Int. J. Open Problems Compt. Math., Vol. 4, No. 1, March 2011
ISSN 1998-6262; Copyright © ICSRS Publication, 2011
www.i-csrs.org

Zero Gravity of Open Channel Flows

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Abstract

The aim of this work is the study of a free surface flow past a submerged triangular obstacle at the bottom of a channel. The flow is assumed to be steady. The fluid is treated as inviscid and incompressible. The effects of gravity and surface tension are not taken into account. We use the method of the free streamline theory based on the hodograph method and the Schwarz-Christoffel transformation technique to obtain the exact solution.

Keywords: Open Channel, Free Surface, Flow, Free Streamline Theory.
AMS Mathematics Subject Classification (2000): 76B07, 76D45, 76M40.

1 Introduction

We consider a steady two-dimensional motion of a fluid past an obstacle at the bottom of a channel. The fluid is assumed to be inviscid, incompressible and the flow is irrotational. Far upstream the flow is uniform with a constant velocity U and a constant depth L (see Fig. 1). The effects of gravity and surface tension are not taken into account. The classical problem of a free streamline flow of an ideal fluid has been studied by many authors [1-7]. Toison et al. [5] used an iterative method but Bloor et al. [3] used a series truncation. The first work in this type of problems is characterized by the use of Schwarz-Christoffel formula. The latter can be used to analyze flows limited by a rectilinear paroi and an unknown free surface.

2 Formulation of the problem

We consider a steady irrotational flow of an incompressible and inviscid fluid over a triangular obstacle (Fig.1). A system of cartesian coordinates is defined with the x-axis along the bottom and the y-axis going through the apex of the triangle. As $|x|\rightarrow \infty$; far upstream and downstream the flow approaches a uniform stream with a constant velocity U and depth L.

We define the discharge $Q=UL$

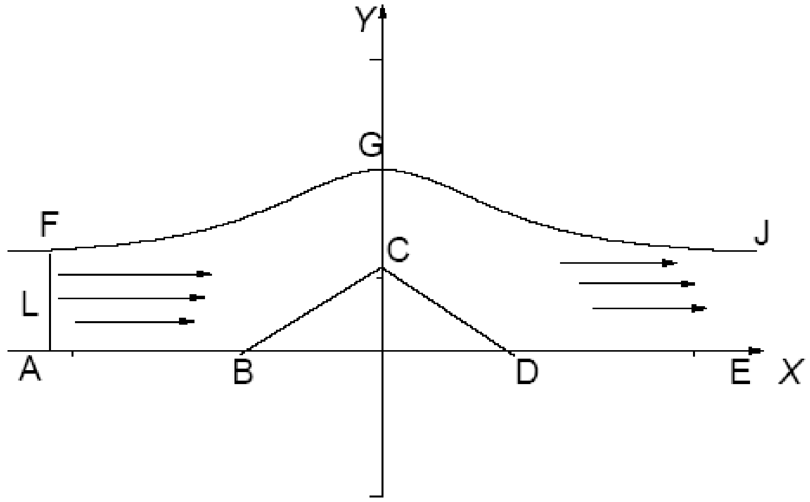

Figure 1: The z-plan.

We consider $\xi = u\text{-}iv$ as the complex velocity such that u and v are its components and $f = \Phi + i\psi$ as the complex function where Φ and ψ are the potential function and the stream function respectively.

The function f transforms the z-plan into an infinite band (see Fig. 2).

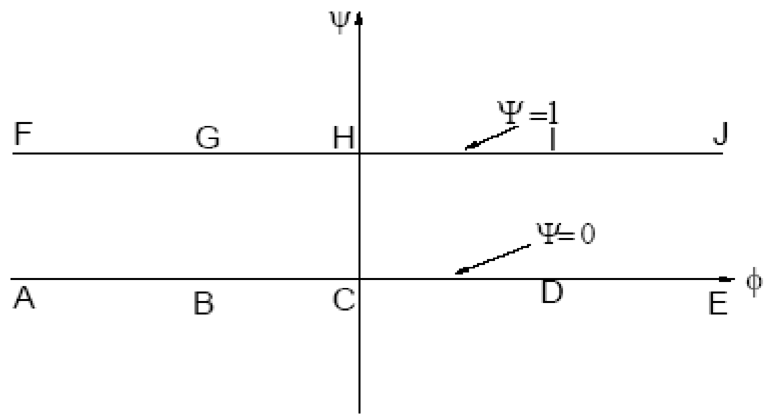

Figure 2: The f-plan.

The physical flow problem as formulated above can be considered as a boundary value problem in the potential function *Φ* (*x,y*) which can be solved numerically [8]:

$$(P_1)\begin{cases} \Delta\phi = 0 & \textit{interior of the field}, \\ \dfrac{\partial \phi}{\partial n} = 0 & \textit{on} \quad BC, CD, \\ \dfrac{\partial \phi}{\partial y} = 0 & \textit{on} \quad AB, DE, \\ \dfrac{1}{2}\left(\dfrac{\partial\phi}{\partial x}\right)^2 + \dfrac{1}{2}\left(\dfrac{\partial\phi}{\partial y}\right)^2 = cte & \text{on the unknown free surface } FHJ. \end{cases}$$

3 Resolution of the problem

To solve the problem (P_1), we initially use the method of the free streamline theory introduced by Kirchoff based on the hodograph transformation to find the form of the free surface. The complex transformation is defined by:

$$\Omega = \log\left(U / \frac{df}{dz}\right) = \log\left(\frac{U}{u - iv}\right) = \log\left(\frac{U}{q}\right) + i\theta, \tag{1}$$

Where z $=x+ iy$, $q = \sqrt{u^2 + v^2}$ and θ are the module speed and the angle between the velocity vector and the x axis, respectively. By this last transformation, the field occupied by the fluid in the z-plan is transformed into an infinite band in the Ω-plan (see Fig. 3).

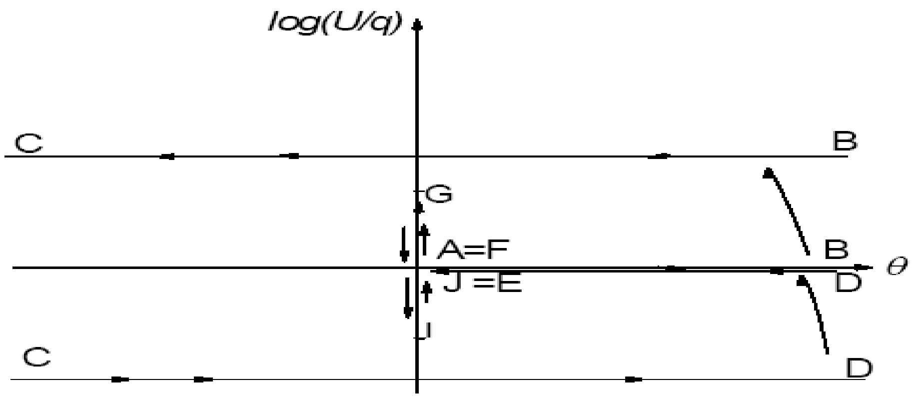

Figure 3: The *Ω*-plan.

The conforms transformation of an infinite band in the plan to the lower half-plan of another complex λ -plan, is given by the theorem of Schwarts-Christoffel, by respecting the direction and the orientation of the flow (see Fig. 4).

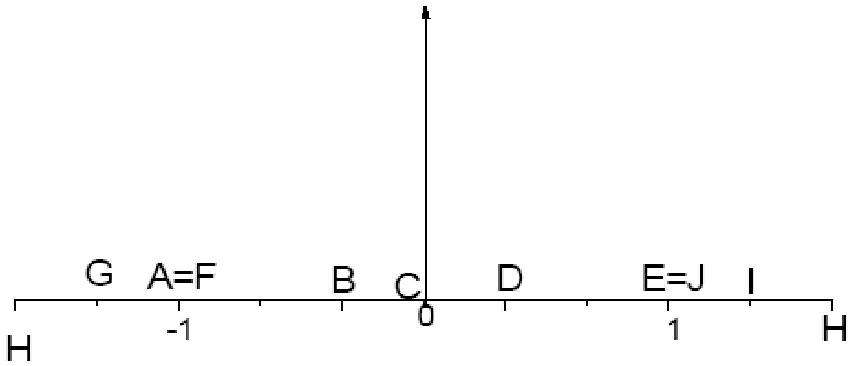

Figure 4:The λ-plan.

This transformation is given by:

$$\Omega = \frac{\beta}{\pi}\log\left[\frac{\left(\sqrt{(\lambda^2-1)}+i\frac{\sqrt{3}}{2}\right)\left(\sqrt{(\lambda^2-1)}-i\right)}{\left(\sqrt{(\lambda^2-1)}-i\frac{\sqrt{3}}{2}\right)\left(\sqrt{(\lambda^2-1)}+i\right)}\right], \tag{2}$$

Here β the angle at the bottom.

The transformation which transforms the interior of the infinite band of the f -plan towards the lower half-plan of the λ -plan is:

$$f = \frac{LU}{\pi}\log\frac{1+\lambda}{1-\lambda}, \tag{3}$$

we obtain

$$\lambda = \frac{e^{\frac{\pi f}{LU}}-1}{e^{\frac{\pi f}{LU}}+1}, \tag{4}$$

we have

$$\frac{df}{d\lambda} = \frac{-2LU}{\pi}\frac{1}{\lambda^2-1}, \tag{5}$$

Using the relation $U\dfrac{dz}{d\lambda}=U\dfrac{dz}{df}\dfrac{df}{d\lambda}$, we obtain .

$$U\frac{dz}{d\lambda}=\left(\frac{-2LU}{\pi}\ \frac{1}{\lambda^2-1}\right)\times\left[\frac{\left(\sqrt{(\lambda^2-1)}+i\dfrac{\sqrt{3}}{2}\right)\left(\sqrt{(\lambda^2-1)}-i\right)}{\left(\sqrt{(\lambda^2-1)}-i\dfrac{\sqrt{3}}{2}\right)\left(\sqrt{(\lambda^2-1)}+i\right)}\right]^{\frac{\beta}{\pi}} \tag{6}$$

By integrating (6)

$$z=\frac{-2L}{\pi}\left[\int\frac{1}{\lambda^2-1}\left[\frac{\left(\sqrt{(\lambda^2-1)}+i\dfrac{\sqrt{3}}{2}\right)\left(\sqrt{(\lambda^2-1)}-i\right)}{\left(\sqrt{(\lambda^2-1)}-i\dfrac{\sqrt{3}}{2}\right)\left(\sqrt{(\lambda^2-1)}+i\right)}\right]^{\frac{\beta}{\pi}}d\lambda+z_0\right], \tag{7}$$

where z_0 is a constant to be determined.

We give the shape of the free surface in the following two cases: $\beta = 0$ and $\beta=+\dfrac{3\pi}{4}$.

Case 1: $\beta = 0$

The relation (7) becomes

$$z(\lambda)=\frac{-2L}{\pi}\left[\int\frac{d\lambda}{\lambda^2-1}+z_0\right], \tag{8}$$

Then

$$z(\lambda)=\frac{-1}{\pi}\log\frac{\lambda-1}{\lambda+1}+z_0$$

z_0 is calculated taking $z(\lambda)=z_H$ when $\lambda\to\infty$ at the point H, then $z_0=iy_H$.

The expression of $z(\lambda)$, for $\beta=0$, is given by

$$z(\lambda)=\frac{-1}{\pi}\ \log\frac{\lambda-1}{\lambda+1}+\ iy_H\,. \tag{9}$$

The form of the free surface for $\beta = 0$ is a straight line ($y = y_H$) in the plan $z = x + iy$ (a uniform flow).

Case 2: $\beta = +\frac{3\pi}{4}$

Considering the relation (7), we obtain

$$z = \frac{-2}{\pi}\left[\int \frac{1}{\lambda^2 - 1}\left[\frac{\left(\sqrt{(\lambda^2-1)} + i\frac{\sqrt{3}}{2}\right)\left(\sqrt{(\lambda^2-1)} - i\right)}{\left(\sqrt{(\lambda^2-1)} - i\frac{\sqrt{3}}{2}\right)\left(\sqrt{(\lambda^2-1)} + i\right)}\right]^{\frac{3}{4}} d\lambda + z_0\right], \tag{10}$$

we have $z_o = (x_o, y_o) = (x(\lambda), y(\lambda)) = (0, y_H)$ when $\lambda \to \infty$, then $(x_o, y_o) = (0, y_H)$.

The integral $\int \frac{1}{\lambda^2 - 1}\left[\frac{\left(\sqrt{(\lambda^2-1)} + i\frac{\sqrt{3}}{2}\right)\left(\sqrt{(\lambda^2-1)} - i\right)}{\left(\sqrt{(\lambda^2-1)} - i\frac{\sqrt{3}}{2}\right)\left(\sqrt{(\lambda^2-1)} + i\right)}\right]^{\frac{3}{4}} d\lambda$ is difficult to evaluate analytically and a numerical integration is necessary.

A numerical integration of (10) leads to the form of the free surface for $\beta = +\frac{3\pi}{4}$ (see Figure 5).

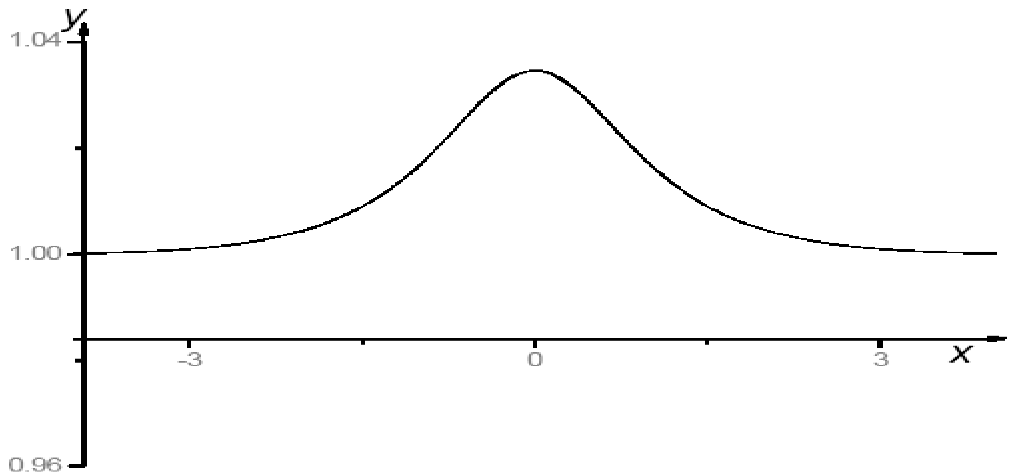

Figure 5: The Form of the free surface

4 Open Problem

The Navier- Stokes equations are a system of nonlinear partial differential equations. These equations are considered to be the pillars of fluid mechanics. They describe the motion of a fluid and used to find solutions of many practical applications in physics and engineering. However, theoretical understanding of the solution of these equations is incomplete especially when including turbulence. This remains one of the greatest unsolved problems in physics despite its immense importance in science and engineering. It would also be very interesting to find the exact solution of these equations using the method of the free streamline theory based on the hodograph method and the Schwarz-Christoffel transformation technique.

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