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Rothe Method Applied to Semilinear Hyperbolic Integro-differential Equation With Integral Conditions

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Abstract

We are interested in the study of hyperbolic integro-differential equation with initial and integral conditions by using Rothe's method. We transform the integral inhomogeneous conditions to homogeneous ones by introducing a new function then prove, under some conditions on the function appearing in the equation, the existence and uniqueness of weak solution

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1 Introduction

Many physical phenomena can be modeled by boundary value problems with non-local conditions. This is when the values of function on the boundary are related to values inside the domain or when direct measurements on the boundary are not possible.

It shows that problems related to non- local conditions have many applications in many problems such as in the theory of heat conduction, thermo elasticity, plasma physics, control theory, etc.. In particular, the introduction of non-local conditions can improve the qualitative and quantitative characteristics of the problem which lead to good results concerning existence, uniqueness and regularity of the solution. The current analysis of these problems has a great interest. and many methods are used to solve such problems as the functional methods, methods of approximation, a priori estimates ....

In this work, we apply an approximation method to study of an hyperbolic integro-differential equation with non local conditions of integral type. The importance of approximation methods is that they don't only prove the existence and uniqueness of the solution but they also allow the construction algorithms for numerical solutions. These methods such Galerkin method and discretization in time methode called also the Rothe method, are a very effective tool in the study of the approximate solution and its convergence to the solution of problem. In general it is difficult to find the exact solution of such problems, the approximation methods provide other ways to find approximate solutions.

The Rothe method has its origins in the work of E. Rothe in 1930 [14], it has also been developed in the works of Rektorys [12, 13] and Kacur [7, 8]. Several other results have been achieved for differential equations with integral conditions in the work of Bouziani and all [4, 10] and Mesloub [9] and for the problems of integro-differential equations in the works of Bahuguna [1-3] and Guezane-Lakoud and all [5, 6].

Inspired by [1, 4, 6], our goal is to extend this technics to hyperbolic integro-differential equations with integral conditions.

2 Description of the Problem and Hypothesis

In this paper we study a semi-linear hyperbolic integro-differential equation

$$\frac{\partial^2\theta}{\partial t^2} - \frac{\partial^2\theta}{\partial x^2} = g\left(x,t\right) + \int_0^t a\left(t-s\right) k\left(s,\theta\left(x,s\right)\right) ds \qquad \left(x,s\right) \in \left(0,1\right) \times \left(0,T\right), \tag{1}$$

with the initial conditions

$$\theta\left(x,0\right) = \theta_0\left(x\right), \ \frac{\partial\theta}{\partial x}\left(x,0\right) = \theta_1\left(x\right) \qquad x \in \left(0,1\right) \tag{2}$$

and the integral conditions

$$\int_0^1 \theta\left(x,t\right) dx = E\left(t\right) \qquad t \in \left[0,T\right], \tag{3}$$

$$\int_0^1 x\theta\left(x,t\right) dx = M\left(t\right) \qquad t \in \left[0,T\right] \tag{4}$$

Using the transformation $u\left(x,t\right) = \theta\left(x,t\right) - r\left(x,t\right)$, where

$r(x,t) = 6(2M(t) - E(t))x - 2(3M(t) - 2E(t))$, then, the equivalently problem is to find a function u satisfying

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = f(x,t) + \int_0^t a(t-s)\,k(s,u(x,s))\,ds \qquad (x,s) \in (0,1) \times (0,T), \tag{5}$$

$$u(x,0) = U_0(x), \ \frac{\partial u}{\partial x}(x,0) = U_1(x) \qquad x \in (0,1), \tag{6}$$

$$\int_0^1 u(x,t)\,dx = \int_0^1 xu(x,t)\,dx = 0 \tag{7}$$

where

$$f(x,t) = g(x,t) - \frac{\partial^2 r}{\partial t^2}, U_0(x) = \theta_0(x) - r(x,0), U_1(x) = \theta_1(x) - \frac{\partial r}{\partial t}(x,0). \tag{8}$$

Hence, instead of looking v we simply look for u. The solution of problem (1)-(4) will be obtained by the formula $\theta(x,t) = u(x,t) + r(x,t)$.

To solve this problem we apply Rothe's approximation, we divide the time interval I into n subintervals $[t_{j-1,}t_j]$, $j = 1....n$, where $t_j = j.h$ and the length $h = \frac{T}{n}$, we denote $u_j = u_j(x) = u_j(x, jh)$ the approximation of u, then we replace $\frac{\partial^2 u}{\partial t^2}$ and $\frac{\partial u}{\partial t}$ at each point $t = t_j$, $j = 1....n$,by the difference quotients respectively $\delta^2 u_j = \dfrac{\delta u_j - \delta u_{j-1}}{h}$ and $\delta u_j = \dfrac{u_j - u_{j-1}}{h}$. Thereafter, we get a system of n differential equations in x with the unknown functions $u_j(x)$ that are the approximative solutions of (5) at the points t_j, we use these functions to construct the Rothe's functions defined by

$$u^n(x,t) = u_{j-1} + \delta u_j(t - t_{j-1}), t \in [t_{j-1}, t_j], j = 1....n,$$

and the corresponding step function

$$\overline{u}^n(t) = \begin{cases} u_j, & t \in [t_{j-1}, t_j], \quad j = 1....n \\ & U_0 \quad t \in [-h, 0]. \end{cases}$$

Then we prove that $u^n(x,t)$ converges in some appropriate sense to the solution of (5).

Let $I = [0,T]$, we denote by $(,)$, and $\|.\|$ the classical inner product and the corresponding norm respectively in $L^2(0,1)$. Let V be the following space

$$V = \left\{ v \in L^2(0,1) \ : \int_0^1 v(x)\,dx = \int_0^1 xv(x)\,dx = 0 \right\} \tag{9}$$

Obviously, V is a closed subset of $L^2(0,1)$ and hence is a Hilbert space that can be embedded in the space B, which is the completion of $C_0(0,1)$, the space of all continuous functions on $(0,1)$ having compact support in $(0,1)$ with the inner product $(u,v)_B = \int_0^1 \left(\int_0^x u(\xi)\,d\xi \int_0^x v(\xi)\,d\xi\right) dx$ and the corresponding norm $\|u\|_B^2 = (u,u)_B$, then the inequality

$$\|u\|_B^2 \leq \frac{1}{2}\|u\|^2 \tag{10}$$

holds for every $u \in L^2(0,1)$.

We identify a function $u : (0,1)\times I \longmapsto u(x,t) \in R$, such that for each $t \in I$, $u(.,t) \in L^2(0,1)$ with the function $u : I \longmapsto L^2(0,1)$ by $u(t)(x) = u(x,t)$, $t \in I$, $x \in (0,1)$. We will use in this paper the classical function spaces $C^{0,1}(I,X)$, $C^{1,1}(I,X)$, $L^2(I,X)$ and $L^\infty(I,X)$ where X is a Banach space.

For solving the problem (5)-(7) we make the following hypothesis:

H_1) $f(t) \in L^2(0,1)$ and $\|f(t) - f(t')\|_B \leq l\,|t-t'|$, for some positive constant l.

H_2) $U_0(x), U_1(x) \in H^2(0,1)$.

H_3)

$$\int_0^1 U_0(x)\,dx = \int_0^1 xU_0(x)\,dx = 0, \tag{11}$$

$$\int_0^1 U_1(x)\,dx = \int_0^1 xU_1(x)\,dx = 0. \tag{12}$$

H_4) The continuous functions a and k are such that

$|a(t) - a(t')| \leq c_1\,|t-t'|$. $k : I\times \mathrm{B} \longrightarrow L^2(0,1)$ is continuous to both the variables and satisfies $\|k(t,u)\|_B \leq \|u(t)\|_B$.

H_5) For $u(t)$, $v(t) \in V$, we have $\|k(t,u) - k(t,v)\| \leq L(t)\,\|u(t) - v(t)\|_B$, for almost all $t \in I$, where $L \in L^1(I)$ is a positive function.

Definition 2.1 *By a weak solution of the problem* $(5)-(7)$ *we mean a function* $u : I \longrightarrow L^2(0,1)$ *such that*:

1) $u \in C^{0,1}(I,V)$

2) *u has (a.e. in I) a strong derivative* $\frac{du}{dt} \in L^\infty(I,V) \cap C^{0,1}(I,B))$ *and* $\frac{\partial^2 u}{\partial t^2} \in L^\infty(I,B)$.

3) $u(0) = U_0$ *in V and* $\frac{du}{dt}(0) = U_1 inB$.

4) *for all $\phi \in V$ and a.e.$t \in I$, the identity*

$$\int_I \left(\frac{d^2u(t)}{dt^2}, \phi\right)_B dt + \int_I (u(t), \phi)\,dt = \int_I \left(f(t) + \int_0^t a(t-s)\,k(s,u)\,ds, \phi\right)_B dt \tag{13}$$

is satisfied.

3 Main Results

Now we expose the main result of this paper.

Theorem 3.1 *Assume that hypotheses $(H_1)-(H_4)$ hold, then there exists a weak solution u for problem (5)-(7) in the sense of definition 2.1. In addition, if (H_5) is also satisfied, then u is unique.*

The prove of this theorem will be done later with the help of several Lemmas.

3.1 Discretization Scheme and A priori Estimates

We divide the interval I into n subintervals of the length $h = \frac{T}{n}$, and denote $u_j = u(t_j)$, with $t_j = jh$, $j = 1, ..., n$. Successively, for $j = 1, ..., n$ we solve the linear stationary boundary value problem

$$\frac{u_j - 2u_{j-1} + u_{j-2}}{h^2} - \frac{d^2 u_j}{dx^2} = f_j + h \sum_{i=0}^{j-1} a_{ji} k_i, \tag{14}$$

$$\int_0^1 u_j(x)\, dx = 0, \tag{15}$$

$$\int_0^1 x u_j(x)\, dx = 0, \tag{16}$$

where $f_j = f(t_j)$, $a_{ji} = a(t_j - t_i)$, and $k_i = k(t_i, u_i)$. Setting $u_{-1}(x) = U_0(x) - hU_1(x)$, $u_0(x) = U_0(x)$, $\quad x \in (0,1)$. Denote $\delta u_j = \frac{u_j - u_{j-1}}{h}$, $\delta^2 u_j = \frac{\delta u_j - \delta u_{j-1}}{h^2}$, $\quad j = 0, ..., n$ and define Rothe's sequence (u_n) of Lipschitz continuous functions from $I \longrightarrow H^2(0,1) \cap V$ by

$$u^{(n)}(t) = u_{j-1} + \delta u_j (t - t_j), \qquad t \in [t_{j-1}, t_j], \quad j = 1, ..., n \tag{17}$$

the auxiliary functions are

$$\delta u^{(n)}(t) = \delta u_{j-1} + \delta^2 u_j (t - t_j), \qquad t \in [t_{j-1}, t_j], \quad j = 1, ..., n, \tag{18}$$

$$\bar{u}^{(n)}(t) = \begin{cases} u_j \text{ for } t \in (t_{j-1}, t_j], & j = 1, ..., n \\ U_0 \quad \text{for} \quad t \in [-h, 0], \end{cases} \tag{19}$$

$$\bar{\delta u}^{(n)}(t) = \begin{cases} \delta u_j \text{ for } t \in (t_{j-1}, t_j], & j = 1, ..., n \\ U_1 \quad \text{for} \quad t \in [-h, 0] \end{cases} \tag{20}$$

Theorem 3.2 *The problem (14)-(16) admits a unique solution* $u_j \in H^2(0,1)$ *for* $n \geq 1,\ j = 1, ..., n.$

Proof. Suppose that u_{j-1} and u_{j-2} are already known in $H^2(0,1)$, then $f_j \in L^2(0,1)$, then the general solution of (14) is given by

$$u_j(x) = k_1(x)\cosh\left(\frac{x}{h}\right) + k_2(x)\sinh\left(\frac{x}{h}\right), \quad x \in (0,1) \tag{21}$$

where k_1 and k_2 are two functions of x such that

$$\left.\begin{array}{c} \frac{dk_1}{dx}(x)\cosh\left(\frac{x}{h}\right) + \frac{dk_2}{dx}(x)\sinh\left(\frac{x}{h}\right) = 0 \\ \frac{dk_1}{dx}(x)\sinh\left(\frac{x}{h}\right) + \frac{dk_2}{dx}(x)\cosh\left(\frac{x}{h}\right) = h\left[\frac{-2u_{j-1}+u_{j-2}}{h} - f_j - h\sum_{i=0}^{j-1} a_{ji}k_i\right]. \end{array}\right\} \tag{22}$$

Remarking that the determinant of (22) is

$$\triangle = \cosh^2\left(\frac{x}{h}\right) - \sinh^2\left(\frac{x}{h}\right) = 1, \tag{23}$$

then

$$\frac{dk_1}{dx}(x) = hF_j(x)\sinh\left(\frac{x}{h}\right), \frac{dk_2}{dx}(x) = hF_j(x)\cosh\left(\frac{x}{h}\right), \tag{24}$$

with

$$F_j = \frac{-2u_{j-1}+u_{j-2}}{h} - f_j - h\sum_{i=0}^{j-1} a_{ji}k_i \tag{25}$$

that is

$$\left.\begin{array}{c} k_1(x) = h\int_0^x F_j(\xi)\sinh\left(\frac{\xi}{h}\right)d\xi + \lambda_1 \\ k_2(x) = h\int_0^x F_j(\xi)\cosh\left(\frac{\xi}{h}\right)d\xi + \lambda_2. \end{array}\right\} \tag{26}$$

From identity (17), we get

$$u_j(x) = h\int_0^x F_j(\xi)\sinh\left(\frac{x-\xi}{h}\right)d\xi + \lambda_1\cosh\left(\frac{x}{h}\right) + \lambda_2\sinh\left(\frac{x}{h}\right). \tag{27}$$

Choosing a pair (λ_1, λ_2) such that conditions (15) and (16) hold, then we obtain

$$\left.\begin{array}{c} \lambda_1\int_0^1 \cosh\left(\frac{x}{h}\right)dx + \lambda_2\int_0^1 \sinh\left(\frac{x}{h}\right)dx = -h\int_0^1\int_0^x F_j(\xi)\sinh\left(\frac{x-\xi}{h}\right)d\xi dx \\ \lambda_1\int_0^1 x\cosh\left(\frac{x}{h}\right)dx + \lambda_2\int_0^1 x\sinh\left(\frac{x}{h}\right)dx = -h\int_0^1\int_0^x xF_j(\xi)\sinh\left(\frac{x-\xi}{h}\right)d\xi dx, \end{array}\right\} \tag{28}$$

therefore

$$\left\{\begin{array}{l} \lambda_1\sinh\frac{1}{h} + \lambda_2\left(\cosh\frac{1}{h} - 1\right) = -\int_0^1\int_0^x F_j(\xi)\sinh\frac{x-\xi}{h}d\xi dx \\ \lambda_1\sinh\frac{1}{h} + \lambda_2\left(\cosh\frac{1}{h} - 1\right) = -\int_0^1\int_0^x F_j(\xi)\sinh\frac{x-\xi}{h}d\xi dx, \end{array}\right. \tag{29}$$

since the determinant of (29)

$$\triangle(h) = 2h - 2h\cosh\left(\frac{1}{h}\right) + \sinh\left(\frac{1}{h}\right) = 2\sinh\left(\frac{1}{2h}\right)\left(\cosh\left(\frac{1}{2h}\right) - 2h\sinh\left(\frac{1}{2h}\right)\right) \tag{30}$$

does not vanish for any $h > 0$, we deduce that the system (29)) admits a unique solution $(\lambda_1, \lambda_2) \in R^2$, which means that problem (14)-(16) is uniquely solvable and obviously $u_j \in H^2(0,1)$ since $F_j \in L^2(0,1)$. In what follow we denote by C a non negative constant not depending on n, j and h.

Lemma 3.3 *There exist a constant $C > 0$ and a natural number $N \in N^*$ such that* $\|\delta u_j\|_B^2 + \|u_j\|^2 \leq C, \qquad j = 1, ..., n, \; n > N.$

Proof. Let $\phi \in V$. It is easy to see that $\int_0^x (x-\xi)\phi(\xi)\,d\xi = \Im_x^2\phi$, $\forall x \in (0,1)$ where $\Im_x^2\phi := \Im_x(\Im_\xi\phi) = \int_0^x d\xi \int_0^\xi \phi(\mu)\,d\mu$. Thus

$$\Im_1^2\phi = \int_0^1 (1-\xi)\phi(\xi)\,d\xi = \int_0^1 \phi(\xi)\,d\xi - \int_0^1 \xi\phi(\xi)\,d\xi. \tag{31}$$

Multiplying (14) by $\Im_x^2\phi$ for all $j = 1, ..., n$, then integrating over $(0,1)$ to get

$$\int_0^1 \delta^2 u_j(x)\,\Im_x^2\phi dx - \int_0^1 \frac{d^2u_j}{dx^2}(x)\,\Im_x^2\phi dx = \int_0^1 \left(f_j(x) + h\sum_{i=0}^{j-1} a_{ji}k_i\right)\Im_x^2\phi dx. \tag{32}$$

Integrating by parts all terms in (32) then using (30), it yields

$$\begin{aligned}\int_0^1 \delta^2 u_j(x)\,\Im_x^2\phi dx &= \Im_x\left(\delta^2 u_j\right)\Im_x^2\phi\Big|_{x=0}^{x=1} - \int_0^1 \Im_x\left(\delta^2 u_j\right)\Im_x\phi dx \\ &= -\left(\delta^2 u_j, \phi\right)_B.\end{aligned}$$

$$\int_0^1 \frac{d^2u_j}{dx^2}(x)\,\Im_x^2\phi dx = -u_j(x)\,\Im_x\phi\Big|_{x=0}^{x=1} + \int_0^1 u_j(x)\,\phi(x)\,dx = (u_j, \phi).$$

$$\begin{aligned}\int_0^1 \left(f_j + h\sum_{i=0}^{j-1} a_{ji}k_i\right)\Im_x^2\phi dx &= \int_0^1 \frac{d}{dx}\Im_x\left(f_j + h\sum_{i=0}^{j-1} a_{ji}k_i\right)\Im_x^2\phi dx \\ &= -\left(f_j + h\sum_{i=0}^{j-1} a_{ji}k_i, \phi\right)_B.\end{aligned} \tag{33}$$

So (32) becomes

$$\left(\delta^2 u_j, \phi\right)_B + (u_j, \phi) = \left(f_j + h\sum_{i=0}^{j-1} a_{ji}k_i, \phi\right)_B, \quad \forall\phi \in V, \; \forall j = 1, ..., n. \tag{34}$$

Setting $\phi = \delta u_j \in V$ in (34), we get $(\delta u_j - \delta u_{j-1}, \delta u_j)_B + (u_j, u_j - u_{j-1}) = h^2 \sum_{i=0}^{j-1} (a_{ji}k_i, \delta u_j)_B + h(f_j, \delta u_j)_B$. Taking account the identities $2(u_j, u_j - u_{j-1}) = \|u_j\|^2 + \|u_j - u_{j-1}\|^2 - \|u_{j-1}\|^2$, $2(\delta u_j - \delta u_{j-1}, \delta u_j)_B = \|\delta u_j\|_B^2 + \|\delta u_j - \delta u_{j-1}\|_B^2 - \|\delta u_{j-1}\|_B^2$, we obtain

$$\|\delta u_j\|_B^2 - \|\delta u_{j-1}\|_B^2 + \|u_j\|^2 - \|u_{j-1}\|^2 \leq 2Ch^2 \sum_{i=0}^{j-1} \|k_i\|_B \|\delta u_j\|_B + 2h\|f_j\|_B \|\delta u_j\|_B. \tag{35}$$

From ε-inequality for $\varepsilon = 1$, we deduce $2\|k_i\| \|\delta u_j\|_B \leq 2\|u_i\| \|\delta u_j\|_B \leq \|u_i\|^2 + \|\delta u_j\|_B^2$, $2h\|f_j\|_B \|\delta u_j\|_B \leq 2h\|f\|_{C(I,B)} \|\delta u_j\|_B \leq h + Ch\|\delta u_j\|_B^2$. Substituting in (35) to get

$$\|\delta u_j\|_B^2 - \|\delta u_{j-1}\|_B^2 + \|u_j\|^2 - \|u_{j-1}\|^2 \leq Ch\|\delta u_j\|_B^2 + Ch^2 \sum_{i=0}^{j-1} \|u_i\|^2 + Ch \tag{36}$$

Now choosing in inequality (36), N such that $C\frac{T}{N} < 1$, then for $n > N$, it yields

$$\begin{aligned}(1 - Ch)\left[\|\delta u_j\|_B^2 + \|u_j\|^2\right] &\leq \left(1 - Ch^2\right)\left[\|\delta u_{j-1}\|_B^2 + \|u_{j-1}\|^2\right] \\ &\quad + Ch^2 \sum_{i=0}^{j-1} \|u_i\|^2 + Ch. \end{aligned} \tag{37}$$

Applying this inequality recursively, we obtain $(1 - Ch)^j \left[\|\delta u_j\|_B^2 + \|u_j\|^2\right] \leq (1 + jCh^2)^j \left[\|\delta u_0\|_B^2 + \|U_0\|^2\right] + jCh$, which implies $\|\delta u_j\|_B^2 + \|u_j\|^2 \leq C$.

Lemma 3.4 *There exist a constant $C > 0$ and a natural number $N \in N^*$ such that* $\|\delta^2 u_j\|_B^2 + \|\delta u_j\|^2 \leq C, \quad j = 1, ..., n > N.$

Proof. We consider the difference $(34)_j - (34)_{j-1}$ so

$$\begin{aligned}\left(\delta^2 u_j, \phi\right)_B + (u_j - u_{j-1}, \phi) &= \left(\delta^2 u_{j-1}, \phi\right)_B + (a_{jj-1}k_{j-1}, \phi)_B \\ &\quad + h \sum_{i=0}^{j-2} ((a_{ji} - a_{j-1i}) k_i, \phi)_B \quad + (f_j - f_{j-1}, \phi)_B. \end{aligned} \tag{38}$$

Setting $\phi = \delta^2 u_j$ in (38) we get

$$\begin{aligned}2\left\|\delta^2 u_j\right\|_B^2 + 2(\delta u_j, \delta u_j - \delta u_{j-1}) &= 2\left(\delta^2 u_{j-1}, \delta^2 u_j\right)_B + 2h\left(a_{jj-1}k_{j-1}, \delta^2 u_j\right)_B \\ &\quad + 2h \sum_{i=0}^{j-2} \left((a_{ji} - a_{j-1i}) k_i, \delta^2 u_j\right)_B + 2\left(f_j - f_{j-1}, \delta^2 u_j\right)_B, \end{aligned}$$

which gives

$$2\left\|\delta^2 u_j\right\|_B^2 + \|\delta u_j\|^2 - \|\delta u_{j-1}\|^2 \le \left\|\delta^2 u_{j-1}\right\|_B^2 + \left\|\delta^2 u_j\right\|_B^2$$

$$+2Ch\|k_{j-1}\|_B\left\|\delta^2 u_j\right\|_B 2Ch^2\sum_{i=0}^{j-2}\|k_i\|_B\left\|\delta^2 u_j\right\|_B 2\|f_j - f_{j-1}\|\left\|\delta^2 u_j\right\|_B. \quad (39)$$

Remarking that $\|k_i\|_B \le \|u_i\| \le C$ and using ε-inequality for $\varepsilon = 1$ we obtain $2Ch\|k_{j-1}\|_B\|\delta^2 u_j\|_B \le Ch\|k_{j-1}\|_B^2 + Ch\|\delta^2 u_j\|_B^2 \le Ch + Ch\|\delta^2 u_j\|_B^2$. On the other hand we have

$$2Ch^2\sum_{i=0}^{j-2}\|k_i\|_B\left\|\delta^2 u_j\right\|_B \le Ch^2\sum_{i=0}^{j-2}\left(\|k_i\|_B^2 + \left\|\delta^2 u_j\right\|_B^2\right)$$

$$\le (j-1)\,Ch^2 + (j-1)\,Ch^2\left\|\delta^2 u_j\right\|_B^2 \le Ch + Ch\left\|\delta^2 u_j\right\|_B^2,$$

the last term on the right (39) is estimate by $2\|f_j - f_{j-1}\|\|\delta^2 u_j\|_B \le 2Ch\|\delta^2 u_j\|_B$ $leq Ch + Ch\|\delta^2 u_j\|_B^2$. Substituting in (39) it yields

$$\begin{aligned}\left\|\delta^2 u_j\right\|_B^2 - \left\|\delta^2 u_{j-1}\right\|_B^2 + \|\delta u_j\|^2 - \|\delta u_{j-1}\|^2 &\le Ch\left\|\delta^2 u_j\right\|_B^2 \\ &\quad + Ch^2\sum_{i=0}^{j-1}\|\delta u_i\|^2 + Ch \quad (40)\end{aligned}$$

Let $N \in N^*$ such that $C\frac{T}{N} < 1$,for $n > N$, then inequality (40) implies:

$$\begin{aligned}(1 - Ch)\left[\left\|\delta^2 u_j\right\|_B^2 + \|\delta u_j\|^2\right] &\le \left(1 + Ch^2\right)\left[\left\|\delta^2 u_{j-1}\right\|_B^2 + \|\delta u_{j-1}\|^2\right] \\ &\quad + Ch^2\sum_{i=0}^{j-1}\|\delta u_i\|^2 + Ch, \quad (41)\end{aligned}$$

similarly to the proof of Lemma 3.3, we get the desired result.

Corollary 3.5 *For all $t, s \in I$ and $n > N$, Lemmas 3.3 and 3.4 imply*

$$\left\|u^{(n)}(t)\right\| + \left\|\bar{u}^{(n)}(t)\right\| + \left\|\delta u^{(n)}(t)\right\| + \left\|\bar{\delta u}^{(n)}(t)\right\| + \left\|\frac{d}{dt}\delta u^{(n)}(t)\right\|_B \le C \quad (42)$$

$$\left\|u^{(n)}(t) - \bar{u}^{(n)}(t)\right\| + \left\|\delta u^{(n)}(t) - \bar{\delta u}^{(n)}(t)\right\|_B \le \frac{C}{n} \quad (43)$$

$$\left\|u^{(n)}(t) - u^{(n)}(s)\right\| + \left\|\delta u^{(n)}(t) - \delta u^{(n)}(s)\right\| \le C|t - s| \quad (44)$$

3.2 Convergence Results and Existence

We define the functions

$$f^n(t) = \begin{cases} f_j, & t \in (t_{j-1}, t_j,], \quad 1 \le j \le n \\ & f_0 = f(0) \end{cases}$$

and

$$K^n(0) = ha_{10}k_0, K^n(t) = h\sum_{i=0}^{j-1} a_{ji}k_i,$$

then (34) becomes

$$\left(\frac{d}{dt}\delta u^{(n)}(t), \phi\right)_B + \left(\bar{u}^{(n)}(t), \phi\right) = (f^n(t) + K^n(t), \phi)_B, \quad \forall \phi \in V \tag{45}$$

which gives

$$\int_I \left(\frac{d}{dt}\delta u^{(n)}(t), \phi\right)_B dt + \int_I \left(\bar{u}^{(n)}(t), \phi\right) dt = \int_I (f^n(t) + K^n(t), \phi)_B \, dt \quad \forall \phi \in V \tag{46}$$

Theorem 3.6 *Under the hypotheses $(H_1) - (H_4)$, then there exists a function $u \in C^{0,1}(I, V)$ such that $\frac{du}{dt} \in L^\infty(I, V) \cap C^{0,1}(I, B)$, $\frac{d^2u}{dt^2} \in L^\infty(I, V)$ and subsequences $\{u^{n_k}\}_k \subset \{u^n\}_n$, $\left\{\bar{u}^{(n_k)}\right\}_k \subset \left\{\bar{u}^{(n)}\right\}_n$ such that*

$$u^{n_k} \rightharpoonup u \quad in\ L^2(I, V), \tag{47}$$

$$\bar{u}^{(n_k)} \rightharpoonup u \quad in\ L^2(I, V), \tag{48}$$

$$\delta u^{(n_k)} \rightharpoonup \frac{du}{dt} \quad in\ L^2(I, V), \tag{49}$$

$$\bar{\delta u}^{(n_k)} \rightharpoonup \frac{du}{dt} \quad in\ L^2(I, V), \tag{50}$$

$$\frac{d}{dt}\delta u^{(n_k)} \rightharpoonup \frac{d^2u}{dt^2} \quad in\ L^2(I, B), \tag{51}$$

where "$\rightharpoonup$" denotes the weak convergence.

Proof. From inequality (42) we deduce that $\left\{u^{(n)}\right\}_n$, $\left\{\bar{u}^{(n)}\right\}_n$ are uniformly bounded in $L^2(I, V)$ with respect to n. Therefore, there exists subsequences

$\{u^{n_k}\}_k$ and $\left\{\overset{-(n_k)}{u}\right\}_k$ that converge weakly to some functions u and $\overline{u}$ respectively. From the estimate (43) it follows that $u = \overline{u}.$ The inequality (42) implies that $\left\{\delta u^{(n)}\right\}_n$, and $\left\{\overset{-}{\delta u}^{(n)}\right\}_n$ are uniformly bounded in $L^2(I,V)$, so we can extract two subsequences $\left\{\delta u^{(n_k)}\right\}_k$, $\left\{\overset{-}{\delta u}^{(n_k)}\right\}_k$ that converge weakly to w and $\overline{w}$ respectively,using (43) we obtain $w = \overline{w}$. Now we prove that $w = \frac{du}{dt}$. From

$$u^{(n_k)}(t) - U_0 = \int_0^t \frac{du^{(n_k)}(s)}{ds}, \quad \forall t' \in I,$$

we get

$$u(t) = U_0 + \int_0^t w(s)ds, \qquad \forall t' \in I, \tag{52}$$

that gives $u \in C(I,B)$ and $w = \frac{du}{dt}$ in $L^2(I,V)$. Taking into account (42) we have$\left\{\frac{d}{dt}\delta u^{(n)}\right\}_n$ is uniformly bounded in $L^2(I,B)$, so it has a subsequence $\left\{\frac{d}{dt}\delta u^{(n_k)}\right\}_k$ such that $\frac{d}{dt}\delta u^{(n_k)} \rightharpoonup S$. To prove that $S = \frac{d^2u}{dt^2}$, we consider the equality

$$\delta u^{(n_k)} - U_1 = \int_0^t \frac{d}{ds}\delta u^{(n_k)}(s)\,ds,$$

so

$$\frac{du}{dt} - U_1 = \int_0^t S(s)\,ds \tag{53}$$

and consequently $\frac{du}{dt} \in C(I,B)$ and $S = \frac{d^2u}{dt^2}$ in I. From corollary 3.5 it follows that $u : I \longrightarrow V$ and $w = \frac{du}{dt} : I \longrightarrow$B are Lipschitz continuous, hence

$$\frac{du}{dt}\ inL^{\infty}(I,V) \text{ and } \frac{dw}{dt} = \frac{d^2u}{dt^2} \in L^{\infty}(I,B)\,. \tag{54}$$

Lemma 3.7 *The sequence $\{K^n(t)\}_n$ is uniformly bounded in $L^2(I,B)$ and has a subsequence $\{K^{n_k}(t)\}_k$ such*

$$K^{n_k} \to K(u)\,,\ \ as\ k \longrightarrow +\infty, in\ L^2(I,B)\,. \tag{55}$$

Proof. See the Proof of Lemma 2.4 in [3].

Theorem 3.8 *Under the hypotheses $(H_1) - (H_4)$ then the limit u is the weak solution for problem (5)-(7) in the sense of definition 2.1. In addition, if (H_5) is satisfied, then u is unique.*

Proof. Remarking that $u \in C^{0,1}(I,V)\,, \frac{du}{dt}\ inL^{\infty}(I,V) \cap C^{0,1}(I,B)\,, \frac{d^2u}{dt^2}\ inL^{\infty}(I,B)$ and u satisfies the integral conditions since $u(t) \in V$ and by virtue of (52),

(53) we deduce that $u(0) = U_0$ and $\frac{du}{dt}(0) = U_1$. From the hypothesis (H_1) we get:

$$\|f^n(t) - f(t)\|_B \leq \frac{C}{n} \quad a.e. \text{ in } I, \tag{56}$$

thus

$$f^n \longrightarrow f \quad \text{in } L^2(I, B) \tag{57}$$

Passing to the limit as $n = n_k \longrightarrow +\infty$ in (46) and by means of the convergence properties (48), (51), (55) and (57) we arrive at

$$\int_I \left(\frac{d^2u}{dt^2}(t), \phi\right)_B dt + \int_I (u(t), \phi)\, dt = \int_I (f(t), \phi)_B\, dt + \int_I (Ku(t), \phi)_B\, dt \quad \forall\phi \in V .$$

Now we prove the uniqueness under the hypothesis (H_5). If u_1 and u_2 are two weak solutions of (5)-(7) then the difference $u := u_1 - u_2$ satisfies:

$$\begin{aligned} \int_I \left(\frac{d^2u}{dt^2}(t), \phi\right)_B dt + \int_I (u(t), \phi)\, dt &= \\ \int_I \left(\int_0^t a(t-s)\left[k(s, u_1) - k(s, u_2)\right] ds, \phi\right)_B dt \quad \forall\phi &\in V, \end{aligned}$$

let

$$w = \max|a(t)| \int_0^T L(t)\, dt \tag{58}$$

we divide the interval I into a finite number of subintervals of lengths p such that

$$w.p < \frac{1}{2}. \tag{59}$$

Substituting in (13) the function ϕ by

$$\phi = \begin{cases} \frac{du}{dt} & t \in [0, p] \\ 0 & t \in \left]p, T\right], \end{cases} \tag{60}$$

we obtain

$$\int_0^p \frac{d}{dt}\left\|\frac{du}{dt}\right\|_B^2 dt + \int_0^p \frac{d}{dt}\|u(t)\|^2\, dt = 2\int_0^p \left(\int_0^t a(t-s)\left[k(s, u_1) - k(s, u_2)\right] ds, \frac{du}{dt}\right)_B dt, \tag{61}$$

therefore

$$\begin{aligned} &\int_0^p \frac{d}{dt}\left\|\frac{du}{dt}\right\|_B^2 dt + \int_0^p \frac{d}{dt}\|u(t)\|^2\, dt \\ \leq\ & 2\int_0^p \left\|\int_0^t a(t-s)\left[k(s, u_1) - k(s, u_2)\right] ds\right\| \left\|\frac{du}{dt}\right\|_B dt \\ \leq\ & 2p\max_I |a(t)| \int_0^T L(t)\, dt\, \|u(t)\| \left\|\frac{du}{dt}\right\|_B \end{aligned}$$

$$\leq wp\left[\left(\max_{t\in[0,p]}\|u(t)\|\right)^2+\left(\max_{t\in[0,p]}\left\|\tfrac{du}{dt}\right\|_B\right)^2\right] (62)$$

Let t_1 and $t_2\in[0,p]$ such that

$$\left\|\frac{du}{dt}(t_1)\right\|_B=\max_{[0,p]}\left\|\frac{du}{dt}(t)\right\|_B \tag{63}$$

$$\|u(t_2)\|=\max_{[0,p]}\|u(t)\| \tag{64}$$

So, taking into account that $\frac{du}{dt}(0)=u(0)=0$ it follows

$$\begin{aligned}&\int_0^{t_1}\frac{d}{dt}\left\|\frac{du}{dt}\right\|_B^2dt+\int_0^{t_2}\frac{d}{dt}\|u(t)\|^2dt\\ =\ &\left\|\frac{du}{dt}(t_1)\right\|_B^2+\|u(t_2)\|^2\\ \leq\ &\int_0^{p}\frac{d}{dt}\left\|\frac{du}{dt}\right\|_B^2dt+\int_0^{p}\frac{d}{dt}\|u(t)\|^2dt\end{aligned} \tag{65}$$

from inequalities (60), (63) and (66), we get

$$\frac{du}{dt}(t)=u(t)=0,\qquad\forall t\in[0,p]. \tag{66}$$

Repeating the above argument for $[kp,(k+1)p]$, $i=1$,..., we deduce $u=0$. this achieves the proof of Theorem 3.8.

4 Open Problem

In this paper we have studied a second order integro-differential equation with integral conditions. One can develop the Rothe's method for a higher order integro-differential equation. Instead of Rothe's method, one can adopt Galerkine method to investigate a complete hyperbolic integro-differential equation with more general integral conditions.

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