neighborhood of U; that is, $\widetilde{U}$ is a large enough local neighborhood that contains U and the integer values of Σ_i for all $1\leq i\leq n$. Let

$$M = \left\{ (x,\beta) : \min_{(x,\beta)\in\widetilde{U}} \Psi(x,\beta) \ = \Psi\left(x^0,\beta^0\right) \right\}$$

which is the set of points where the minimal value of Ψ exists. Now, suppose, to the contrary, that there exists a vector $(y,t) \in \widetilde{V}$ such that $\Psi(y,t)$ is the global minimal value of Ψ which is not a local minimal value. If such a vector $(y,t) \in \widetilde{V}$ exists then for all $(x,\beta) \in M$

$$\Psi\left(x^0,\beta^0\right) < \Psi(y,t)$$
$$\min_{(x,\beta)\in\widetilde{U}} \Psi(x,\beta) < \Psi(y,t).$$

Because M is the set of vectors obtained from $\widetilde{\Psi}$,

$$\min_{(x,\beta)\in\widetilde{U}} \Psi(x,\beta) = \min_{(x,\beta)\in\mathbb{Z}^n\times\mathbb{R}^m} \Psi(x,\beta).$$

Therefore

$$\min_{(x,\beta)\in\mathbb{Z}^n\times\mathbb{R}^m} \Psi(x,\beta) < \Psi(y,t)$$

which is a contradiction. Hence M is the set of local minimum points which is also the set of global minimum points. This argument also shows that the set of global minimum values is the set of local minimum values.

Tokgöz et al. [11] show that for any strict mixed convex function Ψ there exists a unique global minimum value of Ψ. However, a more important result is the case for which we also have a unique global minimum point for the strict mixed convex function Ψ.

Theorem 4.4 Suppose that $\Psi : \mathbb{Z}^n \times \mathbb{R}^m \to \mathbb{R}$ is a strict mixed convex function which has a minimal value. Then there exist a unique global minimum point (α^0,β^0) of Ψ if the global minimum point of $\widetilde{\Psi}$ is obtained when α_i^0 is not an integer, either $\alpha_i^0 \neq \lfloor\alpha_i^0\rfloor + \frac{1}{2}$ or $\alpha_i^0 \neq \lceil\alpha_i^0\rceil - \frac{1}{2}$ holds for all i $(1 \le i \le n)$ and α_i^0 is an integer for $n-i$.

Proof Suppose when α_i^0 is not an integer $\alpha_i^0 = \lfloor\alpha_i^0\rfloor + \frac{1}{2}$ and $\alpha_i^0 = \lceil\alpha_i^0\rceil - \frac{1}{2}$ hold for i, $1 \le i \le n$, and $(n-i)$ number of α_i^0 are integers where there exists a unique global minimum point for the unique global minimum value. It is sufficient to prove the contradiction for a random fixed index i. The proof follows for all the indices. The minimum value is

$$\Psi\left(\alpha_1^0,\alpha_2^0,...,\left\lfloor\alpha_i^0\right\rfloor,...,\alpha_n^0,\beta_1^0,\beta_2^0,...,\beta_m^0\right)$$

and

$$\Psi\left(\alpha_1^0,\alpha_2^0,...,\left\lceil\alpha_i^0\right\rceil,...,\alpha_n^0,\beta_1^0,\beta_2^0,...,\beta_m^0\right)$$

This indicates that the point which gives the minimal value of Ψ occurs when

$$\left(\alpha^0,\beta^0\right) = \left(\alpha_1^0,\alpha_2^0,...,\gamma_i,...,\alpha_n^0,\beta_1^0,\beta_2^0,...,\beta_m^0\right)$$

for $\gamma_i = \lfloor\alpha_i^0\rfloor$ and $\gamma = \lceil\alpha_i^0\rceil$. Therefore the minimal point is not unique. Since the index is chosen randomly, it holds for all the indeces i where α_i^0 is not an integer. The case when α_i^0 is an integer follows simply by the definition of Ψ.

An example which shows that the global minimum value is unique while the point that corresponds to the global minimum value is not necessarily unique is as follows:

Define a function $\Im : \mathbb{Z}^n \times \mathbb{R}^m \to \mathbb{R}$ by

$$\Im(\alpha,\beta) = \sum_{i=1}^{n} (\alpha_i - 1.5)^2 + \sum_{j=1}^{m} (\beta_j - j)^2$$

where $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n) \in \mathbb{Z}^n$ and $\beta = (\beta_1, \beta_2, ..., \beta_m) \in \mathbb{R}^m$. The global minimum points of $\Im$ are $(h, j) \in \mathbb{Z}^n \times \mathbb{R}^m$ with $h = 1, 2$ where the correspoding global minimum value is $0.25n$.

5 An Open Problem for an M/M/s Queueing System

In this section, we resolve the open problem conjectured in [6] and defined in section 2. We begin by calculating the components of the mixed Hessian matrix; the difference of the first derivative, $\nabla\left(\frac{d\Psi}{d\mu}\right)$, the second derivative, $\frac{d^2\Psi}{d\mu^2}$, and the second difference $\nabla_{11}(\Psi)$ of $\Psi(s,\mu)$ given in (3.1). The calculations are as follows:

After algebraic manipulation of the function $\Psi(s,\mu)$ and defining

$$e_n^{\frac{\lambda}{\mu}} = \sum_{j=0}^{n} \frac{\left(\frac{\lambda}{\mu}\right)^j}{j!}$$

(2.1) takes the form

$$\Psi(s,\mu) = c_1 s + c_2\mu + c_3\frac{\lambda}{\mu} + c_3\frac{\lambda\mu}{(s\mu-\lambda)}\frac{1}{(s-1)!\,(s\mu-\lambda)\left(\frac{\lambda}{\mu}\right)^{-s} e_{(s-1)}^{\frac{\lambda}{\mu}} + \mu}$$

which can be further simplified as

$$\begin{aligned}\Psi(s,\mu) &= c_1 s + c_2\mu + c_3\frac{\lambda}{\mu} + c_3\frac{\lambda}{s\mu-\lambda}\;\frac{e_s^{\frac{\lambda}{\mu}} - e_{(s-1)}^{\frac{\lambda}{\mu}}}{\left(\frac{s\mu-\lambda}{s\mu}\right)e_{(s-1)}^{\frac{\lambda}{\mu}} + 1} \qquad (5.1)\\ &= c_1 s + c_2\mu + c_3\frac{\lambda}{\mu} + c_3\frac{\lambda\mu s}{(s\mu-\lambda)^2}\;\frac{e_s^{\frac{\lambda}{\mu}} - e_{(s-1)}^{\frac{\lambda}{\mu}}}{e_{(s-1)}^{\frac{\lambda}{\mu}} + \frac{s\mu}{s\mu-\lambda}}\end{aligned}$$

By (5.1)

$$\frac{d\Psi(s,\mu)}{d\mu} = c_2 - c_3\frac{\lambda}{\mu^2} - c_3\frac{\lambda s(\lambda+\mu s)}{(s\mu-\lambda)^3}\frac{e_s^{\frac{\lambda}{\mu}} - e_{(s-1)}^{\frac{\lambda}{\mu}}}{e_{(s-1)}^{\frac{\lambda}{\mu}} + \frac{s\mu}{s\mu-\lambda}} \qquad (5.2)$$

$$+ \; c_3\frac{\mu s^2\lambda^2}{(s\mu-\lambda)^2}\frac{e_s^{\frac{\lambda}{\mu}} - e_{(s-1)}^{\frac{\lambda}{\mu}}}{\left(e_{(s-1)}^{\frac{\lambda}{\mu}} + \frac{s\mu}{s\mu-\lambda}\right)^2}.\left(\frac{1}{(s\mu-\lambda)^2} - \frac{1}{\mu\,(s\mu-\lambda)}\right)$$

Let

$$\varphi_i(s,\mu) = \frac{e_s^{\frac{\lambda}{\mu}} - e_{(s-1)}^{\frac{\lambda}{\mu}}}{\left(e_{(s-1)}^{\frac{\lambda}{\mu}} + \frac{s\mu}{s\mu-\lambda}\right)^i} \qquad for \quad i = 1,2,3$$

then

$$\frac{d\Psi(s,\mu)}{d\mu} = c_2 - c_3\frac{\lambda}{\mu^2} - c_3\frac{\lambda s(\lambda+\mu s)}{(s\mu-\lambda)^3}\varphi_1(s,\mu)$$
$$+ c_3\frac{\mu s^2\lambda^2}{(s\mu-\lambda)^2}\varphi_2(s,\mu)\left(\frac{1}{(s\mu-\lambda)^2} - \frac{1}{\mu\,(s\mu-\lambda)}\right)$$

Therefore

$$\nabla_1\left(\frac{d}{d\mu}\left(\Psi(s,\mu)\right)\right) = (c_2 - c_3\frac{\lambda}{\mu^2} - c_3\frac{\lambda\,(s+1)\,(\lambda+\mu\,(s+1))}{((s+1)\,\mu-\lambda)^3}\varphi_1(s+1,\mu)$$
$$+ c_3\frac{\mu\,(s+1)^2\,\lambda^2}{((s+1)\,\mu-\lambda)^2}\varphi_2(s+1,\mu)(\frac{1}{((s+1)\,\mu-\lambda)^2}$$
$$- \frac{1}{\mu\,((s+1)\,\mu-\lambda)}) - (c_2 - c_3\frac{\lambda}{\mu^2} - c_3\frac{\lambda s(\lambda+\mu s)}{(s\mu-\lambda)^3}\varphi_1(s,\mu)$$
$$+ c_3\frac{\mu s^2\lambda^2}{(s\mu-\lambda)^2}\varphi_2(s,\mu)\left(\frac{1}{(s\mu-\lambda)^2} - \frac{1}{\mu\,(s\mu-\lambda)}\right))$$

$$= -c_3\frac{\lambda\,(s+1)\,(\lambda+\mu\,(s+1))}{((s+1)\,\mu-\lambda)^3}\varphi_1(s+1,\mu) + c_3\frac{\mu\,(s+1)^2\,\lambda^2}{((s+1)\,\mu-\lambda)^2}\varphi_2(s+1,\mu).$$
$$\left(\frac{1}{((s+1)\,\mu-\lambda)^2} - \frac{1}{\mu\,((s+1)\,\mu-\lambda)}\right) + c_3\frac{\lambda s(\lambda+\mu s)}{(s\mu-\lambda)^3}\varphi_1(s,\mu)$$
$$- c_3\frac{\mu s^2\lambda^2}{(s\mu-\lambda)^2}\varphi_2(s,\mu)\left(\frac{1}{(s\mu-\lambda)^2} - \frac{1}{\mu\,(s\mu-\lambda)}\right)$$

and also

$$\frac{d^2\Psi(s,\mu)}{d\mu^2} = 2c_3\frac{\lambda}{\mu^3} + 2c_3\frac{\mu s^3 + 2s^2\lambda^2}{(s\mu-\lambda)^4}\varphi_1(s,\mu) - c_3\frac{s^2\lambda^2(\lambda+\mu s)}{(s\mu-\lambda)^4}\left(\frac{1}{s\mu-\lambda} - \frac{1}{\mu}\right)\varphi_2(s,\mu)$$
$$+ c_3 s^2\lambda^2\varphi_3(s,\mu)\left(-\frac{\lambda}{\mu^2}\left(e^{\frac{\lambda}{\mu}}_{(s-1)} + \frac{s\mu}{s\mu-\lambda}\right)\right)\cdot\left(\frac{\mu}{(s\mu-\lambda)^4} - \frac{1}{(s\mu-\lambda)^3}\right)$$
$$+ 2c_3 s^2\lambda^2\varphi_3(s,\mu)\left(\frac{\lambda}{\mu^2}e^{\frac{\lambda}{\mu}}_{(s-1)} + \frac{s\lambda}{(s\mu-\lambda)^2}\right)\cdot\left(\frac{\mu}{(s\mu-\lambda)^4} - \frac{1}{(s\mu-\lambda)^3}\right)$$
$$+ c_3 s^2\lambda^2\varphi_2(s,\mu)\left(\frac{-3\mu s-\lambda}{(s\mu-\lambda)^5} + \frac{3s}{(s\mu-\lambda)^4}\right)$$

$$= 2c_3\frac{\lambda}{\mu^3} + 2c_3\frac{\mu s^3 + 2s^2\lambda^2}{(s\mu-\lambda)^4}\varphi_1(s,\mu) + c_3\frac{s^2\lambda^2(\lambda+\mu s)}{(s\mu-\lambda)^4}\left(\frac{1}{\mu} - \frac{1}{s\mu-\lambda}\right)\varphi_2(s,\mu)$$
$$+ c_3 s^2\lambda^2\varphi_3(s,\mu)\left(\frac{\lambda}{\mu^2}e^{\frac{\lambda}{\mu}}_{(s-1)} - \frac{\lambda s}{\mu\,(s\mu-\lambda)}\right)\left(\frac{\mu-\mu s+\lambda}{(s\mu-\lambda)^4}\right)$$
$$+ c_3 s^2\lambda^2\varphi_3(s,\mu)\left(\frac{2\lambda s}{(s\mu-\lambda)^2}\right)\left(\frac{\mu-\mu s+\lambda}{(s\mu-\lambda)^4}\right)$$
$$+ c_3 s^2\lambda^2\varphi_2(s,\mu)\left(\frac{-3\mu s-\lambda+3\mu s^2-3\lambda s}{(s\mu-\lambda)^5}\right) \tag{5.3}$$

The Hessian matrix H given in (3.1) can be evaluated numerically using computer programs. Because of the lengthy calculations, we do not present the determinant of the matrix H in detail; however, it can be shown that the determinant of H is strictly positive for all μ and s with respect to the conditions stated above. In particular, if we consider sufficiently large μ and s, the calculations to find the determinant of H are easier since $e^{\frac{\lambda}{\mu}}_{s} - e^{\frac{\lambda}{\mu}}_{(s-1)} \approx 0$. Hence

$$\frac{d^2\Psi(s,\mu)}{d\mu^2} \approx 2c_3\frac{\lambda}{\mu^3} = \gamma \tag{5.4}$$

where γ is a positive number. Note that the terms at (5.3) other than the term at (5.4) are small enough that they would not effect the sign of the second derivative of the function. Similarly

$$\frac{d\Psi(s,\mu)}{d\mu} \approx c_2 - c_3\frac{\lambda}{\mu^3}$$

for large enough μ and s and by the symmetry,

$$\nabla_1\left(\frac{d}{d\mu}\left(\Psi(s,\mu)\right)\right) = \frac{d}{d\mu}\left(\nabla_1\left(\Psi(s,\mu)\right)\right) = 0 \tag{5.5}$$

Now the only remaining term to calculate in the determinant of the mixed Hessian matrix given in (3.1) is $\nabla_{11}\left(\Psi(s,\mu)\right).$

$$\nabla_1\left(\Psi(s,\mu)\right) = c_1 + c_3\frac{\lambda\mu\left(s+1\right)}{\left(\left(s+1\right)\mu-\lambda\right)^2}\varphi_1(s+1,\mu) - c_3\frac{\lambda\mu}{\left(s\mu-\lambda\right)}\varphi_1(s,\mu)$$

$$\begin{aligned}\nabla_{11}\left(\Psi(s,\mu)\right) = {} & c_3\frac{\lambda\mu\left(s+2\right)}{\left(\left(s+2\right)\mu-\lambda\right)^2}\ \varphi_1(s+2,\mu) \\ & - 2c_3\frac{\lambda\mu\left(s+1\right)}{\left(\left(s+1\right)\mu-\lambda\right)^2}\ \varphi_1(s+1,\mu) + c_3\frac{\lambda\mu}{\left(s\mu-\lambda\right)}\varphi_1(s,\mu)\end{aligned}$$

Since we need the sign of $\nabla_{11}\left(\Psi(s,\mu)\right)$ to be positive,

$$\begin{aligned}\nabla_{11}\left(\Psi(s,\mu)\right) \geq {} & c_3\frac{\lambda\mu\left(s+2\right)}{\left(s\mu-\lambda\right)}\frac{e^{\frac{\lambda}{\mu}}_{(s+2)} - e^{\frac{\lambda}{\mu}}_{(s+1)}}{\left(s\mu-\lambda\right)e^{\frac{\lambda}{\mu}}_{(s-1)} + s\mu} \\ & - 2c_3\frac{\lambda\mu\left(s+1\right)}{s\mu-\lambda}\frac{e^{\frac{\lambda}{\mu}}_{(s+1)} - e^{\frac{\lambda}{\mu}}_{s}}{\left(s\mu-\lambda\right)e^{\frac{\lambda}{\mu}}_{(s-1)} + s\mu} \\ & + c_3\frac{\lambda\mu s}{s\mu-\lambda}\frac{e^{\frac{\lambda}{\mu}}_{s} - e^{\frac{\lambda}{\mu}}_{(s-1)}}{\left(s\mu-\lambda\right)e^{\frac{\lambda}{\mu}}_{(s-1)} + s\mu}\end{aligned} \tag{5.6}$$

$$\begin{aligned}= {} & c_3\frac{\lambda\mu}{\left(s\mu-\lambda\right)}\frac{1}{\left(s\mu-\lambda\right)e^{\frac{\lambda}{\mu}}_{(s-1)} + s\mu}\{\left(s+2\right)\left(e^{\frac{\lambda}{\mu}}_{(s+2)}\right. \\ & \left. - e^{\frac{\lambda}{\mu}}_{(s+1)}\right) - 2\left(s+1\right)\left(e^{\frac{\lambda}{\mu}}_{(s+1)} - e^{\frac{\lambda}{\mu}}_{s}\right) + s\left(e^{\frac{\lambda}{\mu}}_{s} - e^{\frac{\lambda}{\mu}}_{(s-1)}\right)\}\end{aligned}$$

Let

$$\vartheta = c_3\frac{\lambda\mu}{\left(s\mu-\lambda\right)}\frac{1}{\left(s\mu-\lambda\right)e^{\frac{\lambda}{\mu}}_{(s-1)} + s\mu}$$

Applying

$$\begin{aligned}e^{\frac{\lambda}{\mu}}_{(s+2)} &= \sum_{j=0}^{s+1}\frac{\left(\frac{\lambda}{\mu}\right)^j}{j!} + \frac{\left(\frac{\lambda}{\mu}\right)^{s+2}}{(s+2)!} \\ &= e^{\frac{\lambda}{\mu}}_{(s+1)} + \frac{\left(\frac{\lambda}{\mu}\right)^{s+2}}{(s+2)!}\end{aligned} \tag{5.7}$$

several times reduces (5.6) and we have

$$\nabla_{11}\left(\Psi(s,\mu)\right) \geq \vartheta[\frac{\left(\frac{\lambda}{\mu}\right)^{s+2}}{(s+1)!}(\frac{\lambda}{\mu} - 2s + 2 \qquad (5.8)$$

$$+ (3s+2)(s+1)s) + 2(s+1)e^{\frac{\lambda}{\mu}}_{(s-1)}]$$

By applying (5.7) to (5.8)

$$\nabla_{11}\left(\Psi(s,\mu)\right) \geq \vartheta \begin{bmatrix} \frac{\left(\frac{\lambda}{\mu}\right)^{s+2}}{(s+1)!}\left(\frac{\lambda}{\mu} + 2 + 3s^3 + 5s^2\right) \\ +2(s+1)e^{\frac{\lambda}{\mu}}_{(s-1)} \end{bmatrix}$$

$$= \xi > 0 \qquad (5.9)$$

By (5.4), (5.5), (5.9) and (3.1)

$$H = \begin{bmatrix} \nabla_{11}\left(\Psi(s,\mu)\right) & \nabla_1\left(\frac{d}{d\mu}\left(\Psi(s,\mu)\right)\right) \\ \frac{d}{d\mu}\left(\nabla_1\left(\Psi(s,\mu)\right)\right) & \frac{d^2\Psi(s,\mu)}{d\mu^2} \end{bmatrix}$$

$$\approx \begin{bmatrix} \gamma & 0 \\ 0 & \xi \end{bmatrix}$$

hence $det(H) > 0$ with $2c_3\frac{\lambda}{\mu^3} = \gamma > 0$ and $\xi > 0$. By the results of Tokgöz et al. [11], Ψ is a strict mixed convex function.

Another way of deriving the equations for the second difference $\nabla_{11}(\Psi)$, the second derivative $\frac{d^2\Psi(s,\mu)}{d\mu^2}$ and the determinant $det(H)$ is by following the algorithm of Tokgöz [10] given below. Convexity of Ψ for particular values of the unknowns c_1, c_2, c_3, μ, s and λ can be calculated in a certain neighborhood by applying this algorithm.

```
Algorithm:
syms c1 c2 c3 alpha mu rho s
rho = lambda/mu
PSI(s,mu) = c1*s + c2*mu + c3*(lambda/mu + lambda*mu*(rho^s)
/(fact(s-1)*(s*mu-lambda)^2*(sum(rho^j/fact(j),0,s-1)
+((rho^s)/fact(s))*(s*mu/(s*mu-lambda)))
d1M = diff(PSI,mu)
d2M = diff(PSI,mu,2)
Difference1 = PSI(s+1,mu)-PSI(s,mu)
diff_Difference1 = diff(Difference1,mu,1)
Difference2 = PSI(s+2,mu)-2PSI(s+1,mu)+PSI(s,mu)
DetH = d2M*Difference2 - (diff_Difference1)^2
```

The positive definiteness of the equations obtained by this algorithm was checked by using the Mathematica programming language.

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