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On the Computations of Fold Points For Nonlinear Elliptic Eigenvalue Problems

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Abstract

The Adomian decomposition method is used to trace the solution curve of some nonlinear elliptic problems with folds. An approximated series solution to these problems is obtained. Numerical results of the examples presented in this paper conjectured that the value of the simple fold point is the radius of convergence of the series obtained. The validity of the method is verified for Bratu problem which is governed by one parameter λ and a reaction-diffusion problem which is governed by the two parameters λ and α. For the reaction-diffusion problem an implicit relation between α and λ is obtained for the first time. Numerical results obtained indicate the method is efficient and accurate.

1 Introduction

Many problems in science and engineering require the computation of family of solutions of a nonlinear system of the form

$$G(u, \lambda) = 0, u = u(\lambda) \tag{1}$$

where $G : \mathbb{R}^{n+1} \to \mathbb{R}$ is continuously differentiable function, u represents the solution and λ is a real parameter (i.e., Reynold's number, load,etc.) It is required to find the solution for some $\lambda-$intervals, i.e., a path solutions, $(u(\lambda), \lambda)$. Equations of the form (1) are called nonlinear elliptic eigenvalue problems if the operator G with λ fixed is an elliptic differential operator. For more details about this type of operators, see [1].

As a typical example of a nonlinear elliptic eigenvalue problems, we consider the following problem

$$\begin{aligned} G(u, \lambda) &= \Delta u + \lambda f(u) = 0, \quad \text{in } \Omega \\ u &= 0, \qquad\qquad \text{on } \partial\Omega. \end{aligned} \tag{2}$$

Equation (2) arises in many physical problems. For example, in chemical reactor theory, radiative heat transfer, combustion theory, and in modelling the expansion of the universe. The function u could be a function of several variables and the domain Ω is usually taken to be the unit interval $[0,1]$ in $\Re$, or the unite square $[0,1]\times[0,1]$ in $\Re^2$, or the unit cube $[0,1]\times[0,1]\times[0,1]$ in $\Re^3$.

Equation (2) can take several forms, for example, Bratu equation is given by

$$\begin{aligned} \Delta u + \lambda\, e^{u} &= 0, \quad \text{in } \Omega \\ u &= 0, \quad \text{on } \partial\Omega \end{aligned}$$

and a reaction-diffusion problem takes the form

$$\begin{aligned} \Delta u + \lambda \exp\left(\frac{u}{1+\alpha u}\right) &= 0, \quad \text{in } \Omega \\ u &= 0, \quad \text{on } \partial\Omega. \end{aligned}$$

There are no bifurcation points in the two problems above; all singular points are fold points. The behavior of the solution near the singular points has been studied numerically [1], [18], [24] and theoretically [19], [22], [29], [30]

For both the one- and two-dimensional cases, the Bratu problem has exactly one fold point, whereas the three-dimensional case has infinitely many fold points.

In this paper, we will use the Adomian decomposition method (ADM), see [6] , [7] , [17]and [11] to solve Equation (2). We will approximate the solution of this problem by a series of x and λ. Then, the estimated radius of convergence of this series will be used to approximate the folding point.

In the next section we will describe the Adomian decomposition method (Shortly ADM), and section three will discuss the method of calculating the fold point, while in section four, the ADM method will be used to solve some problems given by Equation (2) and summary and conclusions will be presented in section five.

2 Adomian decomposition method

Adomian Decomposition Method (ADM) has been recently used intensively to solve nonlinear ordinary differential equations as well as partial differential equation [2] [5] [6] [7] [14] [8] [9] [10] [12] [28] [33] [34] [25] [26] [27] [35] [36] [37] [38] . It is quantitative rather than qualitative, analytic, requiring neither linearization nor perturbation and continuous with no resort to discretization .

When applying the ADM, we split the given equation into linear and nonlinear parts. Then inverting the linear operator on both sides. Decomposing the nonlinear function in terms of special polynomials called Adomian's polynomials, and finding the successive terms of the series solution by recurrent relation using Adomian polynomials.

In this section we describe how to use the Adomian decomposition method for solving the following problem

$$\begin{aligned} \Delta u + \lambda\, f(u) &= 0, \quad \text{in } \Omega \\ u &= 0, \quad \text{on } \partial\Omega. \end{aligned} \tag{3}$$

The Adomian decomposition method assumes that the solution $u(x)$ of Equation (3) can be written as

$$u(x)=\sum_{n=0}^{\infty}u_n(x) \tag{4a}$$

where $u_n,\ \ n=0,1,2,...$ are polynomials in x, and the nonlinear function $f(u)$ can be written in the series form as

$$f(u)=\sum_{n=0}^{\infty}A_n \tag{4b}$$

where $A_n,\ \ n=0,1,2,...$ are called the Adomian polynomials. These polynomials can be derived by expanding the function $f(u)$ about u_0 as follows

$$f(u)=f(u_0)+f'(u_0)\frac{u-u_0}{1!}+f''(u_0)\frac{(u-u_0)^2}{2!}+...$$

and using Equation(4a) to replace u implies:

$$f(u)=f(u_0)+f'(u_0)\frac{\sum\limits_{n=1}^{\infty}u_n}{1!}+f''(u_0)\frac{\left(\sum\limits_{n=1}^{\infty}u_n\right)^2}{2!}+...$$

Adomian polynomials are obtained by reordering and rearranging the terms given in the last equation. We define the order of the component u_l^m to be ml, and $u_l^m u_j^n$ to be $ml+nj$. Then, the Adomian polynomial A_0 depends on u_0 and collects all terms of order zero, A_1 depends on u_0 and u_1 and collect all terms of order one, ...etc. Therefore, rearranging the terms in the last expansion according to the order, and using the expansion of $f(u)$ given in Equation(4b), will result in the following form of A_n

$$\begin{aligned}
A_0 &= f(u_0)\\
A_1 &= u_1f'(u_0)\\
A_2 &= \frac{u_1^2}{2!}f''(u_0)+u_2f'(u_0)\\
A_3 &= \frac{u_1^3}{3!}f'''(u_0)+\frac{2u_1u_2}{2!}f''(u_0)+u_3f'(u_0)\\
A_4 &= \frac{u_1^4}{4!}f^{(4)}(u_0)+\frac{3u_1^2u_2}{3!}f'''(u_0)+\frac{2u_1u_3+u_2^2}{2!}f''(u_0)+u_4f'(u_0)
\end{aligned}$$

From the previous discussion, one can show that A_k can be derived as

$$A_k=\frac{1}{k!}\frac{d^k}{d\alpha^k}\left[f\left(\sum_{j=0}^{\infty}\alpha^j u_j\right)\right]_{\alpha=0},j\geq 0. \tag{5}$$

Next, define the differential operator L as $L=\frac{d^2}{dx^2}$ on the set of all second order differentiable functions u such that $u=0$ on $\partial\Omega$, say Λ. This operator on Λ is invertible. Then Equation (3) can be written in the form

$$L(u)+\lambda f(u)=0. \tag{6}$$

And defining the inverse operator as $L^{-1}(.) = \int_0^x \int_0^x .dxdx$, then the solution $u(x)$ of Equations (3) can be written in the form:

$$\begin{aligned} u(x) &= -\lambda L^{-1}(f(u)) + c_1 x + c_2 \\ &= -\lambda L^{-1}(\sum_{n=0}^{\infty} A_n) + c_1 x + c_2 \\ &= \sum_{n=0}^{\infty} u_n(x) \end{aligned} \tag{7}$$

where the term $c_1 x + c_2$ is the solution of $L(u) = 0$ and $c_1 x$ and c_2 will be determined later using the boundary conditions. Now equating the terms yields

$$\begin{aligned} u_0(x) &= c_1 x + c_2 \\ u_1(x) &= -\lambda L^{-1}(A_0) \\ u_2(x) &= -\lambda L^{-1}(A_1) \\ &\vdots \\ u_n(x) &= -\lambda L^{-1}(A_{n-1}). \end{aligned} \tag{8}$$

From (8) the sequence $\{u_n\}_{n=0}^{\infty}$ is known, so that the solution of Problem (3) is now given by

$$u(x) = u_0(x) + u_1(x) + u_2(x) + ...$$

3 Computation of fold points

In Equation (1), the parameter λ is often a quantity of physical significance, such as temperature in liquid crystal modeling or the Reynolds number in hydrodynamical flow, and is commonly referred to as the "natural parameter". We are interesting in determining solution path

$$C = \{(u, \lambda) : G(u, \lambda) = 0, u = u(\lambda), a \leq \lambda \leq b\}$$

associated with (1). Here a and b are given bounds for λ.

The solutions $u(\lambda)$, $a \leq \lambda \leq b$, of Equation (1) are commonly computed by a continuation method. In these methods an initial value problem for u is derived by differentiating Equation (1) with respect to λ. Thus, let $u = u(\lambda)$ satisfy Equation (1) yields

$$G_u(u(\lambda), \lambda)\dot{u}(\lambda) + G_\lambda(u(\lambda), \lambda) = 0, \tag{9}$$

where $\dot{u} = \frac{du}{d\lambda}$. Given $u(a)$ and assuming that the Jacobian matrix $J(G)$ is a nonsingular matrix in the neighborhood of the solution path, we can compute $u(\lambda)$ for $a \leq \lambda \leq b$ by solving the initial value problem (9) for $u = u(\lambda)$. Points where the Jacobian matrix

$J(G)$ is nonsingular are referred to as regular points. Points where $J(G)$ is singular are referred to as singular points. The simple singular point is defined as follows.

Definition 1 *A point (u_0, λ_0) is a simple singular point of G, if the null space of $J(G)(u_0, \lambda_0)$ denoted by $\tilde{N}(JG)(u_0, \lambda_0))$ is of dimension one.*

The range is then given by

$$range[J(G)(u_0, \lambda_0)] = \{y : \psi_0^t y = 0\}$$

for some nonzero vector ψ_0.For more details on how to chose ψ_0 we refer the reader to Ref. [4]

Singular points on the solution path are either turning points or bifurcation points of the solution path. The determination of the solution path in a neighborhood of a turning point or bifurcation point requires special care. It is therefore important to detect singular points on the solution path. In this paper, we concentrate on another type of singular point which is the fold (turning) point which can be defined as:

Definition 2 *A singular point (u_0, λ_0) of G is said to be a fold point if it satisfies the condition:*

$$\psi_0^t G_\lambda(u_0, \lambda_0) \neq 0.$$

The augmented system of equations used for calculating the fold point is then given by [30],

$$\begin{pmatrix} G(u, \lambda) \\ G_u(u, \lambda)\varphi \\ l^t\varphi - 1 \end{pmatrix} = 0$$

where l is not in the null space of $G_u(u_0, \lambda_0)$.

Such folds frequently are of intrinsic interest, and there are special algorithms for detecting and calculating them. We refer the reader to [21][22] and [30].

4 Numerical Results

In this section we apply the Adomian decomposition method to solve two special cases of Eq (2), namely, the Bratu problem and the reaction-diffusion problem.

Example 4.1. Consider the one-dimensional Bratu problem

$$u''(x) + \lambda e^u = 0, \qquad x \in (0, 1) \tag{10}$$

with the boundary conditions

$$u(0) = u(1) = 0. \tag{11}$$

The exact solution of the nonlinear boundary value problem given by Eq (10) and (11) is given by the following implicit formula

$$u(x) = -2\ln\left(\frac{\cosh\left(\frac{2x-1}{4}y\right)}{\cosh\left(\frac{y}{4}\right)}\right) \tag{12}$$

where y solves

$$\sqrt{2\lambda}\cosh\left(\frac{y}{4}\right) - y = 0. \tag{13}$$

In Figure 1, we sketch the curves of the function in Eq (13) in the (λ, y)-plane. From this graph we see that Eq. (10) has two solutions when $\lambda < \lambda_*$, one solution when $\lambda = \lambda_*$, and no solution when $\lambda > \lambda_*$. In Figure 2, we sketch the graph of $\|u\|_\infty = \max\{|u(x)| : 0 \le x \le 1\}$ versus λ. From this graph we see that there is a turning point λ_*. To apply the Adomian decomposition method for finding the solution of Eq. (10), we first write Eq. (10) in operator form as

$$L(u) = -\lambda e^u \tag{14}$$

where $L = \frac{d^2}{dx^2}$, then using the inverse operator $L^{-1}(.) = \int\limits_0^x \int\limits_0^s .dtds$. One can write the solution of Eq. (14) as

$$u(x,\lambda) = -\lambda L^{-1}(e^u) + c_1x + c_2 \tag{15}$$

Assuming that $u_0(x) = c_1x + c_2$, and using the boundary conditions at $x = 0$ and $x = 1$ for evaluating c_1x and c_2, implies that $u_0(x) = 0$. Now using formula (5) for writing down the Adomian Polynomials and applying the inverse operator (8) to find the consecutive terms $u_n(x,\lambda)$ we get

$$\begin{aligned}
u_0(x,\lambda) &= 0, A_0 = 1 \\
u_1(x,\lambda) &= \frac{x-x^2}{2}\lambda, A_1 = \frac{x-x^2}{2}\lambda, \\
u_2(x,\lambda) &= \frac{x-2x^3+x^4}{24}\lambda^2, A_2 = \frac{x+3x^2-8x^3+4x^4}{24}\lambda^3, \\
u_3(x,\lambda) &= \frac{9x-10x^3-15x^4+24x^5+8x^6}{1440}\lambda^3, \\
A_3 &= \frac{9x+30x^2-10x^3-165x^4+204x^5-68x^6}{1440}\lambda^3 \\
u_4(x,\lambda) &= \frac{23x-21x^3-35x^4+7x^5+77x^6-68x^7+17x^8}{20160}\lambda^4, \\
A_4 &= \frac{46x+161x^2+42x^3-665x^4-616x^5+2520x^6-1984x^7+496x^8}{40320}\lambda^4 \\
&\vdots
\end{aligned}$$

one can see that the terms $u_n(x,\lambda)$ satisfy the conditions

$$u_n(0,\lambda) = 0 = u_n(1,\lambda), n = 1,2,3,.... \tag{16}$$

Using the above result, we approximate the exact solution by

$$u_{app4}(x,\lambda) = u_0(x,\lambda) + u_1(x,\lambda) + u_2(x,\lambda) + u_3(x,\lambda) + u_4(x,\lambda).$$

The graphs of the exact solution and the approximate solutions $u_{app4}(x,\lambda)$ at $\lambda = 1, 2$ and 3 are given in Figure 3, Figure 4 and Figure 5 respectively. From these figures we see that the approximation function $u_{app4}(x,\lambda)$ is accurate when λ is far from λ_*, as in Figure 3, while it becomes not accurate as we become closer and closer to the folding point λ_*, as it is shown in Figure 5. To reduce the error, more terms are computed and added to the series above. Figure 6 shows the graph of $u_{app30}(x,\lambda) = \sum_{k=0}^{30} u_k(x,\lambda)$ and the exact solution at $\lambda = \lambda_*$.

One can see that each term in $u_{app30}(x,\lambda)$ satisfies the boundary conditions (11). This technique is used to compute the approximate solution a specific value of λ. However, to find the value of λ_*, we need a normalized condition. This means, we should force the solution to satisfy another condition different from the boundary ones. To overcome with this problem, we compute $u_0(x,\lambda), u_1(x,\lambda), u_2(x,\lambda), \ldots$ that satisfy the initial condition at $x = 0$, while the condition at $x = 1$ will be used at the end of the computations. In both approaches, we will get the same solution when n goes to infinity. Thus, $u_0(x,\lambda), u_1(x,\lambda), u_2(x,\lambda), \ldots$ will be as follows

$$\begin{aligned} u_0(x,\lambda) &= 0, u_1(x,\lambda) = -\frac{\lambda x^2}{2}, u_2(x,\lambda) = \frac{\lambda^2 x^4}{24}, \\ u_3(x,\lambda) &= -\frac{\lambda^3 x^6}{180}, u_4(x,\lambda) = \frac{17\lambda^4 x^8}{20160}, \ldots \end{aligned}$$

Thus, the sum of these polynomials has the form

$$\sum_{k=0}^{N} u_k(x,\lambda) = \sum_{k=1}^{N} (-1)^k b_k \lambda^k x^{2k} \tag{17}$$

where $b_1 = \frac{1}{2}, b_2 = \frac{1}{24}, b_3 = \frac{1}{180}, b_4 = \frac{17}{20160}, \ldots$Thus, the summation in Eq (17) is zero at $x = 0$ and $\sum_{k=1}^{N} (-1)^k b_k \lambda^k$ at $x = 1$. Therefore, the approximate solution becomes

$$u_{appN}(x) = -b_0 x + \sum_{k=1}^{N} (-1)^k b_k \lambda^k x^{2k}, \tag{18}$$

where $b_0 = \sum_{k=1}^{N} (-1)^k b_k \lambda^k$.

Recall that the series solution given by Eq. (17) is a series in both x and λ. Thus, as $N \to \infty$, one has to determine the radius of convergence ρ_λ such that the series converges for $0 < \lambda < \rho_\lambda$, provided that the series converges for $0 \le x \le 1$. To find the value of ρ_λ we use the ratio test and find that

$$\begin{aligned} \lim_{n\to\infty} \left| \frac{u_{n+1}(x,\lambda)}{u_n(x,\lambda)} \right| &= \lim_{n\to\infty} \left| \frac{b_{n+1}\lambda x^2}{b_n} \right| \\ &\le \lim_{n\to\infty} \left| \frac{b_{n+1}\lambda}{b_n} \right| < 1 \end{aligned}$$

where the last inequality was obtained using the fact that $0 < x \leq 1$. Then upon solving the last inequality for λ yields the radius of convergence of the series (18) for the parameter λ which will be given by

$$\lambda^* = \rho_\lambda = \lim_{n\to\infty} \left| \frac{b_n}{b_{n+1}} \right| \tag{19}$$

where b_n , $n = 0, 1, 2...$ are the coefficients given by formula (17).

Experimenting with several similar examples showed that the fold point can be calculated using the radius of convergence as calculated above which suggests the following conjecture:

Conjecture 1 *let* $u(x, \lambda) = \sum_{k=0}^{\infty} u_k(x, \lambda) = \sum_{k=0}^{\infty} a_k \lambda^k x^k$ *be the solution of Eq. (10) that converges for* $0 \leq x \leq 1$. *Then the simple folding point will be* $\lambda^* = \rho_\lambda = \lim_{n\to\infty} \left| \frac{b_n}{b_{n+1}} \right|$.

Applying the above result on the problem at hand and after computing the first 30 terms in the series approximation of the solution $u(x, \lambda)$ we find that

$$\rho_\lambda = \lim_{n\to\infty} \left| \frac{b_n}{b_{n+1}} \right| \approx \frac{b_{29}}{b_{30}} = 3.5138307$$

which is the value of the turning point λ^* accurate to 7 decimal places.

Example 4.2.

Consider the reaction-diffusion problem

$$u''(x) + \lambda \exp\left(\frac{u}{1 + \alpha u}\right) = 0, \qquad x \in (0, 1) \tag{20}$$

with the boundary conditions

$$u(0) = u(1) = 0. \tag{21}$$

To apply the Adomian decomposition method for finding the solution of Eq. (19) we first write it in operator form as:

$$L(u) = -\lambda \exp\left(\frac{u}{1 + \alpha u}\right) \tag{22}$$

where $L = \frac{d^2}{dx^2}$, then using the inverse operator $L^{-1}(.) = \int_0^x \int_0^s .dtds$. One can write the solution of Eq. (21) as

$$u(x, \lambda) = -\lambda L^{-1}(\exp\left(\frac{u}{1 + \alpha u}\right)) + c_1 x + c_2 \tag{23}$$

Let $u_0(x) = c_1 x + c_2$, then using the boundary conditions at $x = 0$ and $x = 1$ to evaluate c_1 and c_2, gives $u_0(x) = 0$. Similar analysis to Example 4.1 gives

$$\begin{aligned}
u_0(x,\lambda,\alpha) &= 0, A_0 = 1\\
u_1(x,\lambda,\alpha) &= \frac{x-x^2}{2}\lambda, A_1 = \frac{x-x^2}{2}\lambda,\\
u_2(x,\lambda,\alpha) &= \frac{x-2x^3+x^4}{24}\lambda^2, A_2 = \frac{x+3x^2-8x^3+4x^4}{24}\lambda^2 + \alpha\frac{-x^2+2x^3-x^4}{4}\lambda^2,\\
u_3(x,\lambda,\alpha) &= \left(\frac{1}{2}x(\frac{3-2\alpha}{240}) - \frac{1}{144}x^3 + x^4(\frac{-1+2\alpha}{96}) - x^5(\frac{-2+3\alpha}{120}) + x^6(\frac{-2+3\alpha}{360})\right)\lambda^3,\\
&\vdots
\end{aligned}$$

where the terms $u_n(x,\lambda,\alpha)$ are computed so that they satisfy the boundary conditions

$$u_n(0,\lambda,\alpha) = 0 = u_n(1,\lambda,\alpha), n = 1,2,3,.... \tag{24}$$

Using the above result, we approximate the exact solution by

$$u_{app_n}(x,\lambda,\alpha) = u_0(x,\lambda,\alpha) + u_1(x,\lambda,\alpha) + u_2(x,\lambda,\alpha) + ... + u_n(x,\lambda,\alpha).$$

The graph of the approximate solution $u_{app30}(x,\lambda,\alpha)$ at $\lambda = 5.22949$ and $\alpha = 0.2457804$ is given in Figure 7. To compute the value of the turning point (λ_*,α_*), we compute $u_0(x,\lambda,\alpha), u_1(x,\lambda,\alpha), u_2(x,\lambda,\alpha), ...$ that satisfy the initial condition at $x = 0$, while the condition at $x = 1$ will be used at the end of the computations. Thus, $u_0(x,\lambda,\alpha), u_1(x,\lambda,\alpha), u_2(x,\lambda,\alpha), ...$ will be as follows

$$\begin{aligned}
u_0(x,\lambda,\alpha) &= 0, u_1(x,\lambda,\alpha) = -\frac{\lambda x^2}{2}, u_2(x,\lambda,\alpha) = \frac{\lambda^2x^4}{24},\\
u_3(x,\lambda,\alpha) &= -\frac{\lambda^3x^6}{180} + \frac{\lambda^3x^6}{120}\alpha,\\
u_4(x,\lambda,\alpha) &= \frac{17\lambda^4x^8}{20160} - \frac{63\lambda^4x^8}{20160}\alpha + \frac{45\lambda^4x^8}{20160}\alpha^2, ...
\end{aligned}$$

Thus, the sum of these polynomials has the form

$$\sum_{k=0}^{N} u_k(x,\lambda,\alpha) = \sum_{k=1}^{N}(-1)^k b_k\lambda^k x^{2k} \tag{25}$$

where $b_1 = \frac{1}{2}, b_2 = \frac{1}{24}, b_3 = \frac{1}{180} - \frac{\alpha}{120}, b_4 = \frac{17-63\alpha+45\alpha^2}{20160}, ...$Thus, the summation in Eq (24) is zero at $x = 0$ while at $x = 1$, the sum will be $\sum_{k=1}^{N}(-1)^k b_k\lambda^k$. Therefore, the approximate solution becomes

$$u_{app_N}(x,\lambda,\alpha) = -b_0x + \sum_{k=1}^{N}(-1)^k b_k\lambda^k x^{2k}, \tag{26}$$

where

$$b_0 = \sum_{k=1}^{N}(-1)^k b_k\lambda^k \tag{27}$$

and

$$b_k = \sum_{l=0}^{k-2} c_l \alpha^l. \quad (28)$$

Now using the boundary condition at $x = 1$, gives the following implicit relation between the two parameters α and λ

$$u_{appN}(1) = -b_0 + \sum_{k=1}^{N} (-1)^k b_k \lambda^k = 0.$$

For $N = 4$, the implicit relation between λ and α is given as

$$-\frac{\lambda}{2} + \frac{\lambda^2}{24} - \frac{\lambda^3}{180} + \frac{\lambda^3}{120}\alpha + \frac{17\lambda^4}{20160} - \frac{63\lambda^4}{20160}\alpha + \frac{45\lambda^4}{20160}\alpha^2 = 0.$$

Recall that the series given by Eq. (27) is a series in α while the series solution given by Eq. (25) of x, λ and α. Thus, as $N \to \infty$, one has to determine the radius of convergence of the series (25) and (27). When the ratio test is used to find the radius of convergence of the series (25), the value of $\lambda^* = 5.22949$ is obtained. While the value of $\alpha^* = 0.2457804$ is obtained by calculating the radius of convergence of the series (27). Comparing these two values with the ones obtained in [31] [32] shows that the method is very efficient.

5 Conclusion

In this article, we have discussed the folding point of nonlinear elliptic eigenvalue problems. The Adomian decomposition method is used to derive the series representation of the solution of the problem. The method is tested for the two examples: The Bratu problem and the diffusion-reaction problem. For the Bratu problem, It is shown that the value of the simple folding point is the radius of convergence of the series solution of the problem when it is viewed as a function of the parameter λ. The value of the simple folding point was found to be $\lambda^* = 3.5138307$ which is accurate to 7 decimal places.

For the reaction-diffusion problem, the series solution is a function of the two parameters α and λ. An implicit relation between these two parameters was obtained for the first time. The value of the simple folding point for this problem was found to be $\lambda^* = 5.22949$ and the critical value of α is found to be $\alpha^* = 0.2457804$.

Thus we were able to develop a simple technique to calculate the folding point that requires no discretization nor approximation. The results obtained indicate that the method is reliable and accurate.

6 Open problem

In this article, the ADM method was used to determine the fold point for the Bratu problem. However, the solution of this problem exhibits chaotic behavior for certain values of the parameter λ (See [3] for details). This suggests the following open problem: How one can employ the ADM method or any other analytical method to study the variation of the solution with the parameter λ? and how one can study the chaotic behavior of the solution of this problem? . The second open problem is the proof of the

conjucture presented in section 4.

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λ

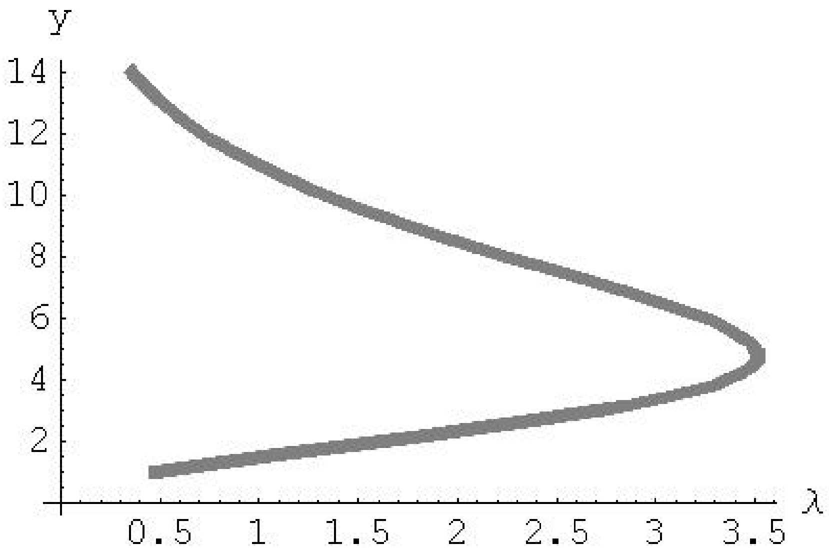

Figure 1: The curves of Equation (13) in the (λ, y) plane

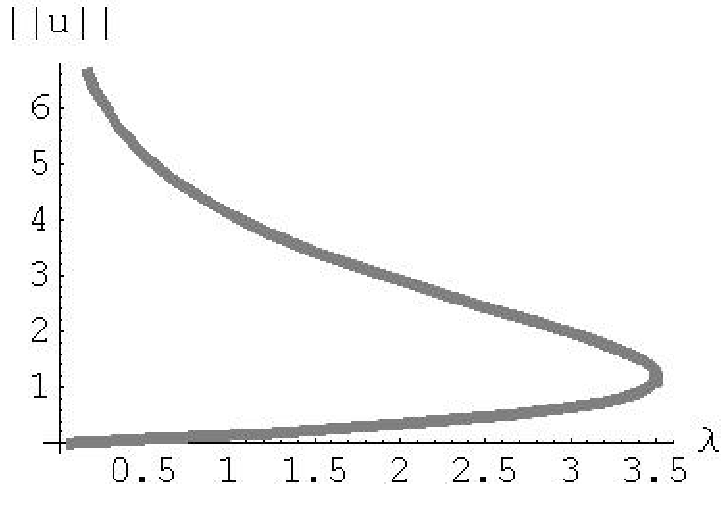

Figure 2: The curve of $||u||_{\infty}$ versus λ

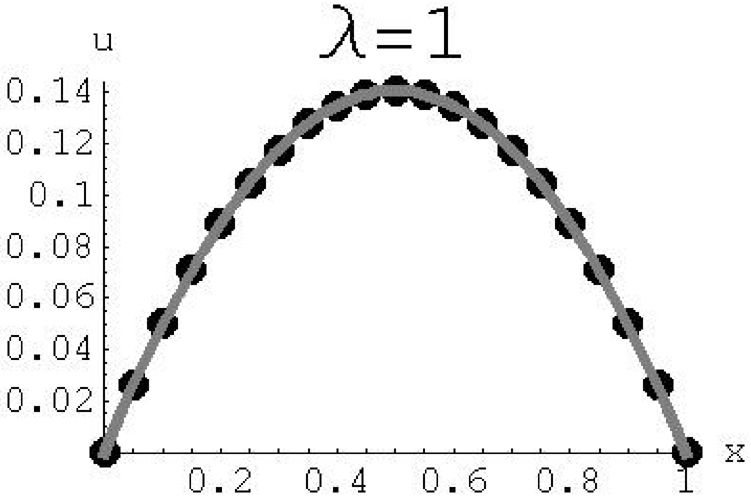

Figure 3: The graph of the exact solution and $u_{app4}(x, \lambda)$ at $\lambda = 1$.

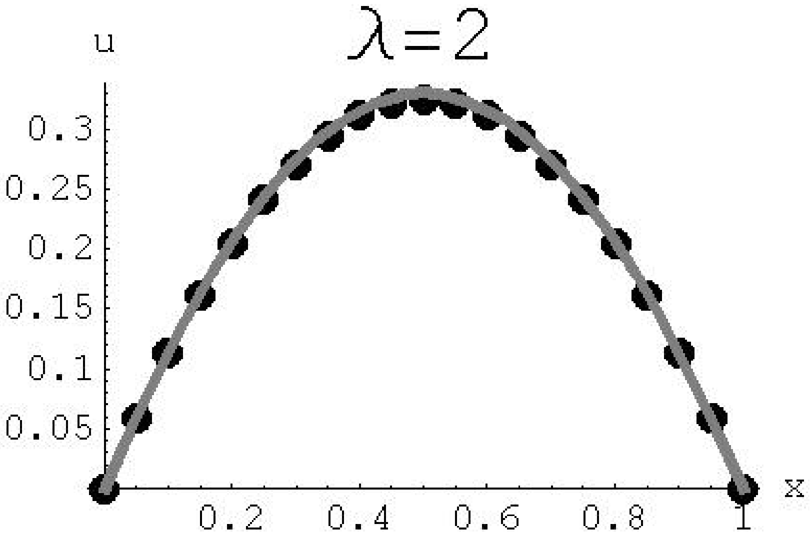

Figure 4: —- Exact solution and .... u_{app4}(x, λ) at $\lambda = 2$

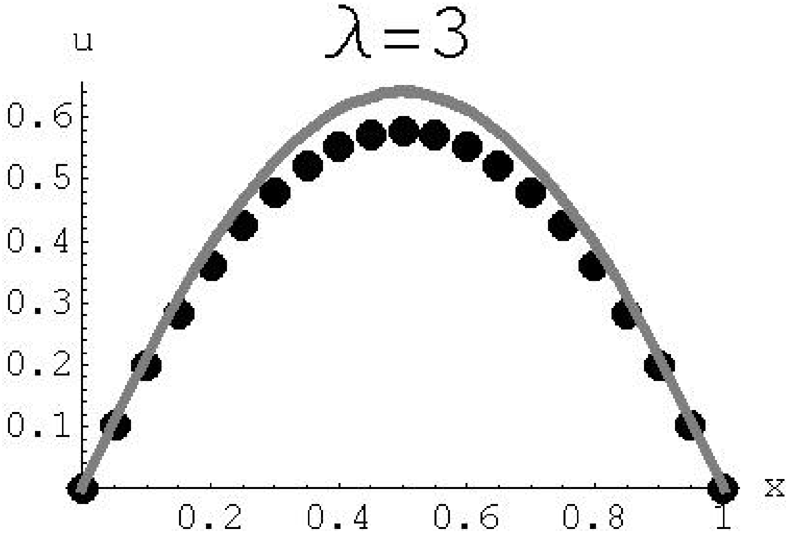

Figure 5: —- Exact solution and .... $u_{app4}(x, \lambda)$ at $\lambda = 3$

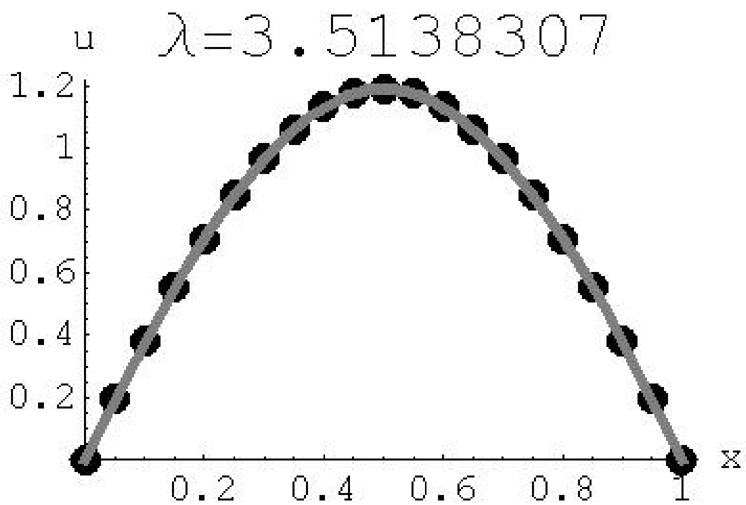

Figure 6: —- Exact solution and .... $u_{app4}(x, \lambda)$ at $\lambda = \lambda_*$

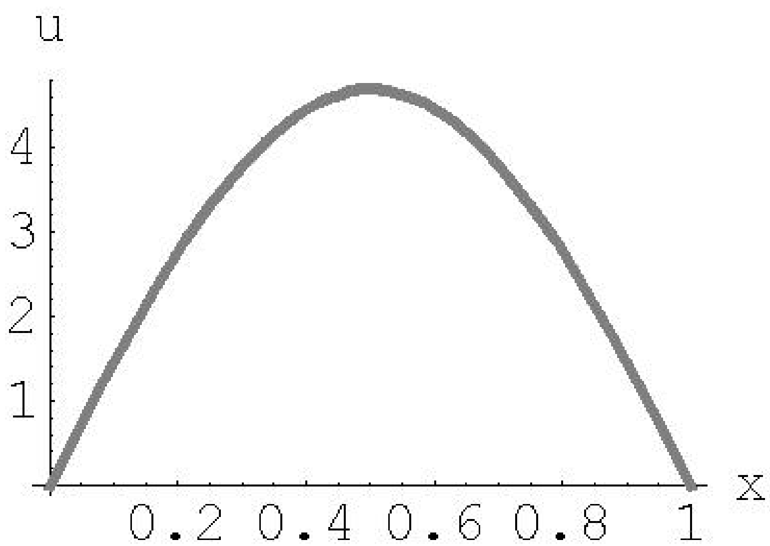

Figure 7: The approximate solution $u_{app30}(x, 5.22949, 0.2457804)$