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Local Estimates for $\tilde{B}_n^{(\alpha)}$ Polynomials

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Abstract

In this paper, we give some local estimations for the $\tilde{B}_n^{(\alpha)}$***-type polynomials by using asymptotic properties of jacobi orthtogonal polynomials, the aim of this paper is to prove the similar results given as those in [1]. It is shown that if*** $\alpha > -\frac{1}{2}, \beta > -1$***, then the orthonormal Gegenbauer-Sobolev type polynomials fulfill the local estimate*** $|\tilde{B}_n^{(\alpha)}(t)| \leq D \frac{1}{\omega_n^{\left(\frac{\alpha}{2}+\frac{1}{4},\frac{\beta}{2}+\frac{1}{4}\right)}(x)}$ ***for all*** $t \in U_n(x)$ ***and each*** $x \in [-1,1]$ ***where*** $U_n(x)$ ***are subintervals of*** $[-1,1]$ ***defined by*** $U_n(x) = \left[x - \frac{\varphi_n(x)}{n}, x + \frac{\varphi_n(x)}{n}\right] \cap [-1,1]$ ***for*** $n \in \mathbb{N}$ ***and*** $x \in [-1,1]$ ***with*** $\varphi_n(x) := \sqrt{1-x^2} + \frac{1}{n}$.

Keyword: Gegenbauer-Sobolev- type polynomials, Local estimation, Inequalities.

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1 Introduction

Let $\omega^{(\alpha,\beta)}(x) = (1-x)^{\alpha} \cdot (1+x)^{\beta}, x \in [-1,1]$, be a Jacobi weight with $\alpha, \beta > -1$.
Let

$p_n(x) = p_n^{(\alpha,\beta)}(x) = \gamma_n^{(\alpha,\beta)} x^n + \ldots, n \in \mathbb{N}_0$

denote the unique Jacobi polynomials of precise degree n, with leading coefficients $\gamma_n^{(\alpha,\beta)} > 0$, fulfilling the orthogonal conditions

$$\int_{-1}^{1} p_n(x) p_m(x) \omega^{(\alpha,\beta)}(x) = \delta_{m,n}, n, m \in \mathbb{N}_0.$$

In [1], M. Felten, introduced modified Jacobi weights as

$$\omega_n^{(\alpha,\beta)}(x) := \left(\sqrt{1-x} + \frac{1}{n}\right)^{2\alpha} \left(\sqrt{1+x} + \frac{1}{n}\right)^{2\beta}, x \in [-1,1], n \in \mathbb{N} \tag{1}$$

He proved the following Theorem (see [1])

Theorem 1.1: Let $\alpha, \beta > -1$ and $n \in \mathbb{N}$. Then

$$| p_n^{(\alpha,\beta)}(x) | \leq C \frac{1}{\omega_n^{\left(\frac{\alpha}{2}+\frac{1}{4},\frac{\beta}{2}+\frac{1}{4}\right)}(x)} \tag{2}$$

for all $x \in [-1,1]$ with a positive constant $C = C(\alpha, \beta)$ being independent of n and x.

The above estimate, firstly appeared in [4].

Then for $\alpha, \beta \geq -\frac{1}{2}$ Felten (see [1]), extended the previous results as follows:

Theorem 1.2: Let $\alpha, \beta \geq -\frac{1}{2}$ and $n \in \mathbb{N}$. Then

$$| p_n^{(\alpha,\beta)}(t) | \leq C \frac{1}{\omega_n^{\left(\frac{\alpha}{2}+\frac{1}{4},\frac{\beta}{2}+\frac{1}{4}\right)}(x)} \tag{3}$$

for all $t \in U_n(x)$ and each $x \in [-1,1]$, where

$$U_n(x) := \left\{ t \in [-1,1] : | t - x | \leq \frac{\varphi_n(x)}{n} \right\} = \left[x - \frac{\varphi_n(x)}{n}, x + \frac{\varphi_n(x)}{n} \right] \cap [-1,1] \tag{4}$$

for $n \in \mathbb{N}$ and $x \in [-1,1]$ with $\varphi_n(x) := \sqrt{1-x^2} + \frac{1}{n}$.

In [3], T. H. Koornwinder, introduced the polynomials $\left(P_n^{(\alpha,\beta,M,N)}(x)\right)_{n=0}^{\infty}$ defined as follows:

Definition 1.3: Fix $M, N \geq 0$ and $\alpha, \beta > -1$. For $n = 0,1,2,\cdots$ define

$$P_n^{(\alpha,\beta,M,N)}(x) = \left(\frac{(\alpha+\beta+1)_n}{n!}\right)^2 \cdot \left[(\alpha+\beta+1)^{-1}(B_n M(1-x) - A_n N(1+x)\frac{d}{dx} + A_n B_n)\right] p_n^{(\alpha,\beta)}(x),$$

where

$$A_n = \frac{(\alpha+1)_n n!}{(\beta+1)_n(\alpha+\beta+1)_n} + \frac{n(n+\alpha\beta+1)M}{(\beta+1)(\alpha+\beta+1)}, \qquad (\alpha)_n \doteq \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} \tag{5}$$

and

$$B_n = \frac{(\beta+1)_n n!}{(\alpha+1)_n(\alpha+\beta+1)_n} + \frac{n(n+\alpha\beta+1)N}{(\alpha+1)(\alpha+\beta+1)}. \tag{6}$$

We call these polynomials the Koornwinder's Jacobi-type polynomials.

The above defined polynomials are orthogonal on the interval $[-1,1]$ with respect to the measure μ defined by (7)

$$\int_{-1}^{1} f(x)d\mu(x) = \frac{\Gamma(\alpha+\beta+2)}{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}\int_{-1}^{1} f(x)(1-x)^{\alpha}(1+x)^{\beta}dx$$

$$+Mf(-1)+Nf(1)$$

where $f \in C([-1,1])$ and $M, N \geq 0, \alpha, \beta > -1$.

Clearly for $M = N = 0$ one has

$$P_n^{(\alpha,\beta,0,0)}(x) = P_n^{(\alpha,\beta)}(x). \tag{8}$$

Some basic properties of $P_n^{(\alpha)}(x)$ are given as belows, (see [5], page 80)

$$P_n^{(\alpha)}(x) = \frac{\Gamma\left(\alpha+\frac{1}{2}\right)}{\Gamma(2\alpha)} \frac{\Gamma(n+2\alpha)}{\Gamma\left(n+\alpha+\frac{1}{2}\right)} P_n^{\left(\alpha-\frac{1}{2},\alpha-\frac{1}{2}\right)}(x) \tag{9}$$

Also ,

$$P_n^{(\alpha)}(-x) = (-1)^n P_n^{(\alpha)}(x). \tag{10}$$

$$\frac{d}{dx}P_n^{(\alpha)}(x) = 2\alpha P_{n-1}^{(\alpha+1)}(x) \tag{11}$$

We summarize some properties of Gegenbauer-Sobolev- type polynomials that we will need in the sequel. In [7] the representation of $\tilde{B}_n^{(\alpha)}$ in terms of gegenbauer orthonormal polynomials is

$$\tilde{B}_n^{(\alpha)}(x) = A_n(1-x^2)p_{n-4}^{(\alpha+4)}(x) + B_n\left(1-x^2\right)p_{n-2}^{(\alpha+2)}\left(x\right) + C_n p_n^{(\alpha)}\left(x\right) \tag{12}$$

where

$a)$ If $M = 0, N = 0,$ then

$$A_n \cong \frac{2^{\alpha+1}\Gamma(\alpha+1)}{\alpha+2}, B_n \cong -2^{\alpha+1}\Gamma(\alpha+1)\sqrt{\frac{\alpha+1}{\Gamma(2\alpha+3)}}, C_n \cong -A_n$$

$b)$ If $M>0, N>0,$ then

$$A_n \cong 2^{\alpha+1}\Gamma(\alpha+1)\sqrt{\frac{\alpha+2}{\Gamma(2\alpha+3)}},\ B_n \sim -n^{-2\alpha-2}, B_n \sim -n^{-2\alpha-2}, C_n \sim -n^{-2\alpha-2}$$

$c)$ If $M>0, N=0,$ then

$$A_n = 0, B_n \cong -2^{\alpha+1}\Gamma(\alpha+1)\sqrt{\frac{\alpha+1}{\Gamma(2\alpha+3)}}, C_n \sim -n^{-2\alpha-2}$$

Theorem 1.4 : ([6])Let $\alpha,\beta>-1, M, N>0.$ For every $x\in[-1,1]$ there exists a unique constant C such that the following relation:

$$\left(h_n^{\alpha,\beta,M,M}\right)^{-\frac{1}{2}}\left|P_n^{(\alpha,\beta,M,N)}(x)\right| \le C\left(1-x+\frac{1}{n^2}\right)^{-\frac{\alpha}{2}-\frac{1}{4}}\left(1+x+\frac{1}{n^2}\right)^{-\frac{\beta}{2}-\frac{1}{4}},$$

holds for every $n\in\mathbb{N}$.

For properties of Jacobian polynomials (see [4],[5]), we get the following estimation for the $\tilde{B}_n^{(\alpha)}$-type polynomials:

$$\left|\tilde{B}_n^{(\alpha)}(\cos\theta)\right| = \begin{cases} 0\left(\theta^{-\alpha-\frac{1}{2}}\right), & if\ \frac{c}{n}\le\theta\le\frac{\pi}{2} \\ 0\left(n^{\alpha+\frac{1}{2}}\right), & if\ 0\le\theta\le\frac{c}{n} \end{cases} \tag{13}$$

for $\alpha>-\frac{1}{2}$ and $n\ge 1$.

The aim of this paper is to prove the similar results as those given in Theorems 1.1 and 1.2 for $\tilde{B}_n^{(\alpha)}$-type polynomials, when $\alpha>-\frac{1}{2}, \beta\ge -1,$

2 Main Results

The following Theorem is the main result of this note.

Theorem 2.1 Let $\alpha>-\frac{1}{2}, \beta>-1$ and $n\in\mathbb{N}$. Then

$$|\tilde{B}_n^{(\alpha)}(x)| \le D \frac{1}{\omega_n^{\left(\frac{\alpha}{2}+\frac{1}{4},\frac{\beta}{2}+\frac{1}{4}\right)}(x)} \tag{14}$$

for all $x \in [-1,1]$ with a positive constant $D = D(\alpha,\beta)$ being independent of n and x.

Proof: Proof of the Theorem is similar to Theorem 2.1 in [1]. Let $x \in [0,1]$, and let $\theta \in \left[0, \frac{\pi}{2}\right]$ such that $x = \cos\theta.$ From (13) one has the following estimation

$$\left|\tilde{B}_n^{(\alpha)}(\cos\theta)\right| \le C \begin{cases} \theta^{-\alpha-\frac{1}{2}}, \; if \; \frac{c}{n} \le \theta \le \frac{\pi}{2} \\ n^{\alpha+\frac{1}{2}}, \; if \; 0 \le \theta \le \frac{c}{n} \end{cases} \tag{15}$$

If in the last relation, we substitute $x = \cos\theta$ we have:

$$\left|\tilde{B}_n^{(\alpha)}(x)\right| \le C \begin{cases} n^{\alpha+\frac{1}{2}}, \; if \; 0 \le \arccos x \le \frac{c}{n} \\ \left(\arccos x\right)^{-\left(\alpha+\frac{1}{2}\right)}, \; if \; \frac{c}{n} \le \arccos x \le \frac{\pi}{2} \end{cases} \tag{16}$$

where C is fixed positive constant being independent of n and θ.

In what follows we will make use of the following estimates:

$$\frac{\pi}{2}\sqrt{1-x} = \frac{\pi}{\sqrt{2}}\sqrt{\frac{1-x}{2}} = \frac{\pi}{\sqrt{2}}\sin\frac{t}{2} \ge \frac{\pi}{\sqrt{2}}\left(\frac{2}{\pi}\cdot\frac{t}{\sqrt{2}}\right) = t = \arccos x \tag{17}$$

and

$$\sqrt{2}\sqrt{1-x} = 2\sqrt{\frac{1-x}{2}} = 2\sin\frac{t}{2} \le 2\cdot\frac{t}{2} = t = \arccos x \tag{18}$$

for $\alpha > -\frac{1}{2}$, we have $-\left(\alpha + \frac{1}{2}\right) < 0.$

If $0 \le \arccos x \le \frac{c}{n}$, then from relations (15) and (17) we obtain:

$$|\tilde{B}_n^{(\alpha)}(x)| \le C_6 n^{\alpha+\frac{1}{2}} = C_6\left(\frac{c}{n}+\frac{\sqrt{2}}{n}\right)^{-\left(\alpha+\frac{1}{2}\right)} \le C_7\left(\sqrt{1-x}+\frac{1}{n}\right)^{-\left(\alpha+\frac{1}{2}\right)}.$$

If $\frac{c}{n} \le \arccos x \le \frac{\pi}{2}$, again according to relations (16) and (18) we have:

$$|\tilde{B}_n^{(\alpha)}(x)| \le C_8(\arccos x)^{-\left(\alpha+\frac{1}{2}\right)} = C_9(\arccos x + \arccos x)^{-\left(\alpha+\frac{1}{2}\right)} \le C_{10}\left(\sqrt{1-x}+\frac{1}{n}\right)^{-\left(\alpha+\frac{1}{2}\right)}.$$

From previous cases we have proved that:

$$|\tilde{B}_n^{(\alpha)}(x)| \le C_{11}(\alpha,\beta)\left(\sqrt{1-x}+\frac{1}{n}\right)^{-\left(\alpha+\frac{1}{2}\right)} \cdot \left(\sqrt{1+x}+\frac{1}{n}\right)^{-\left(\beta+\frac{1}{2}\right)}$$

for all $x \in [0,1], n \in \mathbb{N}$ and $\alpha > -\frac{1}{2} \beta > -1.$ From (9) and (12) we obtain:

$$|\tilde{B}_n^{(\alpha)}(x)| \le C_{11}(\alpha,\beta)\left(\sqrt{1+x}+\frac{1}{n}\right)^{-\left(\alpha+\frac{1}{2}\right)} \cdot \left(\sqrt{1-x}+\frac{1}{n}\right)^{-\left(\beta+\frac{1}{2}\right)}$$

for all $x \in [-1,0), n \in \mathbb{N}$ and $\alpha > -\frac{1}{2}, \beta > -1.$

The proof is completed.

Next, we will show that the local estimates of previous Theorem, can be further extended. We will prove that $|\tilde{B}_n^{(\alpha)}(x)|$ in (14) can be replaced by $|\tilde{B}_n^{(\alpha)}(t)|$, when ever t is in the interval $U_n(x) = \left[x - \frac{\varphi_n(x)}{n}, x + \frac{\varphi_n(x)}{n}\right] \cap [-1,1]$. In order to do that we will make use of the following Lemma (see [1]).

Lemma 2.2
Let $a, b \le 0, n \in \mathbb{N}$ and $x \in [-1,1]$. Then

$$\omega_n^{(a,b)}(t) \le 16^{-(a+b)} \omega_n^{(a,b)}(x) \tag{19}$$

for all $t \in U_n(x)$.

Theorem 2.3 Let $\alpha > -1, \beta \ge -\frac{1}{2}$ and $n \in \mathbb{N}$. Then

$$|B_n^{(\alpha)}(t)| \le D \frac{1}{\omega_n^{\left(\frac{\alpha}{2}+\frac{1}{4},\frac{\beta}{2}+\frac{1}{4}\right)}(x)} \tag{20}$$

for all $t \in U_n(x)$ and each $x \in [-1,1]$ where $D = D(\alpha,\beta)$ is a positive constant independent of n, t and x.

Proof. Since $\alpha > -1, \beta \geq -\frac{1}{2}$

it follows that $\frac{\alpha}{2}+\frac{1}{4}, \frac{\beta}{2}+\frac{1}{4} \geq 0$. Therefore, by Lemma 2.2 with $a = -\frac{\alpha}{2}-\frac{1}{4}$ and $\beta = -\frac{\alpha}{2}-\frac{1}{4}$ we obtain

$$\frac{1}{\omega_n^{\left(\frac{\alpha}{2}+\frac{1}{4},\frac{\beta}{2}+\frac{1}{4}\right)}(x)} = \omega_n^{\left(-\frac{\alpha}{2}-\frac{1}{4},-\frac{\beta}{2}-\frac{1}{4}\right)}(x) \leq \frac{4^{\alpha+\beta+1}}{\omega_n^{\left(\frac{\alpha}{2}+\frac{1}{4},\frac{\beta}{2}+\frac{1}{4}\right)}(x)}$$

for all $t \in U_n(x)$. Applying Theorem 2.1 yields inequality (15) for all $t \in U_n(x)$, as claimed.

Corollary 2.4.

For all $n \in \mathbb{N}$ with $\alpha > -\frac{1}{2}, \beta \geq -\frac{1}{2}, x \in [-1,1]$ holds:

$$\int_{U_n(x)} |B_n^{(\alpha)}(t)|^2\, \omega_n^{(\alpha,\beta)}(t)dt \leq \frac{D(\alpha,\beta)}{n}.$$

Proof: Applying Theorem 2.3 we obtain:

$$\int_{U_n(x)} |\tilde{B}_n^{(\alpha)}(t)|^2\, \omega_n^{(\alpha,\beta)}(t)dt \leq \frac{D}{\omega_n^{\left(\alpha+\frac{1}{2},\beta+\frac{1}{2}\right)}(x)} \int_{U_n(x)} \omega_n^{(\alpha,\beta)}(t)dt.$$

Using the following result from [2] we obtain.

$$\int_{U_n(x)} \omega_n^{(\alpha,\beta)}(t)dt \leq \frac{D}{n}\omega_n^{\left(\alpha+\frac{1}{2},\beta+\frac{1}{2}\right)}(x)$$

and thus, the proof is completed.

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