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Timelike Biharmonic Curves Of AW(k)-Type In The Lorentzian Heisenberg Group $Heis^3$

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Abstract

In this paper, we study non-geodesic timelike biharmonic curves in the Lorentzian Heisenberg group $Heis^3$ and we show that all of them are helices. We also characterize all non-geodesic timelike biharmonic curves in the Lorentzian Heisenberg group $Heis^3$ and prove that all of them are of type $AW(3)$.

Keywords: *Biharmonic curve,curvature, Heisenberg group, helices, torsion.*

1 Introduction

Let $f:(M,g)\to(N,h)$ be a smooth map between two Lorentzian manifolds. The bienergy $E_2(f)$ of f over compact domain $\Omega\subset M$ is defined by

$$E_2(f)=\int_\Omega h(\tau(f),\tau(f))dv_g,$$

where $\tau(f)=\mathrm{trace}_g\nabla df$ is the tension field of f and dv_g is the volume form of M. Using the first variational formula one sees that f is a biharmonic map if and only if its bitension field vanishes identically, i.e.,

$$\tau_2(f):=-\Delta^f(\tau(f))-\mathrm{trace}_g R^N(df,\tau(f))df=0,$$

where

$$\Delta^f=-\mathrm{trace}_g(\nabla^f)^2=-\mathrm{trace}_g\left(\nabla^f\nabla^f-\nabla^f_{\nabla^M}\right)$$

is the Laplacian on sections of the pull-back bundle $f^{-1}(TN)$and R^N is the curvature operator of (N,h) defined by

$$R(X,Y)Z=[\nabla_X,\nabla_Y]Z-\nabla_{X,Y]}Z.$$

A Riemannian submanifold with vanishing Laplacian of mean curvature vector ΔH is defined as a biharmonic submanifold by B.-Y. Chen [7]. In [9], it was proved that the only biharmonic curves in an Euclidean space are straight lines. In [1], the classification of curves satisfying $\Delta H = \lambda H$ and $\Delta^{\perp} H = \lambda H$ in a real space form were given. By looking the Chen's formula (Lemma 4.1, [10]), one sees that the Laplacian in the normal bundle of H, $\Delta^{\perp} H$, is an ingredient of the normal part of ΔH to M and $\Delta^{\perp} H = 0$ is less restrictive than $\Delta H = 0$. However, the condition $\Delta H = \lambda H$ does not imply $\Delta^{\perp} H = \lambda H$. The concepts of submanifolds of type $AW(k)$ are defined ; in particular, curves of type $AW(k)$ were investigated in [2].

In this paper, we study non-geodesic timelike biharmonic curves in the Lorentzian Heisenberg group Heis^3 and we show that all of them are helices. We also characterize all non-geodesic timelike biharmonic curves in the Lorentzian Heisenberg group Heis^3 and prove that all of them are of type $AW(3)$.

2 The Lorentzian Heisenberg Group Heis^3

The Lorentzian Heisenberg group Heis^3 can be seen as the space $\mathbb{R}^3$ endowed with the following multiplication:

$$(\bar{x}, \bar{y}, \bar{z})(x, y, z) = (\bar{x} + x, \bar{y} + y, \bar{z} + z - \bar{x}y + x\bar{y}).$$

Heis^3 is a three-dimensional, connected, simply connected and 2-step nilpotent Lie group.

The Lorentz metric g is given by

$$g = -dx^2 + dy^2 + (xdy + dz)^2.$$

The Lie algebra of Heis^3 has an orthonormal basis

$$e_1 = \frac{\partial}{\partial z}, e_2 = \frac{\partial}{\partial y} - x\frac{\partial}{\partial z}, e_3 = \frac{\partial}{\partial x}$$

for which we have the Lie products

$$e_2, e_3] = 2e_1, [e_3, e_1] = 0, [e_2, e_1] = 0$$

with

$$g(e_1, e_1) = g(e_2, e_2) = 1,\ g(e_3, e_3) = -1.$$

Proposition 2.1. *For the covariant derivatives of the Levi-Civita connection of the left-invariant metric g, defined above, the following is true:*

$$\nabla = \begin{pmatrix} 0 & e_3 & e_2 \\ e_3 & 0 & e_1 \\ e_2 & -e_1 & 0 \end{pmatrix}, \tag{2.1}$$

where the (i, j) *-element in the table above equals* $\nabla_{e_i} e_j$ *for our basis*

$$\{e_k, k = 1,2,3\} = \{e_1, e_2, e_3\}.$$

We adopt the following notation and sign convention for Riemannian curvature operator:

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{X,Y]} Z.$$

The Riemannian curvature tensor is given by

$$R(X,Y,Z,W) = -g(R(X,Y)Z,W).$$

Moreover we put

$$Rabc = R(e_a, e_b)e_c, R_{abcd} = R(e_a, e_b, e_c, e_d),$$

where the indices a, b, c and d take the values $1,2$ and 3.

$$R_{232} = -3R_{131} = -3e_3,$$
$$R_{133} = -R_{122} = e_1,$$
$$R_{233} = -3R_{121} = -3e_2,$$

and

$$R_{1212} = -1,\ R_{1313} = 1,\ R_{2323} = -3.$$

3 Timelike Biharmonic Curves In The Lorentzian Heisenberg Group Heis^3

Let $\gamma : I \to Heis^3$ be a timelike curve on the Lorentzian Heisenberg group Heis^3 parametrized by arc length. Let $\{T, N, B\}$ be the Frenet frame fields tangent to the Lorentzian Heisenberg group Heis^3 along γ defined as follows:

T is the unit vector field γ' tangent to γ, N is the unit vector field in the direction of $\nabla_T T$ (normal to γ), and B is chosen so that $\{T, N, B\}$ is a positively oriented orthonormal basis. Then, we have the following Frenet formulas:

$$\begin{aligned} \nabla_T T &= \kappa N, \\ \nabla_T N &= \kappa T + \tau B, \\ \nabla_T B &= -\tau N, \end{aligned} \tag{3.1}$$

where $\kappa = |\tau(\gamma)| = |\nabla_T T|$ is the curvature of γ and τ is its torsion.

With respect to the orthonormal basis $\{e_1, e_2, e_3\}$ we can write

$$T = T_1 e_1 + T_2 e_2 + T_3 e_3,$$
$$N = N_1 e_1 + N_2 e_2 + N_3 e_3,$$
$$B = T \times N = B_1 e_1 + B_2 e_2 + B_3 e_3.$$

Theorem 3.1. *(see [20]) Let $\gamma : I \to Heis^3$ be a non-geodesic timelike curve on the Lorentzian Heisenberg group $Heis^3$ parametrized by arc length. γ is a timelike non-geodesic biharmonic curve if and only if*

$$\begin{aligned} &\kappa = \text{constant} \neq 0, \\ &\kappa^2 - \tau^2 = -1 + 4B_1^2, \\ &\tau' = -2N_1 B_1. \end{aligned} \tag{3.2}$$

Corollary 3.2. *(see [20]) Let $\gamma : I \to Heis^3$ be a non-geodesic timelike curve on the Lorentzian Heisenberg group $Heis^3$ parametrized by arc length. γ is biharmonic if and only if*

$$\begin{aligned} &\kappa = constant \neq 0, \\ &\tau = constant, \\ &N_1 B_1 = 0, \\ &\kappa^2 - \tau^2 = -1 + 4B_1^2. \end{aligned} \tag{3.3}$$

Theorem 3.3. *(see [20]) Let $\gamma : I \to Heis^3$ be a non-geodesic timelike curve on Lorentzian Heisenberg group $Heis^3$ parametrized by arc length. If $N_1 \neq 0$ then γ is not biharmonic.*

Theorem 3.4. *(see [20]) Let $\gamma : I \to Heis^3$ be a non-geodesic timelike biharmonic curve on the Lorentzian Heisenberg group $Heis^3$ parametrized by arc length. If $N_1 = 0$, then*

$$T(s) = \sinh\phi_0 e_1 + \cosh\phi_0 \sinh\psi(s) e_2 + \cosh\phi_0 \cosh\psi(s) e_3, \tag{3.4}$$

where $\phi_0 \in \mathbb{R}$.

4 Biharmonic Curves of AW(k)-type

Consider a curve in a 3-dimensional Riemannian manifold. Chen [6] proved the following identity:

$$\Delta H = -\nabla_{\gamma'} \nabla_{\gamma'} \nabla_{\gamma'} \gamma', \tag{4.1}$$

where H is the mean curvature vector. Moreover, the Laplacian of the mean curvature in the normal bundle is defined by

$$\Delta^{\perp} H = -\nabla^{\perp}_{\gamma'} \nabla^{\perp}_{\gamma'} \nabla^{\perp}_{\gamma'} \gamma'. \tag{4.2}$$

A curve $\gamma(s)$ in a Riemannian manifold M is called a curve with proper mean curvature vector field [8] if $\Delta H = \lambda H$, where λ is a function.

Lemma 4.1. *Let $\gamma(s)$ be a timelike biharmonic curve in in the Lorentzian Heisenberg group $Heis^3$. Then,*

$$\nabla_{\gamma'}\nabla_{\gamma'}\gamma' = \kappa^2 T + \kappa\tau B, \tag{4.3}$$

$$\nabla_{\gamma'}\nabla_{\gamma'}\nabla_{\gamma'}\gamma' = (\kappa^3 - \kappa\tau^2)N. \tag{4.4}$$

Proof. From (3.1) we have

$$\nabla_{\gamma'}\nabla_{\gamma'}\gamma' = \kappa' N + \kappa^2 T + \kappa\tau B, \tag{4.5}$$

and

$$\nabla_{\gamma'}\nabla_{\gamma'}\nabla_{\gamma'}\gamma' = \left(3\kappa\kappa'\right)T + (\kappa'' + \kappa^3 - \kappa\tau^2)N + \left(2\kappa'\tau + \kappa\tau'\right)B. \tag{4.6}$$

Since $\gamma(s)$ be a timelike biharmonic curve κ and τ are constants. Substituting κ and τ are constants in (4.5) and (4.6) we have(4.3) and (4.4).

Theorem 4.2. *Let $\gamma(s)$ be a biharmonic curve in the Lorentzian Heisenberg group $Heis^3$. Then, γ has parallel mean curvature vector field if and only if $\kappa = 0$.*

Definition 4.3. *(see [2]) A Frenet curve $\gamma(s)$ is said to be*
(i) of type $AW(1)$ if $X_3(s) = 0$,
(ii) of type $AW(2)$ if

$$\|X_2(s)\|^2 X_3(s) = \langle X_3(s), X_2(s)\rangle X_2(s), \tag{4.7}$$

(iii) of type $AW(3)$ if

$$\|X_1(s)\|^2 X_3(s) = \langle X_3(s), X_1(s)\rangle X_1(s), \tag{4.8}$$

where

$$X_1(s) = (\gamma'')^{\perp}(s),\ X_2(s) = (\gamma''')^{\perp}(s),\ X_3(s) = (\gamma'''')^{\perp}(s). \tag{4.9}$$

Let $\gamma(s)$ be a timelike biharmonic curve in the Lorentzian Heis^3. Then from (3.1), (4.3), (4.4) we get

$$X_1(s) = \kappa N, \tag{4.10}$$
$$X_2(s) = \kappa\tau B, \tag{4.11}$$
$$X_3(s) = \kappa(\kappa^2 - \tau^2)N. \tag{4.12}$$

Theorem 4.4. *A timelike biharmonic curve in in the Lorentzian* $Heis^3$ *is type* AW (1) *if and only if*

$$B_1 = \pm\frac{1}{2}. \qquad (4.13)$$

Proof. Now suppose that $\gamma(s)$ is a timelike biharmonic curve of type AW (1). From $X_3(s) = 0$ we obtain

$$\kappa(\kappa^2 - \tau^2) = 0. \qquad (4.14)$$

Since Theorem 3.1 we have $\kappa \neq 0,$ then

$$\kappa^2 - \tau^2 = 0.$$

Using second equation of (3.2) we get

$$-1 + 4B_1^2 = 0 \text{or} B_1 = \pm\frac{1}{2}.$$

Theorem 4.5. *A timelike biharmonic curve in in the Lorentzian* $Heis^3$ *is type* AW (2) *if and only if*

$$B_1 = \pm\frac{1}{2} \text{or} \tau = 0. \qquad (4.15)$$

Proof. Since $\gamma(s)$ is AW(2)-type we have

$$\kappa^3\tau^2(\kappa^2 - \tau^2) = 0.$$

Using second equation of (3.2) we get (4.15).

Theorem 4.6. *All the timelike biharmonic curves in the Lorentzian* $Heis^3$ *are type* AW (3).

Proof. Suppose that $\gamma(s)$ is a timelike biharmonic curve of type AW (3). Equations (4.8) are provided for each s. Hence, the proof is completed.

5 Open Problem

In this work, we study timelike biharmonic curves in the Lorentzian Heisenberg group Heis^3. We have given some explicit characterizations of biharmonic curves. Additionally, problems such as; investigation timelike biharmonic curves or extending such kind curves to higher dimensional Heisenberg group can be presented as further researches.

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