Int. J. Open Problems Compt. Math., Vol. 4, No. 2, June 2011
ISSN 1998-6262;
www.i-csrs.org

Similarity Solutions of a Class of Laminar Three-Dimensional Magnetohydrodynamic Boundary Layer Equations of Power Law Fluids

PATIL VISHWAMBHAR S[1] and TIMOL M. G.[2]

[1]Department of Chemical Technology,

North Maharashtra University, Jalgaon-425001, India.

e mail: Vishme.p@gmail.com

[2]Department of Mathematics, Veer Narmad South Gujarat University,

Magdulla Road, Surat – 395007, India.

e mail: mgtimol@gmail.com

Abstract

Using group theoretic method the similarity solutions for three-dimensional steady incompressible magnetohydrodynamic boundary layer equations in rectangular coordinates for a power-law fluid are investigated. The particular form of restriction to be imposed on free stream velocities and magnetic field strength are derived systematically from the similarity requirement. For the small cross flows, the cross flow component may be generalized and found to be representable as a polynomial of flow variable x. Controlled equations are reduced to those found in literature.

- **Keywords**: *Magnetohydrodynamic (MHD), Non-Newtonian power-law fluid, streamline, Similarity solution, linear group of transformation, spiral group of transformation.*

AMS Subject Classifications 2000: 70G65, 76D05, 54H15, 37J15.

1 Introduction

Magnetohydrodynamics (MHD) deals with the motion of conducting fluids. The research of MHD incompressible viscous flow has many important engineering applications in devices such as power generator, the cooling of reactors, the design of heat exchangers and MHD accelerators. Further, the applications of MHD also cover a wide range of physical areas from liquid metals to cosmic plasmas; for example, the intensely heated and ionized fluids in an electromagnetic field in astrophysics, geophysics, aerodynamics, and plasma physics all deal with magnetohydrodynamics. Because of this, MHD flow problems have attracted the attention of mathematicians and engineers. In these problems, the authors generally assume that the fluid is Newtonian.

Recently non-Newtonian fluids have been found important and useful in technological applications (For details see Refs. [1–3]). A large class of real fluids exhibits non-linear relationships between stress and the rate of strain. Because of this non-linear dependence, the analysis of the behavior of the fluid motion of the non-Newtonian fluids tends to be much more complicated and subtle than that of the Newtonian fluids. Hansen and his co-worker [4-6] are rather first to derive the systematic similarity analysis for three-dimensional boundary layer flow of Newtonian fluids. Further they [7] have extended their work to non-Newtonian power-law fluids and have obtained class of similarity solutions including similarity solutions for small cross flow. Following them [7], Timol et al [8], have used group theoretic technique to derive similarity solutions for the class of three-dimensional boundary layer flow of non-Newtonian fluids of different models including power-law fluids.

In past couple of decades numerous attempts have been made in applying boundary layer theory to the non-Newtonian fluids. The theory makes great simplification in the equation of motion and as a consequence, the equations are much simple to solve. For various non-Newtonian fluid models the progress in such a theoretical simplification is bit slowly. Most work found in literature on the said topic is restricted to simple two dimensional MHD and non MHD boundary layer flow of non-Newtonian power law fluids. In recent years, many investigations have concentrated on the MHD flows because of its important applications in metallurgical industry, such as the cooling of continuous strips and filaments drawn through a quiescent fluid and the purification of molten metals from non-metallic inclusions. The MHD flows of non-Newtonian fluids were initially studied by Sarpkaya [9], and then followed by Djukic [10, 11], Andersson et al. [12], and Liao [13], etc. Xu et al [14] have presented the study of the unsteady magnetohydrodynamic (MHD) viscous flows of non-Newtonian fluids caused by an impulsively stretching plate by means of an analytic technique, namely the homotopy analysis method. Whereas Ishak et al [15] have reported the numerical solution of MHD flow and heat transfer outside a stretching cylinder.

The governing system of partial differential equations is converted into a system of ordinary differential equations using similarity transformation, which is then solved numerically by the Keller-box method. Guedda et al [16] have studied the steady-state laminar boundary layer flow, governed by the Ostwald-de Wale power-law model of an incompressible non-Newtonian fluid past a semi-infinite power-law stretched flat plate with uniform free stream velocity. A generalization of the usual Blasius similarity transformation is used to find similarity solutions. Under appropriate assumptions, they have transformed their partial differential equations into an autonomous third-order nonlinear degenerate ordinary differential equation with boundary conditions. Recently, Ferdows and Olajuwon [17] have derived the Similarity solution of Micropolar Power law fluid over a vertical plate.

Quite rare information is available in the literature about three-dimensional magnetohydrodynamic boundary layer flows of non-Newtonian fluids. This is because the differential equations governing the flow are highly non-linear system of partial differential equations of boundary value type with three independent and three dependent variables. Further the presence of terms due to non-Newtonian nature of fluids and external transverse magnetic field poses extra difficulties while simplifying such flow equations. Timol and Timol [18] are probably first to derive basic equations and similarity solutions of three-dimensional magneto-hydrodynamic boundary layer flow of Newtonian fluids. But in order to meet similarity requirements they have assumed the specific form of outer flow and specific form of imposed magnetic field parameter in priori and hence set of similarity equations so obtained have limited practical applications.

In the present paper using group theoretic method systematic similarity analysis is derived to find similarity equations for steady three-dimensional incompressible boundary layer flow of electrically conducting non-Newtonian power-law fluids past external surface under the influence of transverse magnetic field. We have considered special type of flow situation which is independent of z-coordinate, hence it is essentially quasi two-dimensional. Such flows are characterized by the fact that streamline form a system of translates i.e. entire streamline pattern can be obtained by translating any particular streamline parallel to leading edge of the surface. It is hoped that by omitting dependence of flow quantities in one direction more qualitative information may be obtained on the characteristic of three-dimensional boundary layer flows of power-law fluids. It is observed that for some special cases, the present set of equations is well reduced to past well-known equations like Blasius equation, Falkner-Skan equations etc.

2 Problem Formulations

The well-known Ostwald-de-Wale model of power-law fluid is purely phenomenological; however, it is useful in that approximately describes a great number of real non-Newtonian fluids. This model behaves properly under tensor deformation. Use of this model alone assumes that the fluid is purely viscous. Mathematically it can be represented in the form Bird et al [19], Kapur [20] and Sirohi et al [21] as,

$$\bar{\bar{\tau}} = -\left\{ m \left| \sqrt{\frac{1}{2}\bar{\bar{\Delta}}:\bar{\bar{\Delta}}} \right|^{n-1} \right\} \bar{\bar{\Delta}} \tag{2.1}$$

where $\bar{\bar{\tau}}$ and $\bar{\bar{\Delta}}$ are the stress tensor and the rate of deformation tensor, respectively. Also m and n are called the consistency and flow behavior indices respectively. If n < 1, the fluid is called pseudo plastic power law fluid and if n > 1, it is called dilatants power law fluid since the apparent viscosity decreases or increases with the increase shear of rate according as n < 1 or n > 1, if n =1 the fluid will be Newtonian.

Following Patel and Timol [22], equation (2.1), under the boundary layer assumptions, yields following two non-vanishing components:

$$\tau_{yx} = -\,\mathrm{m}\left\{ \left[\left(\frac{\partial \mathrm{u}'}{\partial \mathrm{y}'} \right)^2 + \left(\frac{\partial w'}{\partial \mathrm{y}'} \right)^2 \right]^{\frac{n-1}{2}} \frac{\partial \mathrm{u}'}{\partial \mathrm{y}'} \right\} \tag{2.2}$$

$$\tau_{yz} = -\,\mathrm{m}\left\{ \left[\left(\frac{\partial \mathrm{u}'}{\partial \mathrm{y}'} \right)^2 + \left(\frac{\partial w'}{\partial \mathrm{y}'} \right)^2 \right]^{\frac{n-1}{2}} \frac{\partial w'}{\partial \mathrm{y}'} \right\} \tag{2.3}$$

Where the absolute sign has been dropped since both terms within the sign are positive.

Consider an incompressible and electrically conducting fluid over a vertical isothermal semi-infinite flat plate in the presence of a strong non-uniform magnetic field. The geometry of the flow domain is illustrated in “Fig. 1”, in which *x*-coordinate is orientated parallel to the plate in upward direction with y-axis is taken normal to it, and the origin located at the leading edge of the plate. Due to a cross flow in z-direction, the flow becomes three-dimensional. To simplify the problem, we assume that all flow quantities are independent of the z-coordinate. A magnetic field is assumed to be applied normal to the vertical plate and the induced magnetic field is neglected due to a very small Reynolds number. It is assumed that U and W are the components of velocity outside the boundary-layer in x and z-direction [8].

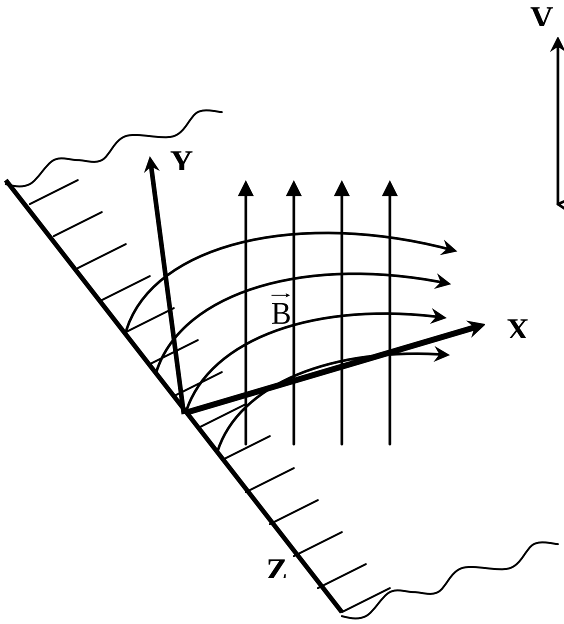

Fig1. Flow over a plate in rectangular coordinate system under the influence of transverse magnetic field.

Following Timol et al [8] and Patel et al [23, 24] for above "equation of state", the dimensionless equations governing the motion of three-dimensional laminar incompressible magnetohydrodynamic boundary layer flow of non-Newtonian power-law fluids can be written as:

Continuity Equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{2.4}$$

Momentum Equations

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left\{\left[\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right]^{\frac{n-1}{2}}\frac{\partial u}{\partial y}\right\} + U\frac{dU}{dx} + S(x)\left(U-u\right) \tag{2.5}$$

$$u\frac{\partial w}{\partial x}+\mathrm{v}\frac{\partial w}{\partial \mathrm{y}}=\frac{\partial}{\partial \mathrm{y}}\left\{\left[\left(\frac{\partial u}{\partial \mathrm{y}}\right)^2+\left(\frac{\partial w}{\partial \mathrm{y}}\right)^2\right]^{\frac{n-1}{2}}\frac{\partial w}{\partial \mathrm{y}}\right\}+\mathrm{U}\frac{dW}{dx}+\mathrm{S(x)}\left(W-w\right) \tag{2.6}$$

with the boundary conditions,

At $y = 0$: $u = v = w = 0$ (2.7)

At $y = \infty$: $u = U(x)$, $w = W(x)$ (2.8)

And the magnetic field strength is

$$\mathrm{S}(x)=\frac{\sigma\,\mathrm{B}^2\,\mathrm{L}}{\rho U_0}\quad \text{where } \mathrm{B}=\mathrm{B}_y(x) \tag{2.9}$$

Where the non-dimensional quantities used are,

$$u=\frac{u'}{U_0},\quad \mathrm{v}=\frac{\mathrm{v}'}{U_0}\,\mathrm{Re}\,n^{\frac{1}{n+1}},\quad \mathrm{w}=\frac{w'}{U_0},\quad \mathrm{U}=\frac{U'}{U_0}$$

$$\mathrm{W}=\frac{W'}{U_0},\quad \mathrm{x}=\frac{\mathrm{x}'}{\mathrm{L}},\quad y=\frac{\mathrm{y}'}{\mathrm{L}}\,\mathrm{Re}\,n^{\frac{1}{n+1}},\quad \mathrm{S(x)}=\frac{\mathrm{L\,S'}}{\mathrm{U_0}}$$

where $\mathrm{Ren}=\dfrac{\rho\,\mathrm{U}_0^{2-n}\,\mathrm{L}^n}{\mathrm{m}}$, $\quad \mathrm{m}=2^{-(n+1)/2}\,\mathrm{K}$ provided $0<\mathrm{n}<2$. (2.10)

Here m is the true consistency of the liquid. Assuming implicitly that B does not depends upon the transverse coordinate y'. The equation of continuity can be satisfied identically by introducing a mathematical function, ψ such that

$$u=\frac{\partial\psi}{\partial \mathrm{y}}\quad \text{and}\quad \mathrm{v}=-\frac{\partial\psi}{\partial \mathrm{x}} \tag{2.11}$$

Equations (2.4-2.8) then becomes,

$$\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\partial y}-\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial \mathrm{y}^2}=\frac{\partial}{\partial \mathrm{y}}\left\{\left[\left(\frac{\partial^2\psi}{\partial \mathrm{y}^2}\right)^2+\left(\frac{\partial w}{\partial \mathrm{y}}\right)^2\right]^{\frac{n-1}{2}}\frac{\partial^2\psi}{\partial \mathrm{y}^2}\right\}+\mathrm{U}\frac{dU}{dx}+\mathrm{S(x)}\left(\mathrm{U}-\frac{\partial\psi}{\partial y}\right) \tag{2.12}$$

$$\frac{\partial \psi}{\partial y}\frac{\partial w}{\partial x}-\frac{\partial \psi}{\partial x}\frac{\partial w}{\partial y}=\frac{\partial}{\partial y}\left\{\left[\left(\frac{\partial^2 \psi}{\partial y^2}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2\right]^{\frac{n-1}{2}}\frac{\partial w}{\partial y}\right\}+U\frac{dW}{dx}+S(x)\left(W-w\right) \tag{2.13}$$

with the boundary conditions,

$$\text{At } y=0: \qquad \frac{\partial \psi}{\partial x}=\frac{\partial \psi}{\partial y}=w=0 \tag{2.14}$$

$$\text{At } y=\infty: \qquad \frac{\partial \psi}{\partial y}=U(x), \quad w=W(x) \tag{2.15}$$

The boundary value problem (2.12)-(2.15) is coupled non-linear partial differential equations whose exact solution is indeed tough. We transform this set of non-linear partial differential equations into ordinary differential equations by the similarity technique so that it’s numerical or closed form solution can be obtained. A group-theoretic analysis is employed in the next section to find the proper similarity transformations and the proper form of U(x) W(x) and S(x) for which similarity solutions will exist.

3 Similarity Analysis

The methods for obtaining similarity transformation were classified into (a) direct methods and (b) group-theoretic methods. The direct methods do not invoke group invariance such as separation of variables method. It is quite straightforward and simple to apply. On the other hand group-theoretic methods are mathematically more elegant and the important concept of invariance under a group of transformations is always invoked. In some group-theoretic procedures the specific form of the group is assumed a priori such as the Birkhoff–Morgan method and the Hellums–Churchill method. On the other hand, procedure such as the finite group method of Moran–Gaggioli is deductive. In this procedure, a general group of transformations is defined and similarity solutions are systematically deduced.

Our method of solution depends on the application of a one-parameter group transformation to the system of highly non-linear partial differential equations (2.12-2.13) along with boundary conditions (2.14-2.15). Under this group transformation the two independent variables (x, y) will be reduced to one i.e. η and the system of partial differential equations (2.12-2.13) along with boundary conditions (2.14-2.15) transforms into system of an ordinary differential equations. Our aim is to make use of group methods to represent the problem in the form of system of ordinary differential equations (similarity representation) in a single independent similarity variable η. In our analysis we search a complete set of absolute invariants of the independent and the dependent variables.

If $\eta = \eta\,(x, y)$ is the absolute invariant of independent variables then,

$F_j\,[\eta\,(x, y)] = g\,(x, y, \psi, w)$ for $j = 1, 2$ are two absolute invariants corresponding to ψ and w.

Two different groups of one-parameter transformation namely the linear and the spiral are usually found to give adequate treatment of boundary layer equations. Each group gives rise to two cases, case-I and case-II that will be separately discussed.

Case I.

A linear group of transformation is chosen as

$$x = A^{p_1}\,\bar{x}, \quad y = A^{p_2}\,\bar{y}, \quad \psi = A^{p_3}\,\bar{\psi}$$
$$w = A^{p_4}\,\bar{w}, \quad U = A^{p_5}\,\bar{U}, \quad W = A^{p_6}\,\bar{W}, \quad S = A^{p_7}\,\bar{S} \tag{3.1}$$

where $p_1, p_2, p_3, p_4, p_5, p_6, p_7$ and A are constants.

We now seek relations among the p_i's such that the equations (2.12-2.15) will be invariant under the group of transformation (3.1). By the application of transformation (3.1), the invariance of (2.12-2.15) implies

$$\frac{p_2}{p_1} = \frac{1 + (n-2)\,p}{(n+1)} \tag{3.2}$$

$$\frac{p_3}{p_1} = \frac{1 + (2n-1)\,p}{(n+1)} \tag{3.3}$$

$$\frac{p_4}{p_1} = \frac{p_5}{p_1} = \frac{p_6}{p_1} = p \tag{3.4}$$

$$\frac{p_7}{p_1} = p - 1 \tag{3.5}$$

We therefore obtain the complete set of absolute invariants for the independent and the dependent variables as,

$$\eta = \frac{y}{x^{\frac{1+(n-2)\,p}{(n+1)}}} \tag{3.6}$$

$$F_1(\eta) = \frac{\psi}{x^{\frac{1+(2n-1)\,p}{(n+1)}}} \tag{3.7}$$

$$G_1(\eta) = \frac{w}{x^p} \tag{3.8}$$

$$S_0 = \frac{S}{x^{(p-1)}} \tag{3.9}$$

$$C_1 = \frac{U}{x^p} \tag{3.10}$$

$$C_2 = \frac{W}{x^p} \tag{3.11}$$

where p is an arbitrary constant.

Here equations (3.10-3.11) are due to the invariance of boundary conditions (2.14-2.15). Substitution from (3.6 -3.11) into (2.12-2.15) yields a set of ordinary differential equations,

$$p\left(\frac{dF_1}{d\eta}\right)^2 - \left(\frac{1+(2n-1)\,p}{(n+1)}\right) F_1 \frac{d^2F_1}{d\eta^2}$$
$$= \frac{d}{d\eta}\left\{\left[\left(\frac{d^2F_1}{d\eta^2}\right)^2 + \left(\frac{dG_1}{d\eta}\right)^2\right]^{\frac{n-1}{2}} \frac{d^2F_1}{d\eta^2}\right\} + p\,C_1^2 + S_0\left(C_1 - \frac{dF_1}{d\eta}\right) \tag{3.12}$$

And

$$p\,G_1\left(\frac{dF_1}{d\eta}\right)-\left(\frac{1+(2n-1)\,p}{(n+1)}\right)F_1\,\frac{dG_1}{d\eta}$$
$$=\frac{d}{d\eta}\left\{\left[\left(\frac{d^2F_1}{d\eta^2}\right)^2+\left(\frac{dG_1}{d\eta}\right)^2\right]^{\frac{n-1}{2}}\frac{dG_1}{d\eta}\right\}+p\,C_1C_2+S_0\left(C_2-G_1\right)$$

(3.13)

with the transformed boundary conditions,

$$\text{At } \eta=0: \qquad F_1=F_1'=G_1=0 \tag{3.14}$$

$$\text{At } \eta=\infty: \qquad F_1'=C_1,\ G_1=C_2 \tag{3.15}$$

Deductions:

- For the non-magnetic case $S_0=0$, equations (3.12-3.15) will reduce to those derived by Na and Hansen [7].
- For the Newtonian case n = 1, equations (3.12-3.15) will reduced to those obtained by Timol et al. [18].
- For two-dimensional case, velocity w in z-direction along with free stream velocity W will vanish and hence equation (3.13) will vanish identically and G_1 along with its derivative will also vanish. Hence equation (3.12) becomes,

$$p\left(\frac{dF_1}{d\eta}\right)^2-\left(\frac{1+(2n-1)\,p}{(n+1)}\right)F_1\,\frac{d^2F_1}{d\eta^2}$$
$$=\frac{d}{d\eta}\left\{\left(\frac{d^2F_1}{d\eta^2}\right)^{n-1}\frac{d^2F_1}{d\eta^2}\right\}+p\,C_1^2+S_0\left(C_1-\frac{dF_1}{d\eta}\right)$$

(3.16)

which is reduced to the equation that is derived by Timol et al [25] and Cobble [26].

Case II.

A one-parameter spiral group of transformation is selected as

$$x = q_1 b + \bar{x}, \qquad y = e^{q_2 b}\,\bar{y}, \qquad \psi = e^{q_3 b}\,\bar{\psi}$$
$$w = e^{q_4 b}\,\bar{w}, \qquad U = e^{q_5 b}\,\bar{U}, \qquad W = e^{q_6 b}\,\bar{W}, \qquad S = e^{q_7 b}\,\bar{S} \tag{3.17}$$

where $q_1, q_2, q_3, q_4, q_5, q_6, q_7$ and b are constants.

Repeating the same procedures as in case-I, the absolute invariants observed are,

$$\xi = \frac{y}{e^{\left(\frac{n-2}{n+1}\right) q x}} \tag{3.18}$$

$$F_2(\xi) = \frac{\psi}{e^{\left(\frac{2n-1}{n+1}\right) q x}} \tag{3.19}$$

$$G_2(\xi) = \frac{w}{e^{q x}} \tag{3.20}$$

$$S_0 = \frac{S}{e^{q x}} \tag{3.21}$$

$$C_3 = \frac{U}{e^{q x}} \tag{3.22}$$

$$C_4 = \frac{W}{e^{q x}} \tag{3.23}$$

where q is an arbitrary constant.

Here equations (3.22-3.23) are due to the invariance of boundary conditions (2.14-2.15). Substituting these quantities from (3.18 – 3.23) in to the basic equations (2.14-2.15), the reduced set of ordinary differential equations found are,

$$q\left(\frac{d\,F_2}{d\xi}\right)^2-\left(\frac{2n-1}{n+1}\right)qF_2\,\frac{d^2F_2}{d\xi^2}$$
$$=\frac{d}{d\xi}\left\{\left[\left(\frac{d^2F_2}{d\xi^2}\right)^2+\left(\frac{d\,G_2}{d\xi}\right)^2\right]^{\frac{n-1}{2}}\frac{d^2F_2}{d\xi^2}\right\}+q\,C_3^2+S_0\left(C_3-\frac{dF_2}{d\xi}\right)\tag{3.24}$$

And

$$q\,G_2\left(\frac{d\,F_2}{d\xi}\right)-\left(\frac{2n-1}{n+1}\right)q\,F_2\,\frac{dG_2}{d\xi}$$
$$=\frac{d}{d\xi}\left\{\left[\left(\frac{d^2F_2}{d\xi^2}\right)^2+\left(\frac{d\,G_2}{d\xi}\right)^2\right]^{\frac{n-1}{2}}\frac{dG_2}{d\xi}\right\}+q\,C_3\,C_4+S_0\left(C_4-G_2\right)\tag{3.25}$$

with the transformed boundary conditions

$$\text{At } \xi=0: \qquad F_2=F_2'=G_2=0 \tag{3.26}$$

$$\text{At } \xi=\infty: \qquad F_2'=C_3,\ G_2=C_4 \tag{3.27}$$

The equations (3.9 – 3.11) and (3.21 – 3.23) shows the mainstream velocities U, W and the magnetic strength S are either powers or exponentials of x for which the similarity solutions may exist. We also observed U (x) = (const.) W (x) .This means the main flow streamlines are straight lines. Although this is a severe restriction, the form of velocity components is somewhat more general than the cases found by Schowalter [27].

Deductions:

- For the non-magnetic case $S_0 = 0$ equations (3.24 -3.27) will reduced to those derived by Na and Hansen [7].
- For the Newtonian case n = 1 equations (3.24 -3.27) will reduced to those obtained by Timol et al. [18]
- For two-dimensional case, velocity w in z-direction along with free stream velocity W will vanish and hence equation (3.25) will vanish identically and G_2 along with its derivative will also vanish. Hence equation (3.24) becomes,

$$q\left(\frac{d F_2}{d\xi}\right)^2 - \left(\frac{2n-1}{n+1}\right) q\, F_2 \frac{d^2F_2}{d\xi^2}$$
$$= \frac{d}{d\xi}\left\{\left(\frac{d^2F_2}{d\xi^2}\right)^{n-1} \frac{d^2F_2}{d\xi^2}\right\} + q\, C_3^2 + S_0\left(C_3 - \frac{dF_2}{d\xi}\right) \tag{3.28}$$

which is reduced to the equations that is derived by Timol et al [25] and Cobble [26].

4. Group Theoretic Solutions for Small Cross Flow

The restriction that U (x) = (const.) W (x) can be relaxed, if the cross-wise velocity is assumed to be small and the mainflow streamlines need not be straight. The principle of superposition of solutions is applied due to the momentum equation in the z-direction is linear in W. The simplifications permitted from the assumption of small cross flow may be made evident by considering the limiting deflection angle, θ of the streamlines within the boundary layer. This angle is the arctangent of the ratio of velocities in the z and x-directions, evaluated at y = 0, i.e.

$$\tan\theta = \lim_{y\to 0} \frac{w}{u} = \lim_{y\to 0} \frac{\left(\frac{\partial w}{\partial y}\right)}{\left(\frac{\partial u}{\partial y}\right)} \tag{4.1}$$

L'Hospital rule is being implemented due to indeterminate form. Therefore, for small cross flow, i.e. small θ, we would expect

$$\frac{\partial w}{\partial y} \leq \frac{\partial u}{\partial y} \tag{4.2}$$

within the boundary layer. The basic equations, (2.12-2.15) may then be simplified to

$$\frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial^2 \psi}{\partial y^2}\right)^n + U\frac{dU}{dx} + S(x)\left(U - \frac{\partial \psi}{\partial y}\right) \tag{4.3}$$

$$\frac{\partial \psi}{\partial y}\frac{\partial w}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial w}{\partial y} = \frac{\partial}{\partial y}\left[\left(\frac{\partial^2 \psi}{\partial y^2}\right)^{n-1}\frac{\partial w}{\partial y}\right] + U\frac{dW}{dx} + S(x)\left(W - w\right) \tag{4.4}$$

with the boundary conditions

$$\text{At } y = 0: \quad \frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial y} = w = 0 \tag{4.5}$$

$$\text{At } y = \infty: \quad \frac{\partial \psi}{\partial y} = U(x), \quad w = W(x) \tag{4.6}$$

By following the same procedures as in the preceding section, the following results are obtained.

Case I.

For the linear group of transformation, the absolute invariants investigated are,

$$\eta = \frac{y}{x^{\frac{1+(n-2)\nu}{(n+1)}}} \tag{4.7}$$

$$F_3(\eta) = \frac{\psi}{x^{\frac{1+(2n-1)\nu}{(n+1)}}} \tag{4.8}$$

$$G_3(\eta) = \frac{w}{x^{\nu}} \tag{4.9}$$

$$S_0 = \frac{S}{x^{(\nu-1)}} \tag{4.10}$$

$$C_5 = \frac{U}{x^{\nu}} \tag{4.11}$$

$$C_6 = \frac{W}{x^{\nu}} \tag{4.12}$$

where ν is an arbitrary constant.

Equations (4.11 - 4.12) are due to the invariance of boundary conditions (4.5-4.6). Thus, equations (4.3 – 4.6) are transformed to

$$\nu\left(\frac{dF_3}{d\eta}\right)^2 - \left(\frac{1+(2n-1)\nu}{(n+1)}\right) F_3 \frac{d^2F_3}{d\eta^2} = \frac{d}{d\eta}\left(\frac{d^2F_3}{d\eta^2}\right)^n + \nu C_5^2 + S_0\left(C_5 - \frac{dF_3}{d\eta}\right) \tag{4.13}$$

$$\nu G_3\left(\frac{dF_3}{d\eta}\right) - \left(\frac{1+(2n-1)\nu}{(n+1)}\right) F_3 \frac{dG_3}{d\eta} = \frac{d}{d\eta}\left\{\left(\frac{d^2F_3}{d\eta^2}\right)^{n-1}\frac{dG_3}{d\eta}\right\} + \nu C_5 C_6 + S_0\left(C_6 - G_3\right) \tag{4.14}$$

with the transformed boundary conditions,

$$\text{At } \eta = 0: \qquad F_3 = F_3' = G_3 = 0 \tag{4.15}$$

$$\text{At } \eta = \infty: \qquad F_3' = C_5, \ G_3 = C_6 \tag{4.16}$$

Deductions:

- For the non-magnetic case $S_0 = 0$ equations (4.13- 4.16) will reduced to those derived by Na and Hansen [7].
- For $\nu = 0$ and n = 1 the above case is reduced to that analyzed by Timol et al [8].
- For two-dimensional case, velocity w in z-direction along with free stream velocity W will vanish and hence equation (4.14) will vanish identically and G_3 along with its derivative will also vanish. Hence equation (4.13) becomes,

$$\nu\left(\frac{dF_3}{d\eta}\right)^2 - \left(\frac{1+(2n-1)\nu}{(n+1)}\right) F_3 \frac{d^2F_3}{d\eta^2} = \frac{d}{d\eta}\left(\frac{d^2F_3}{d\eta^2}\right)^n + \nu C_5^2 + S_0\left(C_5 - \frac{dF_3}{d\eta}\right) \tag{4.17}$$

which is reduced to the equation that is derived by Timol et al [25] and Cobble [26].

Case II.

For the spiral group of transformation, the following absolute invariants are obtained.

$$\xi = \frac{y}{e^{\left(\frac{n-2}{n+1}\right)\delta x}} \tag{4.18}$$

$$F_4(\xi) = \frac{\psi}{e^{\left(\frac{2n-1}{n+1}\right)\delta x}} \tag{4.19}$$

$$G_4(\xi) = \frac{w}{e^{\delta x}} \tag{4.20}$$

$$S_0 = \frac{S}{e^{\delta x}} \tag{4.21}$$

$$C_7 = \frac{U}{e^{\delta x}} \tag{4.22}$$

$$C_8 = \frac{W}{e^{\delta x}} \tag{4.23}$$

where δ is an arbitrary constant.

Equations (4.22 - 4.23) are due to the invariance of boundary conditions (4.5-4.6). Substituting these quantities in equations (4.3 - 4.6) we obtain,

$$\delta\left(\frac{dF_4}{d\xi}\right)^2 - \left(\frac{2n-1}{n+1}\right)\delta F_4 \frac{d^2F_4}{d\xi^2} = \frac{d}{d\xi}\left(\frac{d^2F_4}{d\xi^2}\right)^n + \delta\, C_7^2 + S_0\left(C_7 - \frac{dF_4}{d\xi}\right) \tag{4.24}$$

$$\delta G_4\left(\frac{dF_4}{d\xi}\right) - \left(\frac{2n-1}{n+1}\right)\delta F_4 \frac{dG_4}{d\xi} = \frac{d}{d\xi}\left\{\left(\frac{d^2F_4}{d\xi^2}\right)^{n-1}\frac{dG_4}{d\xi}\right\} + \delta C_7 C_8 + S_0\left(C_8 - G_4\right) \tag{4.25}$$

with the boundary conditions

$$\text{At } \eta = 0: \quad F_4 = F_4' = G_4 = 0 \tag{4.26}$$

$$\text{At } \eta = \infty: \quad F_4' = C_7,\ G_4 = C_8 \tag{4.27}$$

Deductions:

- For the non-magnetic case $S_0 = 0$ equations (4.24 – 4.27) will reduced to those derived by Na and Hansen [7].
- For $\delta = 0$ and n = 1, the above case is reduced to that analyzed by Timol et al [8].
- For two-dimensional case, velocity w in z-direction along with free stream velocity W will vanish and hence equation (4.25) will vanish identically and G_4 along with its derivative will also vanish. Hence equation (4.24) becomes,

$$\delta\left(\frac{dF_4}{d\xi}\right)^2 - \left(\frac{2n-1}{n+1}\right)\delta F_4 \frac{d^2F_4}{d\xi^2} = \frac{d}{d\xi}\left(\frac{d^2F_4}{d\xi^2}\right)^n + \delta C_7^2 + S_0\left(C_7 - \frac{dF_4}{d\xi}\right) \qquad (4.28)$$

which is the same equation derived by Timol et al [25] and Cobble [26].

5 Conclusion

The analysis of the laminar, incompressible three-dimensional MHD boundary layer equations of power law fluids with streamlines forming a system of "Translates" led to solutions for mainstream flows described by equations (3.10-3.11) and (3.22-3.23). By placing the condition of small cross flow, restrictions on mainstream velocity W are considerably relaxed, as it is shown in equations (4.11-4.12) and (4.22-4.23). Furthermore, the linearity of the momentum equation in the crosswise direction makes it possible to generalize the form to any mainstream shape, which can be approximated by a polynomial. Also it is interesting to note that in all cases of flow geometries, the specific form of the transverse magnetic field S(x), for which similarity solution exist is derived from conditions of existence of similarity solution rather than assume it in priori.

6 Open Problem

In the present paper using group theoretic method systematic similarity analysis is derived to find similarity equations for steady three-dimensional incompressible boundary layer flow of electrically conducting non-Newtonian power-law fluids past external surface under the influence of transverse magnetic field. In order to meet the similarity requirements it is observed in the earlier research that the specific form of outer flow and specific form of imposed magnetic field parameter is assumed in priory and hence sets of similarity equations so obtained have limited practical applications. Where as in the present work in all cases of flow geometries, the specific form of the transverse magnetic field S(x), for which similarity solution exist is derived from conditions of existence of similarity solution rather than assuming it in priori.

ACKNOWLEDGEMENTS

The authors are thankful to the referees for their critical review as well as their valuable comments.

Notations

F, G : Dependent variables in the transformed ordinary differential equations.

L : A characteristic length

m, n : Parameters of power-law fluid model

Ren : Reynolds number

u, v, w : Velocity components along the x, y and z-axes

U, W : Main stream velocity in x and z direction

ρ : Density of fluid

ψ : A mathematical function

η, ξ : Independent variables in the transformed ordinary differential equations

τ : Stress tensor

Δ : The rate of deformation tensor

τ_{yx} , τ_{yz} : The two non-vanishing components of the stress tensor.

S : Magnetic field strength

S_0, p, q, ν, δ : Arbitrary constants.

References

[1] Rajagopal K, Mechanics of non-Newtonian fluids, in: G.R Galdi, J. Necas (Eds.), Recent Developments in Theoreical Fluid Mechanics, Vol. 291: Pitman Research Notes in Mathematics Series, Longman, Essex, (1993),

pp. 129-162.

[2] Rajagopal K. R. and Srinivasa A. R., A thermodynamical framework for rate-type fluids, *J. of Non-Newtonian Fluid Mechanics*, 88(3), (2000) pp. 207-227

[3] Y. Wang, T. Hayat and K. Hutter, On nonlinear magnetohydrodynamic problems of an Oldroyd 6-constant fluid, *Int. J. Non-Linear Mechs*. 40, (2005), pp.49-58

[4] Hansen A.G. and Herzig H.Z., On Possible Similarity Solutions for Three-Dimensional Incompressible Laminar Boundary Layers. I - Similarity with Respect to Stationary Rectangular Coordinates, *NACA TN* 3768, (1956).

[5] Herzig H. Z. and Hansen A. G., On Possible Similarity Solutions for Three-Dimensional Incompressible Laminar Boundary Layers. II - Similarity with Respect to Stationary Polar Coordinates, *NACA TN* 3832, (1956).

[6] Herzig H. Z. and Hansen Arthur G, On Possible Similarity Solutions for Three-Dimensional Incompressible Laminar Boundary Layers. III - Similarity with Respect to Stationary Polar Coordinates for Small Angle Variation, *NACA TN* 3890, (1957).

[7] Na T.Y. and Hansen A.G., Similarity solutions of a class of laminar three-dimensional boundary layer equations of power-law fluids, *Int. J. Non-linear Mech.*, vol.2, (1967), pp. 373 – 385.

[8] Timol M.G. and Kalthia N.L., Similarity solutions of three-dimensional boundary layer equations of non-Newtonian fluids, *Int. J. non-Linear Mech.*, vol.21, No.6, (1986), pp. 475 – 481.

[9] T. Sarpkaya, Flow of non-Newtonian fluids in a magnetic filed, *AIChE J*. 7 (1961), pp. 324–328.

[10] D.S. Djukic, On the use of Crocco’s equation for flow of power law fluids in a transverse magnetic field, *AIChE J.* 19 (1973), pp.1159–1163.

[11] D.S. Djukic, Hiemenz magnetic flow of power-law fluids, *J. Appl. Mech-* T ASME 41 (1974), pp. 822–823.

[12] H.I. Andersson, K.H. Bech, B.S. Dandapat, Magnetohydrodynamic flow of a power-law fluid over a stretching sheet, *Int. J. Non-linear Mech.* 27 (1992) pp. 929–936.

[13] S.J. Liao, On the analytic solution of magnetohydrodynamic flows of non-Newtonian fluids over a stretching sheet, *J. Fluid Mech*. 488 (2003) pp.189–212.

[14] Hang Xu, Shi-Jun Liao, Series solutions of unsteady magnetohydrodynamic flows of Non-Newtonian fluids caused by an impulsively stretching plate, *J. Non-Newtonian Fluid Mech.* 129 (2005) 46–55.

[15] Anuar Ishaka, Roslinda Nazar, Ioan Pop, Magnetohydrodynamic (MHD) flow and heat transfer due to a stretching cylinder, *Energy Conversion and Management* 49 (2008) 3265–3269.

[16] Mohamed Guedda, Zakia Hammouch, Similarity Flow Solutions of a Non-Newtonian Power-law Fluid, *Int. J. of Nonlinear Science* Vol.6(2008), No.3, pp.255-264.

[17] M. Ferdows and B. I.Olajuwon, On the Similarity Solution of Micropolar Power Law Fluid over a Vertical Plate, *Int. J. Contemp. Math. Sciences*, Vol. 6, (2011), No. 3, pp.133 – 143.

[18] Timol M.G. and Timol A.G., Three-dimensional magnetofluiddynemic flow with pressure gradient and fluid injection, *Indian J. Pure Appl. Math.*, 19(2), (1988), 208 – 216.

[19] Bird R B, Stewart W E, Lightfoot E N, *Transport Phenomena*, Wiley, Singapore, (1960).

[20] Kapur J N, Note on the boundary layer equations for power-law fluids, *J.Phys.Soc.Japan*, 18 (1963), 144.

[21] V. Sirohi, M.G. Timol and N.L. Kalthia Powell-Eyring model flow near an accelerated plate, *Fluid Dy. Res*. Vol.2, No.3, (1987), pp.193-204

[22] Patel M, Timol M. G., The Stress–Strain Relationship for Visco-Inelastic Non-Newtonian Fluids. *Int. J. of Appl. Math and Mech.*, **6** (12): 79-93, 2010.

[23] Patel Manisha, Timol M G, On the class of similarity solutions for three-dimensional boundary layer flows of non-Newtonian fluids, *Journal of Veer Narmad South Gujarat University*, (2004) II-B: 103-109.

[24] Patel Manisha, Timol M G, Similarity solutions for three-dimensional, laminar, incompressible Boundary Layer equation of non-Newtonian fluids by Generalized dimensional analysis, *Varahmihir J. of Math. Sci.* (2005) Vol 5, No.2, 387-394.

[25] Timol M.G., Group-Theoretic Approach to the Similarity Solutions in Viscoinelastic Flows., Doctoral Thesis, *South Gujarat University, Surat, India,* (1986).

[26] Cobble M. H., Magneto hydrodynamic non-Newtonian power-law fluid with pressure gradient and fluid injection, *J. of Engg. Math*, Vol. 14, No.1, (1980), 1980.

[27] Schowalter W. R, The application of boundary-layer theory to power law pseudoplastic fluids: similar solutions. A*IChE J.* 6, (1960), 24.